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SYSTEMS OF ϕ-LAPLACIAN THREE-POINT BOUNDARY-VALUE PROBLEMS ON THE POSITIVE HALF-LINE

СИСТЕМИ ТРИТОЧКОВИХ ГРАНИЧНИХ ЗАДАЧ ДЛЯ ϕ-ЛАПЛАСІАНА НА ДОДАТНІЙ ПІВОСІ

We study the existence of positive solutions to boundary-value problems for two systems of two second-order nonlinear three-point ϕ-Laplacian equations defined on the positive half-line. The nonlinearities may change sign, exhibit time singularities at the origin, and depend both on the solutions and on their first derivatives. Using the fixed-point theory, we prove some results on the existence of nontrivial positive solutions on appropriate cones in some weighted Banach spaces.

Вивчається існування додатних розв'язків граничних задач для двох систем двох нелінійних триточкових ϕ-лапласових рівнянь другого порядку, що визначені на додатній півосі. Нелінійності можуть змінювати знак, мати часові сингулярності на початку координат та залежати як від розв'язків, так і від їх перших похідних. Теорію нерухомих точок застосовано для доведення деяких результатів щодо існування нетривіальних додатних розв'язків на відповідних конусах в деяких зважених банахових просторах.

1. Introduction. In this paper, we first consider the following general system of nonlinear second-order ϕ-Laplacian three-point boundary-value problem posed on the positive half-line:

$$\begin{aligned} &-(\phi(y_1'))'(t) = m_1(t)f_1(t, y_1(t), y_2(t), y_1'(t), y_2'(t)), \qquad t \in I,\\ &-(\phi(y_2'))'(t) = m_2(t)f_2(t, y_1(t), y_2(t), y_1'(t), y_2'(t)), \qquad t \in I,\\ &y_1(0) = \alpha y_1'(\eta), \qquad \lim_{t\to+\infty} y_1'(t) = 0,\\ &y_2(0) = \alpha y_2'(\eta), \qquad \lim_{t\to+\infty} y_2'(t) = 0, \end{aligned} \tag{1.1}$$

where $\alpha \geq 0$ and $\eta > 0$ are real parameters. For $i = 1, 2$, the nonlinear functions $f_i\colon \mathbb{R}^+ \times \times (\mathbb{R}^+)^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ are nonnegative while $m_i\colon I \to \mathbb{R}^+$ are nonnegative, continuous functions that are allowed to have a singularity at the origin $t = 0$. The interval $I := (0, +\infty)$ denotes the set of positive real numbers and $\mathbb{R}^+ : = [0, +\infty)$.

The nonlinear operator of derivation $\phi\colon \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism such that $\phi(0) = 0$ (ϕ is not necessarily odd), satisfies

$$|\phi^{-1}(x)| \leq \phi^{-1}(|x|) \quad \forall x \in \mathbb{R} \tag{1.2}$$

and is submultiplicative on $\mathbb{R}^+$, i.e.,

$$\phi(\alpha\beta) \leq \phi(\alpha)\phi(\beta) \quad \forall\, \alpha, \beta \in \mathbb{R}^+. \tag{1.3}$$

This implies that the inverse ϕ^{-1} is supermultiplicative:

$$\phi^{-1}(\alpha\beta) \geq \phi^{-1}(\alpha)\phi^{-1}(\beta) \quad \forall\, \alpha, \beta \in \mathbb{R}^+. \tag{1.4}$$

ϕ extends the usual multiplicative p-Laplacian nonlinear operator $\phi(s) = |s|^{p-2}s, \quad p > 1.$

In the second part of the paper, we focus on a system with a particular structure, namely

$$
\begin{aligned}
&-(\phi(y'))'(t) = m(t)f(t, x(t)), \qquad t \in I,\\
&-(\phi(x'))'(t) = m(t)g(t, y(t)), \qquad t \in I,\\
&y(0) = \alpha y'(\eta), \qquad \lim_{t\to+\infty} y'(t) = 0,\\
&x(0) = \alpha x'(\eta), \qquad \lim_{t\to+\infty} x'(t) = 0,
\end{aligned}
\tag{1.5}
$$

where the nonnegative functions $f, g\colon\ I \times \mathbb{R}^+ \to \mathbb{R}$ are continuous but may change sign; the function $m\colon\ I \to \mathbb{R}^+$ is continuous with possible singularity at $t = 0$.

Singular differential systems arise in many branches of applied mathematics and physics such as gas dynamics, Newtonian fluid mechanics, nuclear physics and have attracted many authors during the last couple of years (see, e.g., [1, 8, 9, 12, 13]). The unknowns may represent a density, a temperature, a velocity, . . . hence positivity of solutions is required.

Recently there has been so much work devoted to the investigation of positive solutions to systems of boundary-value problems on finite intervals of the real line (see [11, 14, 18, 19] and the references therein). Also, several methods have been employed to deal with boundary-value problems on the positive half-line; we quote fixed point theorems in special Banach spaces, the fixed point index theory on positive cones of functional Banach spaces, the upper and lower solutions techniques, and the monotone iterative techniques [4, 18].

In the particular case of second-order differential equations corresponding to $\phi = I_d$, system (1.1) has been widely studied in the literature. In 2009, by using the Krasnosel'skii fixed point theorem, Xi, Jia, and Ji [15] studied the existence of positive solutions to the following boundary-value problem with integral boundary conditions on the half-line:

$$
\begin{aligned}
&y_1''(t) + f_1(t, y_1(t), y_2(t)) = 0, \qquad t \in I,\\
&y_2''(t) + f_2(t, y_1(t), y_2(t)) = 0, \qquad t \in I,\\
&y_1(0) = 0, \qquad y_1'(+\infty) = \int\limits_0^{+\infty} g_1(s)y_1(s)ds,\\
&y_2(0) = 0, \qquad y_2'(+\infty) = \int\limits_0^{+\infty} g_2(s)y_2(s)ds.
\end{aligned}
$$

In the same year, Zhang [17] investigated the existence of positive solutions for a singular multipoint system of second-order differential equations posed on an infinite interval; he used the Mönch fixed point theorem and a monotone iterative technique; the system considered reads:

$$
\begin{aligned}
&x''(t) + f(t, x(t), x'(t), y(t), y'(t)) = 0, \quad t \in I,\\
&y''(t) + g(t, x(t), x'(t), y(t), y'(t)) = 0, \quad t \in I,
\end{aligned}
$$

$$x(0) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \quad x'(+\infty) = x_\infty,$$

$$y(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \quad y'(+\infty) = y_\infty.$$

Motivated by the papers [4, 15, 17] and some related results for singular three-point ϕ-Laplacian boundary-value problems posed on the positive half-line [6], we are first concerned in this paper with the existence of positive solutions for the singular system (1.1). This system is different from those considered in [4, 15, 17]. Firstly, the singular system (1.1) is posed on an infinite interval. Secondly, the nonlinear operator of derivation extends the model case of the p-Laplacian nonlinear operator. Finally, the function m presents a time-singularity at $t = 0$.

In this paper, we first prove an existence result of nontrivial nonnegative solutions for system (1.1) under suitable conditions on the positive functions f_i and m_i, $i = 1, 2$, and thus we extend some of the above works. The boundary-value problem is formulated as a fixed point problem of a nonlinear operator. The index fixed point theory on cones of Banach spaces is then employed. This is the object of Section 2. Then Section 3 is devoted to investigating the singular system (1.5) where the nonlinearities are allowed to change sign. We prove the existence of positive solutions on a translate of a cone in a Banach space. Each existence result is illustrated by means of an example of application.

2. The general case of system (1.1). Some preliminaries including the main assumptions and a compactness criterion are presented in Subsection 2.1. We construct a special cone in the Banach space of continuously differentiable functions with vanishing derivatives at positive infinity; then we study the properties of a corresponding fixed point operator. Subsection 2.2 is devoted to proving our main existence theorem. We end this section with an example of application in Subsection 2.3. By a nonnegative solution, we mean a couple of functions $(y_1, y_2) \in C^1[0, +\infty) \times C^1[0, +\infty)$ such that $\phi(y_i') \in C^1(0, +\infty)$ with $y_i(t) \geq 0$ on $[0, +\infty)$ for $i = 1, 2$ and such that the equations in (1.1) are satisfied.

2.1. The general framework. Consider the space

$$\mathbb{X} = \left\{ \begin{array}{c} y = (y_1, y_2) \mid \ y_i \in C^1(\mathbb{R}^+, \mathbb{R}), \ \ y_i(0) = \alpha y_i'(\eta), \\ \text{and } \lim\limits_{t \to +\infty} y_i'(t) = 0, \ \ i = 1, 2 \end{array} \right\}.$$

This is a Banach space with the norm

$$\|y\| = \|y_1\| + \|y_2\|, \ \text{ where } \|y_i\| = \sup_{t \in \mathbb{R}^+} |y_i'(t)|, \ i = 1, 2.$$

In order to transform problem (1.1) into a fixed point problem, the following auxiliary lemma is needed; the proof which is immediate is skipped.

Lemma 2.1. *Let $v \in L^1(I)$. Then $u \in C^1(I)$ is a solution of*

$$\begin{aligned} -(\phi(u'))'(t) &= v(t), \qquad t \in I, \\ u(0) = \alpha u'(\eta), &\qquad \lim_{t \to +\infty} u'(t) = 0 \end{aligned} \tag{2.1}$$

if and only if

$$u(t) = C + \int_0^t \phi^{-1}\left(\int_s^{+\infty} v(\tau)d\tau\right) ds, \qquad t \geq 0, \tag{2.2}$$

where $C = \alpha\phi^{-1}\left(\int_\eta^{+\infty} v(\tau)d\tau\right)$.

In order to study the compactness of the fixed point operator, we first recall a classical result. Given the Banach space

$$C_l([0,+\infty),\mathbb{R}) = \{x \in C([0,+\infty),\mathbb{R}) \mid \lim_{t\to+\infty} x(t) \text{ exists}\}$$

with the norm $\|x\|_l = \sup_{t\in[0,+\infty)} |x(t)|$, we have the following lemma.

Lemma 2.2 [3, p. 62]. *A set $M \subseteq C_l(\mathbb{R}^+,\mathbb{R})$ is relatively compact if the following conditions hold*:

(a) *M is uniformly bounded in $C_l(\mathbb{R}^+,\mathbb{R})$.*

(b) *The functions belonging to M are almost equicontinuous on $\mathbb{R}^+$, i.e., equicontinuous on every compact interval of $\mathbb{R}^+$.*

(c) *The functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(t) - x(\infty)| < \varepsilon$, for all $t \geq T(\varepsilon)$ and $x \in M$.*

From this lemma, we derive the following lemma.

Lemma 2.3. *Let $M \subseteq \mathbb{X}$. Then M is relatively compact in $\mathbb{X}$ if the following conditions hold*:

(a) *M is uniformly bounded in $\mathbb{X}$.*

(b) *The functions belonging to the set $\mathcal{A} = \{z \mid z(t) = y'(t),\ y \in M\}$ are almost equicontinuous on $\mathbb{R}^+$.*

(c) *The functions from $\mathcal{A}$ are equiconvergent at $+\infty$.*

Now, consider the following hypotheses where $i = 1, 2$:

($\mathcal{G}_1$) The functions $f_i\colon \mathbb{R}^+ \times (\mathbb{R}^+)^2 \times \mathbb{R}^2 \to \mathbb{R}^+$ are continuous and when y_1, y_2, z_1, z_2 are bounded $f_i(t,(t+\alpha)y_1,(t+\alpha)y_2,z_1,z_2)$ are bounded on $[0,+\infty)$.

($\mathcal{G}_2$) The functions $m_i\colon\ I \longrightarrow \mathbb{R}^+$ are continuous and do not vanish identically on any subinterval of I. They may be singular at $t = 0$ but are integrably bounded, that is

$$A_i := \int_0^{+\infty} m_i(s)ds < \infty.$$

2.2. Fixed point formulation. Let $\mathbb{P}$ be the nonnegative cone defined by

$$\mathbb{P} = \{y \in \mathbb{X} \mid\ y(t) \geq 0\ \ \forall\, t \geq 0\}.$$

By $y \geq 0$, it is meant $y_i \geq 0$ for each $i = 1, 2$. We start with a simple observation:

***Remark* 2.1.** If, for $i = 1, 2$, $y_i \in \mathbb{P}$, then the mean value theorem yields

$$\sup_{t\geq 0} \frac{y_i(t)}{t+\alpha} \leq \|y_i\|.$$

Now let $\Omega \subset \mathbb{X}$ be a bounded subset. Then, there exists $M > 0$ such that $\|y\| \leq M$, for all $y = (y_1, y_2) \in \Omega$. According to Assumption $(\mathcal{G}_1)$, for $i = 1, 2$, let

$$S_M^{(i)} = \sup \{f_i(t, (t+\alpha)y_1, (t+\alpha)y_2, z_1, z_2)\ ,$$

$$\text{for } \ t \geq 0, \quad (y_1, y_2) \in [0, M]^2, \quad \text{and} \quad (|z_1|, |z_2|) \in [0, M]^2\big\}\,.$$

Then for every $t \geq 0$, $|y_i'(t)| \leq M$ and thus $0 \leq \dfrac{y_i(t)}{t+\alpha} \leq M,\ i = 1, 2$. Hence

$$\int\limits_0^{+\infty} m_i(s) f_i(s, y_1(s), y_2(s), y_1'(s), y_2'(s)) ds =$$

$$= \int\limits_0^{+\infty} m_i(s) f_i\left(s, \frac{(s+\alpha)y_1(s)}{s+\alpha}, \frac{(s+\alpha)y_2(s)}{s+\alpha}, y_1'(s), y_2'(s)\right) ds \leq$$

$$\leq S_M^{(i)} \int\limits_0^{+\infty} m_i(s) ds < \infty, \qquad i = 1, 2.$$

As a consequence, for $i = 1, 2$, the integrals

$$\int\limits_0^{+\infty} m_i(s)) f_i(s, y_1(s), y_2(s), y_1'(s), y_2'(s)) ds$$

are convergent. From Lemma 2.1, we know that the boundary-value problem (1.1) is equivalent to

$$y_i(t) = C_i + \int\limits_0^t \phi^{-1}\left(\int\limits_s^{+\infty} m_i(\tau) f_i\left(\tau, y_1(\tau), y_2(\tau), y_1'(\tau), y_2'(\tau)\right) d\tau\right) ds,$$

where

$$C_i = \alpha\phi^{-1}\left(\int\limits_\eta^{+\infty} m_i(\tau) f_i\left(\tau, y_1(\tau), y_2(\tau), y_1'(\tau), y_2'(\tau)\right) d\tau\right),\ i = 1, 2.$$

Let $\Omega \subset \mathbb{X}$ be a bounded subset and define the integral operators

$$\begin{aligned} F_i :\ \overline{\Omega} \cap \mathbb{P} &\longrightarrow C^1(\mathbb{R}^+, \mathbb{R}^+), \\ y = (y_1, y_2) &\longmapsto F_i y(t), \end{aligned} \tag{2.3}$$

where, for $i = 1, 2$,

$$F_i y(t) = C_i + \int\limits_0^t \phi^{-1}\left(\int\limits_s^{+\infty} m_i(\tau) f_i\left(\tau, y_1(\tau), y_2(\tau), y_1'(\tau), y_2'(\tau)\right) d\tau\right) ds.$$

Next, we study the properties of the fixed point operator F defined by

$$Fy(t) = (F_1 y(t), F_2 y(t))\,.$$

Lemma 2.4. *Under Assumptions* ($\mathcal{G}_1$) *and* ($\mathcal{G}_2$) *the operator* F *maps the set* $\overline{\Omega}\cap\mathbb{P}$ *into* $\mathbb{P}$.

Proof. First we show that $F\colon\overline{\Omega}\cap\mathbb{P}\to\mathbb{X}$ is well defined. Let $y=(y_1,y_2)\in\overline{\Omega}\cap\mathbb{P}$. Then there exists $M>0$ such that $\|y\|\le M$, for all $y=(y_1,y_2)\in\overline{\Omega}\cap\mathbb{P}$. By Assumptions ($\mathcal{G}_1$) and ($\mathcal{G}_2$), we have, for $i=1,2$,

$$\int\limits_0^{+\infty} m_i(\tau)f_i\left(\tau,y_1(\tau),y_2(\tau),y_1'(\tau),y_2'(\tau)\right)d\tau\le A_iS_M^{(i)}.$$

Hence for every $t\in[0,+\infty)$

$$\int\limits_0^t\phi^{-1}\left(\int\limits_s^{+\infty} m_i(\tau)f_i\left(\tau,y_1(\tau),y_2(\tau),y_1'(\tau),y_2'(\tau)\right)d\tau\right)ds\le\int\limits_0^t\phi^{-1}\left(A_iS_M^{(i)}\right)ds<\infty.$$

In addition, we can easily prove that for all $y\in\overline{\Omega}\cap\mathbb{P}$,

$$F_iy\in C^1([0,+\infty),\mathbb{R}),\ F_iy(t)\ge 0,\ t\in[0,+\infty),$$

$$F_iy(0)=C_i=\alpha\phi^{-1}\left(\int\limits_\eta^{+\infty} m_i(\tau)f_i\left(\tau,y_1(\tau),y_2(\tau),y_1'(\tau),y_2'(\tau)\right)\right)=\alpha(F_iy)'(\eta),$$

and

$$\lim_{t\to+\infty}(F_iy)'(t)=\lim_{t\to+\infty}\phi^{-1}\left(\int\limits_t^{+\infty} m_i(\tau)f_i\left(\tau,y_1(\tau),y_2(\tau),y_1'(\tau),y_2'(\tau)\right)\right)=\phi^{-1}(0)=0,$$

ending the proof of the lemma.

The proof of the following lemma is a consequence of Lemma 2.3. The proof is omitted.

Lemma 2.5. *Assume that* ($\mathcal{G}_1$) *and* ($\mathcal{G}_2$) *hold. Then, the mapping* $F\colon\overline{\Omega}\cap\mathbb{P}\to\mathbb{P}$ *is completely continuous.*

2.3. Existence result. Two auxiliary lemmas are needed in this section. More details concerning the theory of the fixed point index on cones of Banach spaces can be found for instance in [5, 10, 16].

Lemma 2.6. *Let* Ω *be a bounded open subset of a real Banach space* E, $\mathcal{P}$ *a cone of* E, $\theta\in\Omega$, *and* $A\colon\overline{\Omega}\cap\mathcal{P}\to P$ *a completely continuous operator. Suppose that*

$$Ax\ne\lambda x\quad\forall\,x\in\partial\Omega\cap\mathcal{P},\ \lambda\ge 1.$$

Then the fixed point index $i\,(A,\Omega\cap\mathcal{P},\mathcal{P})=1$.

Lemma 2.7. *Let* Ω *be a bounded open set in a real Banach space* E, $\mathcal{P}$ *be a cone of* E, *and* $A\colon\overline{\Omega}\cap\mathcal{P}\to\mathcal{P}$ *be a completely continuous mapping. Assume that*

$$Ax\not\le x\quad\forall\,x\in\partial\Omega\cap\mathcal{P}.$$

Then the fixed point index $i\,(A,\Omega\cap\mathcal{P},\mathcal{P})=0$.

We are now in position to prove the main existence result of this section.

Theorem 2.1. *Assume that the following assumptions hold for* $i=1,2$:

$$(\mathcal{G}_3)\quad \begin{cases} 0 \le f_i(t, y_1, y_2, z_1, z_2) \le \\ \le a_i(t)\left(\phi\left(\dfrac{y_1}{t+\alpha}\right) + \phi\left(\dfrac{y_2}{t+\alpha}\right)\right) + b_i(t)\left(\phi(z_1) + \phi(z_2)\right) + c_i(t), \\ \textit{for all } (t, y_1, y_2, z_1, z_2) \in I \times (\mathbb{R}^+)^2 \times \mathbb{R}^2, \textit{ with} \\ m_i a_i,\ m_i b_i,\ m_i c_i \in L^1(\mathbb{R}^+). \end{cases}$$

($\mathcal{G}_4$) *There exists* $R > 0$ *such that*

$$\phi^{-1}\left(2\left(|m_i a_i|_{L^1} + |m_i b_i|_{L^1}\right)\phi(R) + |m_i c_i|_{L^1}\right) < R/2. \tag{2.4}$$

($\mathcal{G}_5$) *There exist* $0 < \gamma < \delta$ *such that for all* $(t, y_1, y_2, z_1, z_2) \in [\gamma, \delta] \times (\mathbb{R}^+)^2 \times \mathbb{R}^2$, *we have*

$$f_i(t, y_1, y_2, z_1, z_2) \ge g_i\left(t, \frac{y_1}{t+\alpha}, \frac{y_2}{t+\alpha}\right),$$

where $g_i \in C([\gamma, \delta] \times (\mathbb{R}^+)^2)$ *satisfies*

$$\liminf_{y_1+y_2 \to 0}\ \min_{t \in [\gamma, \delta]} \frac{g_i\left(t, \dfrac{y_1}{t+\alpha}, \dfrac{y_2}{t+\alpha}\right)}{\phi(y_1 + y_2)} \ge \ell_i \tag{2.5}$$

with constants ℓ_i *satisfying* $\phi^{-1}\left(\ell_i \displaystyle\int_\gamma^\delta m_i(t)dt\right) > 1/\gamma$.

Then, problem (1.1) *has at least one nonnegative solution* $y = (y_1, y_2)$ *such that*

$$0 < \|y\| < R.$$

***Remark* 2.2.** It is easily seen that Assumption ($\mathcal{G}_3$) is a substitution of Assumptions ($\mathcal{G}_1$), ($\mathcal{G}_2$) and Lemmas 2.4 and 2.5 still remain valid.

Proof. Define the open ball

$$\Omega_R = \{y \in \mathbb{X} \colon \|y\| < R\}.$$

From Lemma 2.5, $F \colon \overline{\Omega}_R \cap \mathbb{P} \to \mathbb{P}$ is completely continuous.

Claim 1. $Fy \ne \lambda y$, for $y \in \partial\Omega_R \cap \mathbb{P}$ and $\lambda \ge 1$. To see this, let $y = (y_1, y_2) \in \partial\Omega_R \cap \mathbb{P}$. By Assumption ($\mathcal{G}_3$) and Remark 2.1, the following estimates hold for positive t and $i = 1, 2$:

$$|(F_i y)'(t)| = \phi^{-1}\left(\int_t^{+\infty} m(\tau) f_i(\tau, y_1(\tau), y_2(\tau), y_1'(\tau), y_2'(\tau))d\tau\right) \le$$

$$\le \phi^{-1}\left(\int_0^{+\infty} m_i(\tau)\left(a_i(\tau)\left(\phi\left(\frac{y_1(\tau)}{\tau+\alpha}\right) + \phi\left(\frac{y_2(\tau)}{\tau+\alpha}\right)\right) + \right.\right.$$

$$\left.\left. + b_i(\tau)\left(\phi(y_1'(\tau)) + \phi(y_2'(\tau))\right) + c_i(\tau)\right)d\tau\right) \le$$

$$\le \phi^{-1}\left(|m_i a_i|_{L^1}\left[\phi(\|y_1\|) + \phi(\|y_2\|)\right] + |m_i b_i|_{L^1}\left[\phi(\|y_1\|) + \phi(\|y_2\|)\right] + |m_i c_i|_{L^1}\right) \le$$

$$\leq \phi^{-1}\left(2|m_i a_i|_{L^1}\phi(\|y\|) + 2|m_i b_i|_{L^1}\phi(\|y\|) + |m_i c_i|_{L^1}\right) \leq$$

$$\leq \phi^{-1}\left(2\left(|m_i a_i|_{L^1} + |m_i b_i|_{L^1}\right)\phi(R) + |m_i c_i|_{L^1}\right) < \frac{R}{2}.$$

Taking the least upper bound over t yields $\|F_i y\| < \|y\|/2$, for all $y \in \partial\Omega_R \cap \mathbb{P}$ and all $i = 1, 2$. Moreover

$$\|Fy\| = \|F_1 y\| + \|F_2 y\| < \|y\| \quad \forall\, y \in \partial\Omega_R \cap \mathbb{P}. \tag{2.6}$$

As a consequence

$$Fy \neq \lambda y \qquad \forall\, y \in \partial\Omega_R \cap \mathbb{P} \quad \forall\, \lambda \geq 1. \tag{2.7}$$

If not there would exist some $y_0 \in \partial\Omega_R \cap \mathbb{P}$ and $\lambda_0 \geq 1$ such that $Fy_0 = \lambda_0 y_0$. Hence $\|Fy_0\| = \lambda_0\|y_0\| \geq \|y_0\|$, contradicting (2.6). This implies that (2.7) holds. Therefore, Lemma 2.6 guarantees

$$i\left(F, \Omega_R \cap \mathbb{P}, \mathbb{P}\right) = 1. \tag{2.8}$$

Finally (2.8) and the existence property of the fixed point index imply that the operator F has a fixed point $y = (y_1, y_2)$ which belongs to $\Omega_R \cap \mathbb{P}$ with $0 \leq \|y\| < R$.

Claim 2. By (2.5), there exists an $r_0 > 0$ such that for $i = 1, 2$, we have

$$g_i\left(t, \frac{y_1}{t+\alpha}, \frac{y_2}{t+\alpha}\right) \geq \ell_i \phi(y_1 + y_2), \ \text{ for } 0 \leq y_1 + y_2 \leq r_0 \ \text{ and } \ t \in [\gamma, \delta]. \tag{2.9}$$

Let $0 < r < \min\left(R, \dfrac{r_0}{\delta+\alpha}\right)$ and consider the open set

$$\Omega_r = \Big\{y \in \mathbb{X} \colon \|y\| < r\Big\}.$$

We claim that $Fy \not\leq y$, for every $y \in \partial\Omega_r \cap \mathbb{P}$. Otherwise, let $y_0 = (y_{0,1}, y_{0,2}) \in \partial\Omega_r \cap \mathbb{P}$ be such that

$$Fy_0 \leq y_0. \tag{2.10}$$

Then, by virtue of (1.4), (2.9), and (2.10), for all $t \in [\gamma, \delta]$ and $i = 1, 2$, we have the estimates:

$$y_{0,i}(t) = C_i + \int\limits_0^t \phi^{-1}\left(\int\limits_s^{+\infty} m_i(\tau) f_i(\tau, y_{0,1}(\tau), y_{0,2}(\tau), y'_{0,1}(\tau), y'_{0,2}(\tau))d\tau\right) ds \geq$$

$$\geq \int\limits_0^\gamma \phi^{-1}\left(\int\limits_s^{+\infty} m_i(\tau) f_i(\tau, y_{0,1}(\tau), y_{0,2}(\tau), y'_{0,1}(\tau), y'_{0,2}(\tau))d\tau\right) ds \geq$$

$$\geq \int\limits_0^\gamma \phi^{-1}\left(\int\limits_\gamma^{\delta} m_i(\tau) f_i(\tau, y_{0,1}(\tau), y_{0,2}(\tau), y'_{0,1}(\tau), y'_{0,2}(\tau))d\tau\right) ds \geq$$

$$\geq \int\limits_0^\gamma \phi^{-1}\left(\int\limits_\gamma^{\delta} m_i(\tau) g_i(\tau, y_{0,1}(\tau)/(\tau+\alpha), y_{0,2}(\tau)/(\tau+\alpha))d\tau\right) ds \geq$$

$$\geq \gamma\,\phi^{-1}\left(\int\limits_{\gamma}^{\delta} m_i(\tau)\ell_i\phi(y_{0,1}(\tau)+y_{0,2}(\tau)))d\tau\right) \geq$$

$$\geq \gamma\,\phi^{-1}\left(\phi(\min_{t\in[\gamma,\delta]}(y_{0,1}(t)+y_{0,2}(t))\right)\phi^{-1}\left(\ell_i\int\limits_{\gamma}^{\delta} m_i(\tau)d\tau\right) \geq$$

$$\geq \gamma\,\phi^{-1}\left(\ell_i\int\limits_{\gamma}^{\delta} m_i(\tau)d\tau\right)\min_{t\in[\gamma,\delta]}(y_{0,1}(t)+y_{0,2}(t)) >$$

$$> \min_{t\in[\gamma,\delta]}(y_{0,1}(t)+y_{0,2}(t)) \geq \min_{t\in[\gamma,\delta]} y_{0,i}(t),$$

contradicting the continuity of the functions $y_{0,i}$, $i = 1, 2$, on the compact interval $[\gamma, \delta]$, where C_i is given by (2.2). Therefore, Lemma 2.7 yields

$$i\,(F, \Omega_r \cap \mathbb{P}, \mathbb{P}) = 0. \tag{2.11}$$

Combining (2.8), (2.11), and the fact that $\overline{\Omega}_r \subset \Omega_R$, we find

$$i\,(F, (\Omega_R \setminus \overline{\Omega}_r) \cap \mathbb{P}, \mathbb{P}) = 1. \tag{2.12}$$

Finally, the fixed point $y = (y_1, y_2) \in \mathbb{P}$ satisfies $r < \|y\| < R$.

Theorem 2.1 is proved.

2.4. Example. Let

$$a_1(t) = e^{-90t}, \qquad a_2(t) = e^{-500t}, \qquad b_1(t) = e^{-200t}, \qquad b_2(t) = e^{-80t},$$

$$c_1(t) = \frac{1}{100}, \qquad c_2(t) = 50, \qquad m_1(t) = e^{-3t}, \qquad m_2(t) = e^{-10t},$$

and let the increasing homeomorphism ϕ be defined by $\phi(x) = x^3$. In order to check the inequality (2.4) in Assumption $(\mathcal{G}_4)$, take $\alpha = 3/2$ and $\eta = 10$. Then we can choose $\theta = 1$ and $R = 15$ and so we have

$$\phi^{-1}\left(2\left(|m_1a_1|_{L^1} + |m_1b_1|_{L^1}\right)\phi(R) + |m_1c_1|_{L^1}\right) = \frac{3569}{743} < \frac{R}{2}$$

and

$$\phi^{-1}\left(2\left(|m_2a_2|_{L^1} + |m_2b_2|_{L^1}\right)\phi(R) + |m_2c_2|_{L^1}\right) = \frac{4403}{971} < \frac{R}{2}.$$

Consider the nonlinearities f_i, $i = 1, 2$, defined in $(\mathbb{R}^+)^3 \times \mathbb{R}^2$ by

$$f_i(t, y_1, y_2, z_1, z_2) = a_i(t)\left(\phi\left(\frac{y_1}{t+\alpha}\right) + \phi\left(\frac{y_2}{t+\alpha}\right)\right) + b_i(t)\left(\phi(z_1) + \phi(z_2)\right) + c_i(t).$$

Then for all $0 < \gamma < \delta$ and all $(t, y_1, y_2, z_1, z_2) \in [\gamma, \delta] \times (\mathbb{R}^+)^2 \times \mathbb{R}^2$, we have

$$f_i(t, y_1, y_2, z_1, z_2) \geq a_i(t)\left(\phi\left(\frac{y_1}{t+\alpha}\right) + \phi\left(\frac{y_2}{t+\alpha}\right)\right) + c_i(t) =$$

$$= g_i\left(t, \frac{y_1}{t+\alpha}, \frac{y_2}{t+\alpha}\right),$$

where

$$\frac{g_i\left(t, \frac{y_1}{t+\alpha}, \frac{y_2}{t+\alpha}\right)}{\phi(y_1+y_2)} \geq \frac{c_i(t)}{\phi(y_1+y_2)} \longrightarrow +\infty, \qquad \text{as} \qquad y_1+y_2 \to 0.$$

Therefore Assumptions $(\mathcal{G}_3)-(\mathcal{G}_5)$ in Theorem 2.1 are satisfied. All the computations have been undertaken using Matlab 7.9. As a consequence, we have proved that the problem

$$-((y_1')^3)'(t) = \frac{e^{-93t}}{(t+3/2)^3}(y_1^3+y_2^3) + e^{-203t}((y_1')^3+(y_2')^3) + \frac{e^{-3t}}{100}, \qquad t>0,$$

$$-((y_2')^3)'(t) = \frac{e^{-510t}}{(t+3/2)^3}(y_1^3+y_2^3) + e^{-90t}((y_1')^3+(y_2')^3) + 50e^{-10t}, \qquad t>0,$$

$$y_1(0) = \frac{3}{2}y_1'(10), \qquad \lim_{t\to+\infty} y_1'(t) = 0,$$

$$y_2(0) = \frac{3}{2}y_2'(10), \qquad \lim_{t\to+\infty} y_2'(t) = 0,$$

has at least one nontrivial nonnegative solution $y=(y_1,y_2)\in\mathbb{X}$ satisfying $r<\|y\|<15$, for some $0<r<15$.

3. The particular case of system (1.5). In this section, we prove an existence result of positive solutions for problem (1.5) under new conditions on the functions f, g, and m. In particular the nonlinearities f, g may change sign but we still obtain existence of positive solutions with precise information on the lower bounds. By a positive solution we mean a solution $(y,x)\in$ $\in C^1[0,+\infty)\times C^1[0,+\infty)$ such that $\phi(y')\in C^1(0,+\infty)$ and $\phi(x')\in C^1(0,+\infty)$ with $x(t)>0$ and $y(t)>0$ on $[0,+\infty)$ and the equations in (1.5) are satisfied. Some preliminaries including the main assumptions, the problem transformation, and a compactness criterion of a fixed point operator are collected in Subsection 3.1. Then we prove an existence theorem by constructing a special cone in a weighted Banach space in Subsection 3.2. We end the section with an example of application in Subsection 3.3.

3.1. Problem setting and main assumptions. For some real parameter $\theta>0$, let

$$\mathbb{Y} = \left\{y\in C([0,+\infty),\mathbb{R}) \mid\ y(0)=\alpha y'(\eta),\ \lim_{t\to+\infty} e^{-\theta t}y(t)=0\right\}$$

which is a weighted Banach space with the Bielecki-type norm [2]

$$\|y\|_\theta = \sup_{t\in[0,+\infty)} e^{-\theta t}|y(t)|\cdot$$

***Remark* 3.1.** Notice that if $\lim_{t\to+\infty} y'(t)=0$ then y has a sublinear growth at positive infinity and thus for each positive θ, $\lim_{t\to+\infty} e^{-\theta t}y(t)=0$. Thus $\mathbb{Y}$ is larger than the space $\mathbb{X}$ used in Section 2. However, we will still obtain existence of positive solutions with vanishing derivatives at positive infinity.

The nonlinearities are assumed to satisfy the following hypotheses:

($\mathcal{H}_1$) The functions $f: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ and $g: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are continuous and when y, z are bounded, the function $g(t, e^{\theta t} y)$ is bounded on $[0, +\infty)$ and f satisfies $\int_0^{+\infty} m(s) f(s, k(\alpha + s)) ds < \infty$, for any positive constant k. For instance, one may take $g(t, y) = 1 + \frac{e^{-\theta t}}{1+t} y$.

($\mathcal{H}_2$) The function $m: I \longrightarrow \mathbb{R}^+$ is continuous and does not vanish identically on any subinterval of I. It may be singular at $t = 0$ but is integrably bounded, i.e.,

$$A := \int_0^{+\infty} m(s) ds < \infty.$$

Let

$$K = \begin{cases} \alpha, & \text{if} \quad \theta\alpha \geq 1, \\ \frac{1}{\theta} e^{\theta\alpha - 1}, & \text{if} \quad 0 < \theta\alpha < 1, \end{cases} \tag{3.1}$$

and, for positive t, define the function

$$\omega(t) = \alpha \phi^{-1} \left(\int_\eta^{+\infty} m(\tau) d\tau \right) + \int_0^t \phi^{-1} \left(\int_s^{+\infty} m(\tau) d\tau \right) ds. \tag{3.2}$$

***Remark* 3.2.** The properties of ϕ and m imply that ω is positive. Moreover ω is the unique solution of problem (2.1) for $v \equiv m$.

This function ω is now used to define the translate of the positive natural cone as:

$$\mathcal{K} = \mathcal{P} + \omega = \{ y \in \mathbb{Y} \mid \ y(t) \geq \omega(t) \ \ \forall t \geq 0 \},$$

where

$$\mathcal{P} = \{ y \in \mathbb{Y} \mid \ y(t) \geq 0 \ \ \forall t \geq 0 \}.$$

In [7], we have proved that the classical fixed point theory for compact mappings defined on cones of Banach spaces is still valid on translates of cones. In particular, Lemmas 2.6 and 2.7 hold when the cone $\mathcal{P}$ is replaced by its translate $\mathcal{K}$ and thus we have the following lemma.

Lemma 3.1 ([7], Proposition 4)**.** *Let Ω be a bounded open subset of a real Banach space E, $\mathcal{P}$ a cone of E, $\mathcal{K} = \mathcal{P} + \omega$ a translate of $\mathcal{P}$ with $\omega \in \Omega$. Let $A : \overline{\Omega} \cap \mathcal{K} \to \mathcal{K}$ be a completely continuous mapping satisfying*

$$Ax - \omega \neq \lambda(x - \omega) \qquad \forall x \in \partial\Omega \cap \mathcal{K}, \quad \lambda \geq 1.$$

Then the index $i(A, \Omega \cap \mathcal{K}, \mathcal{K}) = 1$.

Let $\Omega = B(\omega, r)$ be an open ball centered at ω with radius r in a real Banach space E.

Lemma 3.2 ([7], Lemma 3)**.** *Let $\mathcal{P}$ a cone of E, $\mathcal{K} = \mathcal{P} + \omega$ a translate of $\mathcal{P}$. Let $A : \overline{\Omega} \cap \mathcal{K} \to \mathcal{K}$ be a completely continuous mapping satisfying*

$$Fx \not\leq x \qquad \forall x \in \partial\Omega \cap \mathcal{K}.$$

Then the fixed point index $i(A, \Omega \cap \mathcal{K}, \mathcal{K}) = 0$.

For $\Omega \subset \mathbb{Y}$ a bounded subset, define the nonlinear operator

$$F : \overline{\Omega} \cap \mathcal{K} \longrightarrow C([0, +\infty), \mathbb{R}^+)$$

by

$$Fy(t) = C + \int_0^t \phi^{-1} \left(\int_s^{+\infty} m(\tau) f(\tau, Ty(\tau))\, d\tau \right) ds, \qquad t \in \mathbb{R}^+, \tag{3.3}$$

where

$$C = \alpha\phi^{-1} \left(\int_\eta^{+\infty} m(\tau) f(\tau, Ty(\tau)) d\tau \right),$$

$$Ty(\tau) = D + \int_0^\tau \phi^{-1} \left(\int_s^{+\infty} m(\sigma) g(\sigma, y(\sigma)) d\sigma \right) ds,$$

$$D = \alpha\phi^{-1} \left(\int_\eta^{+\infty} m(\sigma) g(\sigma, y(\sigma)) d\sigma \right).$$

As in Lemma 2.1, it is easy to see that if F has a fixed point y, then problem (1.5) admits the couple

$$\left(y, D + \int_0^t \phi^{-1} \left(\int_s^{+\infty} m(\tau) g(\tau, y(\tau)) d\tau \right) ds \right)$$

as a solution. In the following three lemmas, we study some properties of the fixed point operator F.

Lemma 3.3. *Under Assumptions* $(\mathcal{H}_1), (\mathcal{H}_2)$ *and*

$$(\mathcal{H}_3) \quad \begin{cases} f(t, x) \geq 1, \text{ for } t \in \mathbb{R}^+, x \geq D, \\ g(t, y) \geq 0, \text{ for } t \in \mathbb{R}^+, y \geq \omega(t), \end{cases} \tag{3.4}$$

F maps the set $\overline{\Omega} \cap \mathcal{K}$ into $\mathcal{K}$.

Proof. First we show that the mapping $F : \overline{\Omega} \cap \mathcal{K} \to \mathbb{Y}$ is well defined. For $y \in \overline{\Omega} \cap \mathcal{K}$, there exists $M > 0$ such that $\|y\|_\theta \leq M$. Using Assumption $(\mathcal{H}_1)$, let

$$S_M = \sup\{g(t, e^{\theta t} y) \mid t \in \mathbb{R}^+, y \in [0, M]\}.$$

For any $t \geq 0$, we have $0 \leq y(t) e^{-\theta t} \leq M$, and so Assumption $(\mathcal{H}_1)$ implies

$$\int_0^{+\infty} m(\tau) g(\tau, y(\tau)) d\tau = \int_0^{+\infty} m(\tau) g\left(\tau, e^{\theta\tau} y(\tau) e^{-\theta\tau}\right) d\tau \leq A S_M.$$

Hence for each fixed $t \in [0, +\infty)$, we have

$$\int_0^t \phi^{-1} \left(\int_s^{+\infty} m(\tau) f(\tau, Ty(\tau)) d\tau \right) ds \leq$$

$$\leq \int_0^t \phi^{-1}\left(\int_0^{+\infty} m(\tau)f(\tau,\phi^{-1}(AS_M)(\alpha+\tau))d\tau\right)ds<\infty.$$

In addition, it is easily proved that, for all $y\in\overline{\Omega}\cap\mathcal{K}$, we obtain

$$Fy\in C([0,+\infty),\mathbb{R}),\qquad Fy(t)\geq 0,\quad t\in\mathbb{R}^+,\quad Fy(0)=\alpha(Fy)'(\eta),$$

and then (see Remark 2.1 in [6])

$$0\leq \lim_{t\to+\infty} e^{-\theta t}Fy(t)\leq \lim_{t\to+\infty}\frac{t+\alpha}{e^{\theta t}}\sup_{t\in\mathbb{R}^+}|(Fy)'(t)|\leq$$

$$\leq \lim_{t\to+\infty}\frac{t+\alpha}{e^{\theta t}}\,\phi^{-1}\left(\int_0^{+\infty} m(\tau)f\left(\tau,Ty(\tau)\right)d\tau\right)=0.$$

Now, we claim that $Fy(t)\geq\omega(t)$ on $\mathbb{R}^+$. On the contrary we assume

$$\sup_{t\in\mathbb{R}^+}\{\omega(t)-Fy(t)\}>0$$

and consider two cases:

Case 1. $\sup_{t\in\mathbb{R}^+}\{\omega(t)-Fy(t)\}=\lim_{t\to+\infty}\{\omega(t)-Fy(t)\}>0$. From (3.4), we have

$$\lim_{t\to+\infty}\{\omega(t)-Fy(t)\}=$$

$$=\alpha\phi^{-1}\left(\int_\eta^{+\infty} m(\tau)d\tau\right)+\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} m(\tau)d\tau\right)ds-$$

$$-\alpha\phi^{-1}\left(\int_\eta^{+\infty} m(\tau)f(\tau,Ty(\tau))d\tau\right)-$$

$$-\int_0^{+\infty}\phi^{-1}\left(\int_s^{+\infty} m(\tau)f(\tau,Ty(\tau))d\tau\right)ds\leq 0,$$

leading to a contradiction.

Case 2. There exists a real number $t_1\geq 0$ such that

$$\sup_{t\in\mathbb{R}^+}\{\omega(t)-Fy(t)\}=\omega(t_1)-Fy(t_1)>0.$$

Arguing as in Case 1, we can see that $\omega(t_1)-Fy(t_1)\leq 0$ which is again a contradiction, ending the proof of the lemma.

To prove the compactness of the operator F, we need the following result which is a direct consequence of Lemma 2.2.

Lemma 3.4. *Let $M \subseteq \mathbb{Y}$. Then M is relatively compact in $\mathbb{Y}$ if the following conditions hold*:

(a) *M is uniformly bounded in $\mathbb{Y}$.*

(b) *The functions belonging to the set $\mathcal{A} = \{x \mid x(t) = y(t)/e^{\theta t},\ y \in M\}$ are almost equicontinuous on $\mathbb{R}^+$.*

(c) *The functions from $\mathcal{A}$ are equiconvergent at $+\infty$.*

Since m may be singular at the origin, we first consider the regular case; then we argue by approximation.

Lemma 3.5. *Assume that $m\colon (0,+\infty) \to [0,+\infty)$ is continuous and bounded at the origin. Then, the mapping $F\colon \overline{\Omega} \cap \mathcal{K} \to \mathcal{K}$ is completely continuous.*

Proof. *Claim* 1. F is continuous on $\mathcal{K}$. Let $(y_n)_n$ be some sequence converging to some limit y in $\mathcal{K}$; then there exists $N > 0$ independent of n such that

$$\max\left\{\|y\|_\theta,\ \sup_{n\ge 1}\|y_n\|_\theta\right\} \le N.$$

Letting

$$S_N = \sup\left\{g(t, e^{\theta t}y) \mid t \in [0,+\infty), y \in [0,N]\right\},$$

we get

$$\int_0^{+\infty} m(s)[g(s,y_n(s)) - g(s,y(s))]ds \le 2AS_N.$$

Then, the Lebesgue dominated convergence theorem together with the continuity of f, g, and ϕ^{-1} yield the estimates:

$$e^{-\theta t}|(Fy_n)(t) - (Fy)(t))| =$$

$$= e^{-\theta t}\left|\alpha\phi^{-1}\left(\int_\eta^{+\infty} m(\tau)f(\tau, Ty_n(\tau))d\tau\right) + \right.$$

$$+ \int_0^t \phi^{-1}\left(\int_s^{+\infty} m(\tau)f\left(\tau, Ty_n(\tau)\right)d\tau\right)ds -$$

$$-\alpha\phi^{-1}\left(\int_\eta^{+\infty} m(\tau)f(\tau, Ty(\tau))d\tau\right) -$$

$$\left. -\int_0^t \phi^{-1}\left(\int_s^{+\infty} m(\tau)f\left(\tau, Ty(\tau)\right)d\tau\right)ds\right| \longrightarrow 0, \quad \text{as} \quad n \to +\infty.$$

Consequently,

$$\|Fy_n - Fy\|_\theta \longrightarrow 0, \quad \text{as} \quad n \to +\infty,$$

which proves the claim.

Claim 2. F is completely continuous, i.e., it maps bounded sets into relatively compact sets. Let B be a bounded subset of $\mathbb{Y}$; then there exists $M > 0$ such that $\|y\|_\theta \le M$, for all $y \in \overline{B} \cap \mathcal{K}$. On one hand, we obtain

$$\|Fy\|_\theta = \sup_{t\in[0,+\infty)} \left(e^{-\theta t} \left(C + \int_0^t \phi^{-1} \left(\int_s^{+\infty} m(\tau) f\left(\tau, Ty_n(\tau)\right) d\tau \right) ds \right) \right) \le$$

$$\le K\, \phi^{-1} \left(\int_0^{+\infty} m(\tau) f\left(\tau, \phi^{-1}(AS_M)(\alpha+\tau)\right) d\tau \right) < \infty \qquad \forall\, y \in \overline{B} \cap \mathcal{K},$$

which implies that the set $F(\overline{B} \cap \mathcal{K})$ is uniformly bounded. On the other hand, for all $y \in \overline{B} \cap \mathcal{K}$, $\beta \in (0, +\infty)$ and $t_1, t_2 \in [0, \beta]$ $(t_1 < t_2)$, we have the estimates

$$\left| \frac{Fy(t_2)}{e^{\theta t_2}} - \frac{Fy(t_1)}{e^{\theta t_1}} \right| =$$

$$= \left| C\left(e^{-\theta t_2} - e^{-\theta t_1}\right) + e^{-\theta t_2} \int_0^{t_2} \phi^{-1} \left(\int_s^{+\infty} m(\tau) f(\tau, Ty(\tau)) d\tau \right) ds - \right.$$

$$\left. - e^{-\theta t_1} \int_0^{t_1} \phi^{-1} \left(\int_s^{+\infty} m(\tau) f(\tau, Ty(\tau)) d\tau \right) ds \right| \le$$

$$\le \left| e^{-\theta t_2} - e^{-\theta t_1} \right| \left(C + \int_0^{t_1} \phi^{-1} \left(\int_0^{+\infty} m(\tau) f(\tau, \phi^{-1}(AS_M)(\alpha+\tau))\, d\tau \right) ds \right) +$$

$$+ e^{-\theta t_1} \left| \int_{t_1}^{t_2} \phi^{-1} \left(\int_0^{+\infty} m(\tau) f(\tau, \phi^{-1}(AS_M)(\alpha+\tau)\, d\tau \right) ds \right|.$$

The terms in the last two lines tend to 0, as $|t_1 - t_2| \to 0$. Hence $F(\overline{B} \cap \mathcal{K})$ is equicontinuous. Finally, Assumption $(\mathcal{H}_1)$ yields

$$\int_0^{+\infty} m(s) f(s, \phi^{-1}(AS_M)(\alpha+s)) ds < \infty,$$

then

$$\lim_{t\to+\infty} \left| \frac{Fy(t)}{e^{\theta t}} - \lim_{s\to+\infty} \frac{Fy(s)}{e^{\theta s}} \right| \le$$

$$\le \lim_{t\to+\infty} \frac{t+\alpha}{e^{\theta t}}\, \phi^{-1} \left(\int_0^{+\infty} m(\tau) f(\tau, \phi^{-1}(AS_M)(\alpha+\tau)) d\tau \right) = 0.$$

This means that $F(\overline{B}\cap\mathcal{K})$ is equiconvergent at $+\infty$. Using Lemma 3.4, we conclude that $F(\overline{B}\cap\mathcal{K})$ is relatively compact, ending the proof of the lemma.

Lemma 3.6. *Let m be singular at $t=0$. Then the mapping F given by* (3.3) *is completely continuous.*

Proof. Let $B\subset\mathbb{Y}$ be a bounded subset. For each $n\geq 1$, define the approximating operator F_n on $\overline{B}\cap\mathcal{K}$ by

$$F_n y(t)=C+\int\limits_{\frac{1}{n}}^{t}\phi^{-1}\left(\int\limits_{s}^{+\infty}m(\tau)f(\tau,Ty(\tau))d\tau\right)ds,\qquad t\in I.$$

It suffices to prove that F_n converges uniformly to F on $\overline{B}\cap\mathcal{K}$. For every $t\in I$ and $y\in\overline{B}\cap\mathcal{K}$ satisfying $\|y\|_\theta\leq M$, the following estimates follow from ($\mathcal{H}_1$) and ($\mathcal{H}_2$):

$$e^{-\theta t}|F_n y(t)-Fy(t))|=$$

$$=\left|\int\limits_{0}^{\frac{1}{n}}e^{-\theta t}\phi^{-1}\left(\int\limits_{s}^{+\infty}m(\tau)f(\tau,Ty(\tau))d\tau\right)ds\right|\leq$$

$$\leq\frac{1}{n}\phi^{-1}\left(\int\limits_{0}^{+\infty}m(\tau)f(\tau,\phi^{-1}(AS_M)(\alpha+\tau))d\tau\right).$$

Consequently, Assumptions ($\mathcal{H}_1$) and ($\mathcal{H}_2$) together with the Cauchy criterion for convergence of integrals imply that

$$\|F_n y-Fy\|_\theta\longrightarrow 0,\ \text{ as } n\to+\infty.$$

Since, from Lemma 3.5, for each $n\geq 1$, the operator $F_n:\ \overline{B}\cap\mathcal{K}\to\mathcal{K}$ is completely continuous and F_n converges to F uniformly on closed bounded subsets of $\overline{B}\cap\mathcal{K}$, the uniform limit operator F is completely continuous, ending the proof of the lemma.

3.2. Existence result. Let K be defined by (3.1) and for each $M>0$

$$S_M=\sup\{g(t,e^{\theta t}y)\mid\ t\in\mathbb{R}^+,\ y\in[0,M]\}.$$

Our main existence result in this section is the following theorem.

Theorem 3.1. *Assume that Assumptions* ($\mathcal{H}_1$), ($\mathcal{H}_2$) *hold together with*

($\mathcal{H}_4$) $$\begin{cases}f(t,e^{\theta t}y)\leq a_1(t)\,(\phi(y)+y)+b_1(t),\\ g(t,e^{\theta t}y)\leq a_2(t)\,(\phi(y-\omega)+y-\omega)+b_2(t),\end{cases}$$

for all $(t,y)\in\mathbb{R}^+\times\mathbb{R}^+$, where ma_i, mb_i belong to $L^1(I)$, $i=1,2$.

($\mathcal{H}_5$) *There exists $R>0$ such that*

$$K\phi^{-1}\left\{|ma_1|_{L^1}\left(\phi(KR)+KR\right)+|mb_1|_{L^1}\right\}+\|\omega\|_\theta<R,\tag{3.5}$$

$$\phi^{-1}\left\{|ma_2|_{L^1}\left(\phi(R)+R\right)+|mb_2|_{L^1}\right\}<R.\tag{3.6}$$

($\mathcal{H}_6$) *There exist positive constants* $\|\omega\|_\theta < \rho < R$ *and* γ, δ $(\eta < \gamma < \delta)$ *such that, for all* $(t, y) \in [\gamma, \delta] \times [\omega(t), \rho]$,

$$g(t, e^{\theta t} y) > \ell \phi^2(e^{\theta t} y),$$

and for all $t \in [\gamma, \delta], \quad y \geq \mu,$

$$f(t, y) \geq \lambda y,$$

where

$$\ell := \frac{\phi^2(1/\gamma)}{\int\limits_\gamma^{+\infty} m(t)dt}, \qquad \lambda = \left(\gamma \int\limits_\gamma^{+\infty} m(t)dt \right)^{-1},$$

and

$$\mu = (\alpha + \gamma)\phi^{-1} \left(\Lambda \int\limits_\gamma^\delta m(t)dt \right),$$

where $\Lambda = \min_{t \in [\gamma,\delta],\, y \in [0,\rho]} g(t, e^{\theta t} y)$.

Then problem (1.5) *has at least one positive solution* (y, x) *such that*

$$0 < \|y - \omega\|_\theta < R, \qquad \|x\|_\theta \leq K\phi^{-1}(AS_R),$$

and for all $t \in [0, +\infty)$,

$$y(t) > \alpha\phi^{-1} \left(\int\limits_\gamma^\delta m(\tau)d\tau \right) + \int\limits_0^t \phi^{-1} \left(\int\limits_s^{+\infty} m(\tau)d\tau \right) ds,$$

$$x(t) > \alpha\phi^{-1} \left(\int\limits_\gamma^\delta m(s)g(s, y(s))ds \right).$$

Proof. Consider the open ball

$$\Omega_R = \{y \in \mathbb{Y} \colon \|y - \omega\|_\theta < R\}.$$

From Lemma 3.5, the operator $F \colon \overline{\Omega}_R \cap \mathcal{K} \to \mathcal{K}$ is completely continuous.

Claim 1. $Fy - \omega \neq \lambda(y - \omega)$, for all $y \in \partial\Omega_R \cap \mathcal{K}$ and $\lambda \geq 1$. Let $y \in \partial\Omega_R \cap \mathcal{K}$. Using ($\mathcal{H}_4$) and the inequality (3.6), for all positive t, the following estimates hold true:

$$Ty(t) = \alpha\phi^{-1} \left(\int\limits_\eta^{+\infty} m(\sigma)g(\sigma, y(\sigma))d\sigma \right) +$$

$$+ \int\limits_0^t \phi^{-1} \left(\int\limits_s^{+\infty} m(\sigma)g(\sigma, y(\sigma))d\sigma \right) ds \leq$$

$$\leq \alpha\phi^{-1}\left(\int\limits_0^{+\infty} m(\sigma)\left(a_2(\sigma)\left(\phi(e^{-\theta\sigma}(y(\sigma)-\omega(\sigma)))\ +\right.\right.\right.$$

$$\left.\left.\left.+e^{-\theta\sigma}(y(\sigma)-\omega(\sigma))\right)+b_2(\sigma)\right)d\sigma\right)+$$

$$+\int\limits_0^t \phi^{-1}\left(\int\limits_0^{+\infty} m(\sigma)\left(a_2(\sigma)\left(\phi(e^{-\theta\sigma}(y(\sigma)-\omega(\sigma)))\ +\right.\right.\right.$$

$$\left.\left.\left.+e^{-\theta\sigma}(y(\sigma)-\omega(\sigma))\right)+b_2(\sigma)\right)d\sigma\right)ds\leq$$

$$\leq (t+\alpha)\phi^{-1}\left(|ma_2|_{L^1}\left(\phi(\|y-\omega\|_\theta)+\|y-\omega\|_\theta)\right)+|mb_2|_{L^1}\right)=$$

$$=(t+\alpha)\phi^{-1}\left(|ma_2|_{L^1}\left(\phi(R)+R\right)+|mb_2|_{L^1}\right)<R(t+\alpha).$$

Hence

$$e^{-\theta t}Ty(t)\leq\frac{t+\alpha}{e^{\theta t}}\,R\leq KR\qquad\forall\, t\geq 0.$$

Using (3.5), for all positive t, we deduce that

$$e^{-\theta t}|(Fy)(t)-\omega(t)|\leq$$

$$\leq \alpha e^{-\theta t}\phi^{-1}\left(\int\limits_\eta^{+\infty} m(\tau)f(\tau,Ty(\tau))d\tau\right)+$$

$$+e^{-\theta t}\int\limits_0^t\phi^{-1}\left(\int\limits_s^{+\infty}\phi^{-1}\int\limits_t^{+\infty} m(\tau)f(\tau,Ty(\tau),)d\tau\right)ds+e^{-\theta t}|\omega(t)|\leq$$

$$\leq \alpha e^{-\theta t}\phi^{-1}\left(\int\limits_0^{+\infty} m(\tau)\left(a_1(\tau)\left(\phi(e^{-\theta\tau}Ty(\tau))+e^{-\theta\tau}Ty(\tau)\right)+b_1(\tau)\right)d\tau\right)+$$

$$+e^{-\theta t}\int\limits_0^t\phi^{-1}\left(\int\limits_0^{+\infty} m(\tau)\left(a_1(\tau)\left(\phi(e^{-\theta\tau}Ty(\tau))+e^{-\theta\tau}Ty(\tau)\right)+b_1(\tau)\right)d\tau\right)ds+\|\omega\|_\theta\leq$$

$$\leq\frac{t+\alpha}{e^{\theta t}}\,\phi^{-1}\left(\left(|ma_1|_{L^1}\left(\phi(KR)+KR\right)+|mb_1|_{L^1}\right)\right)+\|\omega\|_\theta\leq$$

$$\leq K\phi^{-1}\left(\left(|ma_1|_{L^1}\left(\phi(KR)+KR\right)+|mb_1|_{L^1}\right)\right)+\|\omega\|_\theta<R.$$

Taking the least upper bound over t, we get

$$\|Fy - \omega\|_\theta < \|y - \omega\|_\theta \qquad \forall\, y \in \partial\Omega_R \cap \mathcal{K}. \tag{3.7}$$

As a consequence

$$Fy - \omega \neq \lambda(y - \omega) \qquad \forall\, y \in \partial\Omega_R \cap \mathcal{K} \qquad \forall\, \lambda \geq 1. \tag{3.8}$$

Indeed, on the contrary there would exist some $y_0 \in \partial\Omega_R \cap \mathcal{K}$ and $\lambda_0 \geq 1$ such that $Fy_0 - \omega = \lambda_0(y_0 - \omega)$. Hence

$$\|Fy_0 - \omega\|_\theta = \lambda_0\|y_0 - \omega\|_\theta \geq \|y_0 - \omega\|_\theta = R,$$

contradicting (3.7). This implies that (3.8) holds. Therefore, Lemma 3.1 yields

$$i\,(F, \Omega_R \cap \mathcal{K}, \mathcal{K}) = 1. \tag{3.9}$$

Finally (3.9) and the existence property of the fixed point index imply that the operator F has at least one nonnegative fixed point y which belongs to $\Omega_R \cap \mathcal{K}$.

Claim 2. Let $0 < \widehat{R} < \rho - \|\omega\|_\theta$ and consider the open ball

$$\Omega_{\widehat{R}} := \{y \in \mathbb{Y} \colon \|y - \omega\|_\theta < \widehat{R}\}.$$

We claim that

$$Fy \not\leq y, \ \text{ for all } \ y \in \partial\Omega_{\widehat{R}} \cap \mathcal{K}. \tag{3.10}$$

Otherwise, let $y_0 \in \partial\Omega_{\widehat{R}} \cap \mathcal{K}$ be such that $Fy_0 \leq y_0$. Then, in one hand,

$$0 \leq e^{-\theta t} y_0(t) \leq \widehat{R} + \|\omega\|_\theta < \rho \qquad \forall\, t \in [\gamma, \delta].$$

In the other one, by $(\mathcal{H}_6)$ and the definition of Λ, we obtain the estimates

$$Ty_0(t) = D + \int\limits_0^t \phi^{-1}\left(\int\limits_s^{+\infty} m(\sigma) g(\sigma, y_0(\sigma)) d\sigma\right) ds =$$

$$= \alpha\phi^{-1}\left(\int\limits_\eta^{+\infty} m(\sigma) g(\sigma, y_0(\sigma)) d\sigma\right) +$$

$$+ \int\limits_0^t \phi^{-1}\left(\int\limits_s^{+\infty} m(\sigma) g(\sigma, y_0(\sigma)) d\sigma\right) ds \geq$$

$$\geq \alpha\phi^{-1}\left(\int\limits_\gamma^{\delta} m(\sigma) g(\sigma, y_0(\sigma)) d\sigma\right) +$$

$$+ \int\limits_0^\gamma \phi^{-1}\left(\int\limits_\gamma^{\delta} m(\sigma) g(\sigma, y_0(\sigma)) d\sigma\right) ds \geq$$

$$\geq (\alpha+\gamma)\phi^{-1}\left(\Lambda\int\limits_{\gamma}^{\delta} m(\sigma)d\sigma\right) = \mu.$$

Making use of Assumption $(\mathcal{H}_6)$, we have also the following estimates valid for every $t \in [\gamma,\delta]$:

$$y_0(t) \geq C + \int\limits_0^t \phi^{-1}\left(\int\limits_s^{+\infty} m(\tau)f(\tau, Ty_0(\tau))d\tau\right) ds \geq$$

$$\geq \int\limits_0^{\gamma} \phi^{-1}\left(\int\limits_{\gamma}^{+\infty} m(\tau)f(\tau, Ty_0(\tau))d\tau\right) ds \geq$$

$$\geq \int\limits_0^{\gamma} \phi^{-1}\left(\int\limits_{\gamma}^{+\infty} m(\tau)\lambda\, Ty_0(\tau)d\tau\right) ds \geq$$

$$\geq \int\limits_0^{\gamma} \phi^{-1}\left(\int\limits_{\gamma}^{+\infty} m(\tau)\lambda\,\gamma\phi^{-1}\left(\int\limits_{\gamma}^{+\infty} m(\sigma)g(\sigma, y_0(\sigma))d\sigma\right) d\tau\right) ds >$$

$$> \gamma\phi^{-1}\left(\lambda\gamma\int\limits_{\gamma}^{+\infty} m(\tau)d\tau\phi^{-1}\left(\int\limits_{\gamma}^{+\infty} m(\sigma)\ell\phi^2(y_0(\sigma))d\sigma\right)\right).$$

Using the property (1.4) of ϕ^{-1} and the definition of λ, ℓ, we find a lower bound for y_0:

$$y_0(t) > \gamma\,\phi^{-1}\left(\phi^{-1}\left(\phi^2(\min_{t\in[\gamma,\delta]} y_0(t))\right)\phi^{-1}\left(\ell\int\limits_{\gamma}^{+\infty} m(\sigma)d\sigma\right)\right) \geq$$

$$\geq \gamma \min_{t\in[\gamma,\delta]} y_0(t)\ (\phi^{-1})^2\left(\ell\int\limits_{\gamma}^{+\infty} m(\sigma)d\sigma\right) = \min_{t\in[\gamma,\delta]} y_0(t).$$

Hence for every $t \in [\gamma,\delta]$, $y_0(t) > \min_{t\in[\gamma,\delta]} y_0(t)$, contradicting the continuity of the function y_0 on the compact interval $[\gamma,\delta]$. This implies that (3.8) holds. As a consequence, Lemma 3.2 yields

$$i\,(F, \Omega_{\widehat{R}} \cap \mathcal{K}, \mathcal{K}) = 0. \tag{3.11}$$

To sum up, from (3.10), (3.11), and the fact that $\overline{\Omega}_{\widehat{R}} \subset \Omega_R$, we conclude that

$$i\,(F, (\Omega_R \setminus \overline{\Omega}_{\widehat{R}}) \cap \mathcal{K}, \mathcal{K}) = 1.$$

Therefore, there exists at least one positive fixed point $y \in \mathcal{K}$ satisfying

$$\widehat{R} < \|y-\omega\|_\theta < R \quad \text{and} \quad y(t) \geq \omega(t), \quad t \geq 0.$$

Moreover, for positive t,

$$x(t) = Ty(t) = \alpha\phi^{-1}\left(\int\limits_{\eta}^{+\infty} m(\sigma)g(\sigma, y(\sigma))d\sigma\right) +$$

$$+\int\limits_{0}^{t}\phi^{-1}\left(\int\limits_{s}^{+\infty} m(\sigma)g(\sigma, y(\sigma))d\sigma\right)ds \leq$$

$$\leq (\alpha + t)\phi^{-1}\left(AS_{R+\|\omega\|_\theta}\right).$$

Hence

$$\frac{x(t)}{e^{\theta t}} \leq \frac{\alpha + t}{e^{\theta t}}\,\phi^{-1}\left(AS_{R+\|\omega\|_\theta}\right).$$

Passing to the least upper bound over t yields

$$\|x\|_\theta \leq K\phi^{-1}\left(AS_{R+\|\omega\|_\theta}\right).$$

Finally, for all positive t,

$$x(t) \geq \alpha\,\phi^{-1}\left(\int\limits_{\eta}^{+\infty} m(\sigma)g(\sigma, y(\sigma))d\sigma\right) > \alpha\,\phi^{-1}\left(\int\limits_{\gamma}^{\delta} m(\sigma)g(\sigma, y(\sigma))d\sigma\right),$$

which completes the estimates of the solution (y, x).

Theorem 3.1 is proved.

3.3. Example. Let

$$a_1(t) = \begin{cases} e^{-\frac{4}{5}}, & \text{if } 0 \leq t \leq 1, \\ e^{-\frac{4}{5}t}, & \text{if } t \geq 1, \end{cases} \qquad a_2(t) = \begin{cases} \dfrac{1}{1000}e^{-200000}, & \text{if } 0 \leq t \leq 1, \\ \dfrac{1}{1000}e^{-200000t}, & \text{if } t \geq 1, \end{cases}$$

$$b_1(t) = \frac{1}{9}, \quad b_2(t) = 20, \quad \text{and} \quad m(t) = \begin{cases} t^{-\frac{1}{10}}, & \text{if } 0 < t \leq 1, \\ t^{-\frac{12}{10}}, & \text{if } t \geq 1, \end{cases}$$

and define the increasing homeomorphism ϕ by

$$\phi(x) = \begin{cases} \dfrac{1}{3}x^2, & \text{if } x \geq 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

In order to check the inequality (3.5) in Assumption ($\mathcal{H}_5$), choose $\alpha = \dfrac{1}{2}$ and $\eta = \dfrac{1}{5}$. Thus we can take $\theta = 1$ and $R = 20$. So $K = \dfrac{1213}{2000}$ and $\omega(t) = \dfrac{1}{6}\sqrt{165 - 6\sqrt[10]{5} + \dfrac{10\sqrt{15}}{9}\sqrt[10]{t^9}}$, for $t \geq 0$. Then $\|\omega\|_\theta = \dfrac{2267}{849}$,

$$K\phi^{-1}\left[|ma_1|_{L^1}\left(\phi(KR)+KR\right)+|mb_1|_{L^1}\right]+\|\omega\|_\theta=\frac{3544}{355}<R,$$

and

$$\phi^{-1}\left[|ma_2|_{L^1}\left(\phi(R)+R\right)+|mb_2|_{L^1}\right]=\frac{1934}{101}<R.$$

Now consider the nonlinearities f, g defined for $(t,y)\in(\mathbb{R}^+)^2$ by

$$f(t,y)=a_1(t)\left(\phi(e^{-\theta t}y)+e^{-\theta t}y\right)+b_1(t)\geq 1,$$

$$g(t,y)=a_2(t)\left(\phi(e^{-\theta t}(y-\omega))+e^{-\theta t}(y-\omega)\right)+b_2(t)\geq 0.$$

Then, for every positive constant k, we have

$$\int\limits_0^{+\infty} m(s)f(s,k(\alpha+s))ds<\infty.$$

If we let $\gamma=\dfrac{2}{5}$ and $\delta=\dfrac{1}{2}$, then $\gamma>\eta$ and

$$\int\limits_\gamma^{+\infty} m(t)dt=\frac{8287}{1250}.$$

For $\|\omega\|_\theta<\rho=3<R$, we obtain

$$\Lambda=\min_{t\in[\gamma,\delta],\,y\in\mathbb{R}^+} g(t,e^{\theta t}y)=b_2(t)=20.$$

Then

$$\mu=(\alpha+\gamma)\phi^{-1}\left(\Lambda\int\limits_\gamma^\delta m(t)dt\right)=\frac{1402}{611}.$$

Consequently

$$\frac{f(t,y)}{y}\geq a_1(t)\left(\frac{1}{3}e^{-2\theta t}y+e^{-\theta t}\right)\geq e^{-\frac{4}{5}}\left(\frac{1}{3}e^{-2\theta\delta}\mu+e^{-\theta\delta}\right)=$$

$$=\frac{399}{1000}\geq\lambda=\frac{853}{2262},\quad\text{for all }\ t\in[\gamma,\delta],\quad y\geq\mu.$$

Finally, for all $(t,y)\in[\gamma,\delta]\times[\omega(t),\rho]$, we have the estimates

$$\frac{g(t,e^{\theta t}y)}{\phi^2(e^{\theta t}y)}\geq\frac{b_2(t)}{\phi^2(e^{\theta t}y)}\geq\frac{20}{\phi^2(e^{\theta\delta}\rho)}=369/409>\frac{\phi^2(1/\gamma)}{\displaystyle\int_\gamma^{+\infty}m(\tau)d\tau}=\frac{657}{3011}.$$

Therefore Assumptions $(\mathcal{H}_3)-(\mathcal{H}_6)$ in Theorem 3.1 are fulfilled. All the above computations have been undertaken using Matlab 7.9. Therefore, the singular problem

$$-(\phi(y'))'(t) = m(t)f(t, x(t)), \qquad t > 0,$$

$$-(\phi(x'))'(t) = m(t)g(t, y(t)), \qquad t > 0,$$

$$y(0) = y'\left(\frac{1}{5}\right), \quad \lim_{t\to+\infty} y'(t) = 0,$$

$$x(0) = x'\left(\frac{1}{5}\right), \quad \lim_{t\to+\infty} x'(t) = 0,$$

has at least one positive solution $(y, x) \in \mathbb{Y}^2$ satisfying $y(t) \geq \omega(t) \quad \forall\, t \in \mathbb{R}^+$.

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