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t-GENERALIZED SUPPLEMENTED MODULES

t-УЗАГАЛЬНЕНІ ДОПОВНЕНІ МОДУЛІ

In this paper, t-generalized supplemented modules are defined by starting from the generalized $\oplus$-supplemented modules. In addition, we present examples separating the t-generalized supplemented modules, supplemented modules, and generalized $\oplus$-supplemented modules and also show the equality of these modules for projective and finitely generated modules. Moreover, we define cofinitely t-generalized supplemented modules and give the characterization of these modules. Furthermore, for any ring R, we show that any finite direct sum of t-generalized supplemented R-modules is t-generalized supplemented and an arbitrary direct sum of cofinitely t-generalized supplemented R-modules is a cofinitely t-generalized supplemented module.

Доведено, що t-узагальнені доповнені модулі визначені на основі узагальнених $\oplus$-доповнених модулів. Крім того, наведено приклади, що відокремлюють t-узагальнені доповнені модулі, доповнені модулі та узагальнені $\oplus$-доповнені модулі, а також доведено рівність цих модулів для проективних та скінченнопороджених модулів. Також визначено кофінітно t-узагальнені доповнені модулі та наведено характеристику цих модулів. Більш того, для кожного кільця R доведено, що будь-яка скінченна пряма сума t-узагальнених доповнених R-модулів є t-узагальненою доповненою, а також будь-яка пряма сума кофінітно t-узагальнених доповнених R-модулів є кофінітно t-узагальненим доповненим R-модулем.

1. Introduction. Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R-module. We will denote a submodule N of M by $N \leq M$ and a proper submodule K of M by $K < M$. Let M be an R-module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* submodule of M and denoted by $N \ll M$. Let M be an R-module and $N \leq M$. If there exists a submodule K of M such that $M = N + K$ and $N \cap K = 0$, then N is called a *direct summand* of M and it is denoted by $M = N \oplus K$. For any module M we have $M = M \oplus 0$. $\operatorname{Rad} M$ indicates the radical of M. An R-module M is said to be *simple* if M have no proper submodules with distinct zero. Let M be an R-module. M is called a (*semi*) *hollow* module if every (*finitely generated*) proper submodule of M is small in M. M is called *local* module if M has a largest submodule, i.e., a proper submodule which contains all other proper submodules. A module M is called *distributive* [10] if for every submodules K, L, N of M, $N+(K\cap L) = (N+K)\cap(N+L)$, or equivalently, $N\cap(K+L) = (N\cap K)+(N\cap L)$ holds. Let U and V be submodules of M. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* [12] of U in M. M is called a *supplemented* module if every submodule of M has a supplement. M is called $\oplus$-*supplemented* [6, 8] module if every submodule of M has a supplement that is a direct summand of M. Let M be an R-module and U, V be submodules of M. V is called a *generalized supplement* [2, 11, 13] of U in M if $M = U + V$ and $U \cap V \leq \operatorname{Rad} V$. M is called *generalized supplemented* or briefly *GS-module* if every submodule of M has a generalized supplement and clearly that every supplement submodule is a generalized supplement. M is called a *generalized* $\oplus$-*supplemented* [4, 5, 9, 10] module if every submodule of M has a generalized supplement that is a direct summand in M. In this paper we generalize these modules. A submodule N of an R-module M is called *cofinite* if M/N is finitely generated. M is called *cofinitely supplemented* [1] if every cofinite submodule of M has a supplement in M. M is called *semiperfect* module if every factor module of M has a projective cover.

In the next section, we will define t-generalized supplemented modules and examine the relationship between these modules, supplemented modules and generalized $\oplus$-supplemented modules. For any ring R, we will show that any finite direct sum of t-generalized supplemented modules is a t-generalized supplemented module and find conditions for t-generalized supplemented modules which make factor modules of these t-generalized supplemented modules.

In the last section, we will define cofinitely t-generalized supplemented modules and investigate the relationship with cofinitely supplemented modules. We also show that any direct sum of cofinitely t-generalized supplemented R-modules is also a cofinitely t-generalized supplemented R-module for any ring R.

Lemma 1.1. *Let M be an R-module and N, K be submodules of M. If $N+K$ has a generalized supplement X in M and $N \cap (K+X)$ has a generalized supplement Y in N, then $X+Y$ is a generalized supplement of K in M.*

Proof. See [4], Lemma 3.2.

Lemma 1.2. *Let M be a projective module. Consider the following conditions*:

(i) *M is a semiperfect module.*

(ii) *M is a generalized $\oplus$-supplemented module.*

Then (i) $\Rightarrow$ (ii) *holds and if M is a finitely generated module then* (ii) $\Rightarrow$ (i) *also holds.*

Proof. See [10], Lemma 2.2.

2. *t*-Generalized supplemented modules.

Definition 2.1. *Let M be an R-module. M is called a t-generalized supplemented module if every submodule of M has a generalized supplement which is also a supplement in M. Clearly generalized $\oplus$-supplemented modules are t-generalized supplemented. But the converse implication fails to be true. This will be shown in Example* 2.4.

It is also clear that although every supplemented module is a t-generalized supplemented the converse of this statement is not always true. We will show this situation in Examples 2.1 – 2.3. Since hollow and local modules are supplemented, they are t-generalized supplemented modules.

It is well-known that every $\oplus$-supplemented module is generalized $\oplus$-supplemented (see [4], Example 3.11). Now we will give a situation when the converse is true.

Lemma 2.1. *If M is a finitely generated R-module then M is generalized $\oplus$-supplemented if and only if M is $\oplus$-supplemented.*

Proof. ($\Rightarrow$) Let N be a submodule of M. Since M is generalized $\oplus$-supplemented, there exists a generalized supplement K of N such that K is a direct summand in M. Hence there exists submodules K and L of M such that $M = N + K$, $N \cap K \leq \operatorname{Rad} K$ and $M = K \oplus L$. Since M is finitely generated, we have K is finitely generated and $\operatorname{Rad} K \ll K$. Therefore $N \cap K \ll K$ and K is a supplement of N in M. As a result M is $\oplus$-supplemented.

($\Leftarrow$) Clear.

The following lemma will be used to prove Theorem 2.1.

Lemma 2.2. *Let M be an R-module with $M = M_1 \oplus M_2$ and K, L be submodules of M_1 such that K is a supplement of L in M_1. Then K is a supplement of $M_2 + L$ in M.*

Proof. Let $M_2 + L + N = M$ with $N \leq K$. Hence $M_1 = M_1 \cap M = M_1 \cap (L + N + M_2) = L + N + (M_1 \cap M_2) = L + N$. Since $N \leq K$ and K is a supplement of L in M_1, we get $N = K$. Therefore K is a supplement of $M_2 + L$ in M.

Lemma 2.3. *Let $M = M_1 \oplus M_2$. If K is a supplement submodule in M_1 and T is a supplement submodule in M_2, then $K + T$ is a supplement submodule in M.*

Proof. Suppose that K is a supplement of U in M_1 and T is a supplement of V in M_2. In this case $M_1 = U + K$, $U \cap K \ll K$ and $M_2 = V + T$, $V \cap T \ll T$. Since $M_1 = U + K$ and $M_2 = V + T$, $M = M_1 + M_2 = U + V + K + T$. It is easy to check that $(U + K + V) \cap T \ll T$ and $(V + T + U) \cap K \ll K$. Hence $(U + V) \cap (K + T) \subseteq (U + V + T) \cap K + (U + V + K) \cap T \ll K + T$. Therefore $K + T$ is a supplement of $U + V$ in M.

The next result generalizes Lemma 2.3 which is easily proved.

***Corollary* 2.1.** *Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. For $1 \leq i \leq n$, if K_i is a supplement submodule in M_i, $K_1 + K_2 + \ldots + K_n$ is a supplement submodule in M.*

Theorem 2.1. *For any arbitrary ring R, the finite direct sum of t-generalized supplemented R-modules is t-generalized supplemented.*

Proof. Let n be any positive integer, $\{M_i\}_{1 \leq i \leq n}$ be any finite collection of t-generalized supplemented R-modules and $M = M_1 \oplus M_2 \oplus \ldots \oplus M_n$. Assume that $n = 2$. Let $M = M_1 \oplus M_2$ and N be any submodule of M. Then $M = M_1 + M_2 + N$. Since M_2 is t-generalized supplemented, we can say that $M_2 \cap (M_1 + N)$ has a generalized supplement K in M_2 such that K is a supplement in M_2. So K is a generalized supplement of $M_1 + N$ in M. Since M_1 is t-generalized supplemented, $M_1 \cap (K + N)$ has a generalized supplement L in M_1 such that L is a supplement in M_1. Thus we get $K + L$ is a generalized supplement of N in M (see [4]). Since K is a supplement in M_2 and L is a supplement in M_1, then by Lemma 2.3 $K + L$ is a supplement in M. Therefore M is t-generalized supplemented. The rest of the proof can be completed by induction on n.

The relationship between the concepts "t-generalized supplemented" and "supplemented" is expressed in the following lemma.

Lemma 2.4. *Let M be a finitely generated module. Then M is t-generalized supplemented if and only if M is supplemented.*

Proof. ($\Rightarrow$) Let N be any submodule of M. Since M is t-generalized supplemented then there exists $K \leq M$ such that $M = N + K$, $N \cap K \subseteq \operatorname{Rad} K$ and K is a supplement in M. Since M is finitely generated, we obtain $\operatorname{Rad} M \ll M$. Hence $N \cap K \subseteq \operatorname{Rad} K \subseteq \operatorname{Rad} M \ll M$ and it follows that $N \cap K \ll K$. This means that K is a supplement of N in M and so M is supplemented.

($\Leftarrow$) Clear from definitions.

Lemma 2.5. *Let M be an R-module. If $\operatorname{Rad} M = M$, then M is t-generalized supplemented.*

Proof. Let N be any submodule of M. Since $N + M = M$ and $N \cap M \subseteq M = \operatorname{Rad} M$, we get that M is a generalized supplement of N. On the other hand M is a supplement of 0. Hence M is a t-generalized supplemented.

It is easy to see that every semihollow module is t-generalized supplemented. Now we give some examples of modules, which is t-generalized supplemented but not supplemented. Thus the following examples are given to separate the structures of t-generalized supplemented, supplemented and generalized $\oplus$-supplemented.

***Example* 2.1.** Consider the $\mathbb{Z}$-module $\mathbb{Q}$. Since $\mathbb{Q}$ has no maximal submodule, we have $\operatorname{Rad} \mathbb{Q} = \mathbb{Q}$. By Lemma 2.5, $\mathbb{Q}$ is t-generalized supplemented module. But it is well known that $\mathbb{Q}$ is not supplemented (see [3], Example 20.12).

***Example* 2.2.** Let M be a non-torsion $\mathbb{Z}$-module with $\operatorname{Rad} M = M$. Since $\operatorname{Rad} M = M$ then M is t-generalized supplemented. But M is not supplemented [14].

***Example* 2.3.** Consider the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \mathbb{Z}/p\mathbb{Z}$, for any prime p. In this case $\operatorname{Rad} M \neq$ $\neq M$. Moreover, M is t-generalized supplemented but not supplemented [4].

***Example* 2.4.** Let R be a commutative local ring which is not a valuation ring. Let a and b be elements of R, where neither of them divides the other. By taking a suitable quotient ring, we may assume that $(a) \cap (b) = 0$ and $am = bm = 0$ where m is the maximal ideal of R. Let F be a free R-module with generators x_1, x_2, and x_3, K be the submodule generated by $ax_1 - bx_2$ and $M = F/K$. Thus,

$$M = \frac{Rx_1 \oplus Rx_2 \oplus Rx_3}{R\,(ax_1 - bx_2)} = (R\,\overline{x_1} + R\,\overline{x_2}) \oplus R\,\overline{x_3}.$$

Here M is not $\oplus$-supplemented. But $F = Rx_1 \oplus Rx_2 \oplus Rx_3$ is completely $\oplus$-supplemented [6].

Since F is completely $\oplus$-supplemented, F is supplemented. Since a factor module of a supplemented module is supplemented, we have M is supplemented. So M is t-generalized supplemented. Separately, since M is finitely generated and not $\oplus$-supplemented, M is not generalized $\oplus$-supplemented by Lemma 2.1.

Lemma 2.6. *Let $M = M_1 \oplus M_2$. Then M_2 is t-generalized supplemented if and only if for every submodule N/M_1 of M/M_1, there exists a supplement K in M such that $K \leq M_2$, $M = K + N$ and $N \cap K \subseteq \operatorname{Rad} M$.*

Proof. $(\Rightarrow)$ Assume that M_2 is t-generalized supplemented. Let $N/M_1 \leq M/M_1$. Since M_2 is t-generalized supplemented, there exists a generalized supplement module K of $N \cap M_2$ such that K is a supplement in M_2. Hence there exists $K' \leq M_2$ such that $M_2 = N \cap M_2 + K$, $N \cap M_2 \cap K \subseteq \operatorname{Rad} K$ and $M_2 = K + K'$, $K \cap K' \ll K$. The equality $M = M_1 + M_2$ implies that $M = M_1 + N \cap M_2 + K = N + K$. On the other hand $N \cap K \subseteq \operatorname{Rad} M$. Since K is a supplement of K' in M_2 and $M = M_1 \oplus M_2$, we obtain that K is a supplement of $M_1 + K'$ in M by Lemma 2.2. Therefore K is a supplement in M.

$(\Leftarrow)$ Suppose that M/M_1 satisfies hypothesis properties. Let $H \leq M_2$. Consider the submodule $(H \oplus M_1)/M_1 \leq M/M_1$. By hypothesis, there exists a supplement L in M such that $L \leq M_2$, $M = (L + H) \oplus M_1$ and $L \cap (H + M_1) \subseteq \operatorname{Rad} M$. Since $L \cap H \leq L \cap (H + M_1) \subseteq \operatorname{Rad} M$ and $L \cap H \leq L$, we have $L \cap H \leq L \cap \operatorname{Rad} M = \operatorname{Rad} L$. Hence L is a generalized supplement of H in M_2.

Suppose that L is a supplement of T in M. In case $M = T + L$ and $T \cap L \ll L$. Note that $M_2 = M_2 \cap M = M_2 \cap (L + T) = L + M_2 \cap T$. Since $M_2 \cap T \cap L \leq T \cap L \ll L$, it is easy to see that L is a supplement of $M_2 \cap T$ in M_2.

The following theorem can be written as a consequence of Lemma 2.6.

Theorem 2.2. *Let $M = M_1 \oplus M_2$ be a t-generalized supplemented module and $K \cap M_2$ be a supplement in M for every supplement K in M with $M = K + M_2$. Then M_2 is t-generalized supplemented.*

Proof. Assume that $N/M_1 \leq M/M_1$. Consider the submodule $N \cap M_2$ of M. Since M is t-generalized supplemented, there exists a generalized supplement K' of $N \cap M_2$ such that K' is a supplement in M, i.e., there exists a supplement K' in M such that $M = (N \cap M_2) + K'$ and $(N \cap M_2) \cap K' \leq \operatorname{Rad} K'$. Since $M = (N \cap M_2) + K'$, we get $M = M_2 + K'$. Let $K = M_2 \cap K'$. Then $M_2 = (N \cap M_2) + (M_2 \cap K') = (N \cap M_2) + K$. From $M = M_1 + M_2$ and $M_1 \leq N$, we have $M = N + M_2 = N + (N \cap M_2) + (M_2 \cap K') = N + K$. Since $M = M_2 + K'$ and K' is a supplement in M, $K = K' \cap M_2$ is a supplement in M by hypothesis. Therefore M_2 is t-generalized supplemented by Lemma 2.6.

Now we will investigate some conditions which will ensure that a factor module of a (distributive) t-generalized supplemented module is t-generalized supplemented.

Lemma 2.7. *Let M be a t-generalized supplemented module and $N \leq M$. If $(N+K)/N$ is a supplement submodule in M/N for every supplement submodule K in M, then M/N is a t-generalized supplemented.*

Proof. For any submodule X of M containing N, since M is t-generalized supplemented, there exists $D' \leq M$ such that $M = X + D = D + D'$, $X \cap D \leq \operatorname{Rad} D$ and $D \cap D' \ll D$ for some submodule D of M. Since $M = X + D$ and $N \leq X$, $M/N = (X+D)/N = X/N + (D+N)/N$. Note that $X \cap D \leq \operatorname{Rad} D$, $X/N \cap (D+N)/N = (X \cap D + N)/N \leq (\operatorname{Rad} D + N)/N \leq$ $\leq \operatorname{Rad}((D+N)/N)$. This implies that $(D+N)/N$ is a generalized supplement of X/N in M/N. On the other hand, D is a supplement in M and $(D+N)/N$ is a supplement in M/N by hypothesis. Therefore $(D+N)/N$ is a generalized supplement of X/N in M/N such that $(D+N)/N$ is a supplement in M/N. Hence M/N is t-generalized supplemented.

Theorem 2.3. *Let M be a distributive t-generalized supplemented module. Then for every submodule N of M, M/N is t-generalized supplemented.*

Proof. Let D be a supplement submodule in M. Then there exists $D' \leq M$ such that $M = D + D'$ and $D \cap D' \ll D$. Since $M = D + D'$, we can write that $M/N = (D+N)/N + (D'+N)/N$. From M is distributive, $N + (D \cap D') = (N+D) \cap (N+D')$. This implies that $(D+N)/N \cap$ $\cap (D'+N)/N = [(D+N) \cap (D'+N)]/N = (N + (D \cap D'))/N$. Note that $D \cap D' \ll D$. So, we get $(D+N)/N \cap (D'+N)/N = (D \cap D' + N)/N \ll (D+N)/N$. Hence for every supplement submodule D in M, $(D+N)/N$ is a supplement submodule in M/N. Therefore by Lemma 2.7 M/N is t-generalized supplemented.

3. Cofinitely t-generalized supplemented modules.

Definition 3.1. *Let M be an R-module. We say that M is called cofinitely t-generalized supplemented module if every cofinite submodule of M has a generalized supplement such that it is a supplement in M.*

Clearly every cofinitely generalized $\oplus$-supplemented modules are cofinitely t-generalized supplemented.

Lemma 3.1. *Let M be a finitely generated module. Then M is t-generalized supplemented if and only if M is cofinitely t-generalized supplemented.*

Proof. Since M is finitely generated, the proof is clear.

Lemma 3.2. *Let M be an R-module and* $\operatorname{Rad} M \ll M$. *Then M is cofinitely t-generalized supplemented if and only if M is cofinitely supplemented.*

Proof. $(\Rightarrow)$ Let N be any cofinite submodule of M. Since M is cofinitely t-generalized supplemented, there exists $K \leq M$ such that $M = N + K$, $N \cap K \subseteq \operatorname{Rad} K$ and K is a supplement in M. Since $N \cap K \subseteq \operatorname{Rad} K \subseteq \operatorname{Rad} M$ and $\operatorname{Rad} M \ll M$, $N \cap K \ll M$. Hence we get $N \cap K \ll K$. So K is a supplement of N in M. Therefore M is cofinitely supplemented.

$(\Leftarrow)$ Since M is cofinitely supplemented, for any cofinite submodule N of M, there exists $K \leq M$ such that $M = N + K$ and $N \cap K \ll K$. From $N \cap K \subseteq \operatorname{Rad} K$, K is a generalized supplement of N in M. Therefore M is cofinitely t-generalized supplemented.

As a result of Lemma 3.2, we can obtain the following corollary.

***Corollary* 3.1.** *Let M be a finitely generated R-module. M is cofinitely t-generalized supplemented if and only if M is cofinitely supplemented.*

The Corollary 2.1 together with Lemma 2.2 gives the following important theorem.

Theorem 3.1. *For any ring R, the arbitrary direct sum of cofinitely t-generalized supplemented R-modules is cofinitely t-generalized supplemented.*

Proof. Let $\{M_i\}_{i\in I}$ be any collection of cofinitely t-generalized supplemented R-modules and $M = \underset{i\in I}{\oplus} M_i$. Let N be any cofinite submodule of M. In this case M/N is finitely generated and there exists $k \in \mathbb{Z}^+$, $x_i \in M$, $1 \leq i \leq k$, such that $M/N = \left\langle \{x_1 + N,\ x_2 + N, \ldots, x_k + N\} \right\rangle$. So $M = Rx_1+Rx_2+\ldots+Rx_k+N$. In here, there exists finitely subset $F = \{i_1, i_2, \ldots, i_n\}$ of I such that $x_i \in \underset{j\in F}{\oplus} M_j$ for every $1 \leq i \leq k$. Hence it is clear that $M = M_{i_1} + \left(N + \sum_{j=2}^{n} M_{i_j}\right)$ has trivially a generalized supplement 0 in M. Consider the submodule $M_{i_1} \cap \left(N + \sum_{j=2}^{n} M_{i_j}\right) \leq M_{i_1}$. Since $M_{i_1}/\left[M_{i_1} \cap \left(N + \sum_{j=2}^{n} M_{i_j}\right)\right] \cong M/\left(N + \sum_{j=2}^{n} M_{i_j}\right) \cong (M/N)/\left(\left(N + \sum_{j=2}^{n} M_{i_j}\right)/N\right)$, $M_{i_1} \cap \left(N + \sum_{j=2}^{n} M_{i_j}\right)$ is a cofinite submodule of M_{i_1}. From M_{i_1} is cofinitely t-generalized supplemented then $M_{i_1} \cap \left(N + \sum_{j=2}^{n} M_{i_j}\right)$ has a generalized supplement S_{i_1} such that S_{i_1} is a supplement in M_{i_1}. By Lemma 1.1, S_{i_1} is a generalized supplement of $N + \sum_{j=2}^{n}$ in M. Similarly we can show that for $1 \leq j \leq n$, N has a generalized supplement $S_{i_1} + S_{i_2} + \ldots + S_{i_n}$ such that S_{i_j} is a supplement in M_{i_j}. In this case by Corollary 2.1, $S_{i_1} + S_{i_2} + \ldots + S_{i_n}$ is a supplement in $M_{i_1} \oplus M_{i_2} \oplus \ldots \oplus M_{i_n}$. Since $M_{i_1} \oplus M_{i_2} \oplus \ldots \oplus M_{i_n}$ is direct summand of M, then by Lemma 2.2, $S_{i_1}+S_{i_2}+\ldots+S_{i_n}$ is a supplement in M. Consequently, $\underset{i\in I}{\oplus} M_i$ is cofinitely t-generalized supplemented.

Theorem 3.2. *Let M be a projective and finitely generated R-module. Then the following assertions are equivalent*:

(i) *M is a semiperfect module,*

(ii) *M is generalized $\oplus$-supplemented,*

(iii) *M is cofinitely generalized $\oplus$-supplemented,*

(iv) *M is t-generalized supplemented,*

(v) *M is cofinitely t-generalized supplemented.*

Proof. (i) $\Leftrightarrow$ (ii) Clear by Lemma 1.2.

(ii) $\Rightarrow$ (iv) Clear from definitions.

(iv) $\Rightarrow$ (ii) Since M is t-generalized supplemented and finitely generated, by Lemma 2.4 M is supplemented. On the other hand, since M is projective then M is $\oplus$-supplemented. Hence M is generalized $\oplus$-supplemented.

Since M is finitely generated then (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (v) are clear.

The following remark shows that a cofinitely t-generalized supplemented module need not to be cofinitely generalized $\oplus$-supplemented.

***Remark* 3.1.** In Example 2.2, from M is t-generalized supplemented, M is cofinitely t-generalized supplemented. But since M is finitely generated and not generalized $\oplus$-supplemented, M is not cofinitely generalized $\oplus$-supplemented.

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