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İ. Eryılmaz, C. Duyar (Ondokuz Mayis Univ., Turkey)

A STUDY OF FUNCTIONAL PROPERTIES AND MULTIPLIERS SPACES OF GROUP $L\left(p,q\right)\left(G\right)$-ALGEBRAS

ВИВЧЕННЯ ФУНКЦІОНАЛЬНИХ ВЛАСТИВОСТЕЙ ТА ПРОСТОРІВ МНОЖНИКІВ ДЛЯ ГРУП $L\left(p,q\right)\left(G\right)$-АЛГЕБР

Let G be a locally compact Abelian group (noncompact and nondiscrete) with Haar measure and suppose that $1 < p < \infty$ and $1 \leq q \leq \infty$. The purpose of the paper is to define temperate Lorentz spaces and study the spaces of multipliers on Lorentz spaces and characterize them as the spaces of multipliers of certain Banach algebras.

Нехай G є локальною компактною абелівською групою (некомпактною та недискретною) з мірою Хаара. Припустимо, що $1 < p < \infty$ та $1 \leq q \leq \infty$. Наша мета — визначити помірні простори Лоренца, вивчити простори множників на просторах Лоренца та охарактеризувати їх як простори множників на деяких банахових алгебрах.

1. Introduction and preliminaries. In this paper, we are interested in the relationship between the spaces of multipliers on $L\left(p,q\right)\left(G\right)$ and the spaces of multipliers $L^{t}\left(p,q\right)\left(G\right)$ on group $L\left(p,q\right)\left(G\right)$-algebra. The multipliers of type (p,p) and multipliers of the group L_p-algebras $\left(L_p^t\left(G\right)\right)$ were studied and developed by Mckennon [11 – 13] and Griffin [6] where the multipliers are identified as the operators commuting with the translation operators. The ideas in these papers are used frequently in our paper for the generalization of the results of Mckennon and Griffin concerning multipliers of type (p,p) to the Lorentz spaces $L\left(p,q\right)\left(G\right)$ for $1 \leq p < \infty$, $1 \leq q \leq \infty$.

For the convenience of the reader, we now review briefly what we need from the theory of $L\left(p,q\right)\left(G\right)$ Lorentz spaces. Let (G,Σ,μ) be a measure space and let f be a measurable function on G. For each $y > 0$, let

$$\lambda_f\left(y\right) = \mu\left\{x \in G\colon\ \left|f\left(x\right)\right| > y\right\}.$$

The function λ_f is called the distribution function of f. The rearrangement of f is defined by

$$f^{*}\left(t\right) = \inf\left\{y > 0\colon \lambda_f\left(y\right) \leq t\ \right\} = \sup\left\{y > 0\colon \lambda_f\left(y\right) > t\ \right\}, \quad t > 0,$$

where $\inf \varnothing = \infty$. Also the average function of f is defined by

$$f^{**}(t) = \frac{1}{t}\int\limits_0^t f^{*}\left(s\right)ds, \quad t > 0.$$

Note that $\lambda_f\left(\cdot\right), f^{*}\left(\cdot\right)$ and $f^{**}(\cdot)$ are nonincreasing and right continuous on $(0,\infty)$ [2, 9]. For $p,q \in (0,\infty)$ we define

$$\|f\|_{p,q}^{*} = \left(\frac{q}{p}\int\limits_0^\infty \left[f^{*}\left(t\right)\right]^q t^{q/p-1}dt\right)^{1/q}, \quad \|f\|_{p,q} = \left(\frac{q}{p}\int\limits_0^\infty \left[f^{**}\left(t\right)\right]^q t^{q/p-1}dt\right)^{1/q}.$$

Also, if $0 < p, q = \infty$ we define

$$\|f\|_{p,\infty}^{*} = \sup_{t>0} t^{1/p} f^{*}\left(t\right) \quad \text{and} \quad \|f\|_{p,\infty} = \sup_{t>0} t^{1/p} f^{**}\left(t\right).$$

For $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz spaces are denoted by $L(p,q)(G,\mu)$ (or shortly $L(p,q)(G)$) is defined to be the vector space of all (equivalence classes of) measurable functions f on G such that $\|f\|_{p,q}^{*} < \infty$. We know that $\|f\|_{p,p}^{*} = \|f\|_{p}$ and so $L^{p}(G) = L(p,p)(G)$, where $L^{p}(G)$ is the usual Lebesgue space. It is also known that if $1 < p < \infty$ and $1 \leq q \leq \infty$, then

$$\|f\|_{p,q}^{*} \leq \|f\|_{p,q} \leq \frac{p}{p-1}\|f\|_{p,q}^{*}$$

for each $f \in L(p,q)(G)$ and $\left(L(p,q)(G), \|\cdot\|_{p,q}\right)$ is a Banach space [1, 2, 9].

2. Main results. Let G be a local compact Abelian group, λ be a Haar measure on G and $C_c(G)$ be the space of complex-valued, continuous functions with compact support. For $1 < p < \infty$, $1 \leq q \leq \infty$, symbols $B_{p,q}$ and $M_{p,q}$ will stand for the following spaces:

$$B_{p,q} = \{\, T \mid T\colon L(p,q)(G) \to L(p,q)(G),\ T \text{ is bounded and linear} \,\},$$

$$M_{p,q} = \{T \in B_{p,q} \mid T(L_x) = L_x(T) \quad \text{for all } x \in G\},$$

where $L_x(\cdot)$ is the translation operator. It is so easy to show by usual techniques that the space $B_{p,q}$ is a Banach algebra with respect to composition under operator norm and the space $M_{p,q}$ is a complete subspace of $B_{p,q}$.

Let $\mathrm{Hom}_{L^1(G)}(L(p,q)(G), L(p,q)(G)) = \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$ denote the space of all module homomorphisms of $L^1(G)$-module $L(p,q)(G)$, that is, the space of operators in $B_{p,q}$ satisfying $T(f*g) = f*T(g)$ for all $f \in L^1(G)$ and $g \in L(p,q)(G)$. The module homomorphisms space is a Banach $L^1(G)$-module by $(f \circ T)(g) = f*T(g) = T(f*g)$ for all $g \in L(p,q)(G)$.

We briefly describe the content of this paper. In Subsection 2.1, we will construct pq-temperate functions spaces for Lorentz spaces and give some properties. In Subsection 2.2, we will characterize the multipliers space of $L(p,q)(G)$ as a certain Banach algebra and generalize the results of Mckennon to $L(p,q)(G)$. Finally in Subsection 2.3, we will examine the multipliers spaces $L^t(p,q)(G)$ on group $L(p,q)(G)$-algebras.

2.1. The space $L^t(p,q)(G)$ and its basic properties.

Definition 1. *Let $f \in L(p,q)(G)$ and $(h*f)(x) = \int_G h(t) f(t^{-1}x)\, d\lambda(t)$ be defined for all $h \in L(p,q)(G)$ and almost all $x \in G$. If $(h*f) \in L(p,q)(G)$ for all $h \in L(p,q)(G)$ and one of the following conditions*:

$$\sup\left\{\|h*f\|_{p,q} : h \in L(p,q)(G),\ \|h\|_{p,q} \leq 1\right\} < \infty, \tag{1}$$

$$\sup\left\{\|h*f\|_{p,q} : h \in C_c(G),\ \|h\|_{p,q} \leq 1\right\} < \infty \tag{2}$$

is satisfied, then the function f is called a pq-temperate and the space of these functions is showed by $L^t(p,q)(G)$. The spaces $L^t(p,q)(G)$ can be renamed as group $L(p,q)(G)$-algebras. The value in (1) *or* (2) *will be used for the norm of $f \in L^t(p,q)(G)$ and showed by $\|f\|_{p,q}^{t}$.*

Proposition 1. *$C_c(G)$ is a subspace of $\left(L^t(p,q)(G), \|\cdot\|_{p,q}^{t}\right)$. Also, $L^t(p,q)(G)$ is a dense subspace of $L(p,q)(G)$.*

***Remark* 1.** The space $L^1(G) \cap L(p,q)(G)$ is also contained by $L^t(p,q)(G)$. So we have another evidence for nonempty property of the space $L^t(p,q)(G)$.

Proposition 2. *For each $f \in L^t(p,q)(G)$, there is an operator W_f on $L(p,q)(G)$ by $W_f(\cdot) = (\cdot) * f$ which also belongs to $B_{p,q}$.*

Proof. Let's take any $f \in L^t(p,q)(G)$. Since $W_f(g) = g * f$ for all $g \in L(p,q)(G)$, W_f is well-defined by Definition 1. Also, the following equality:

$$\|W_f\| = \sup_{\|g\|_{p,q} \le 1} \|W_f(g)\|_{p,q} = \sup_{\|g\|_{p,q} \le 1} \|g * f\|_{p,q} = \|f\|_{p,q}^t$$

says that W_f is bounded on $L(p,q)(G)$. The linearity property of W_f can be seen easily. Thus $W_f \in B_{p,q}$.

Proposition 2 is proved.

We have also another good result here. That is, for all $f, g \in L^t(p,q)(G)$

$$W_{f*g} = W_g \circ W_f = W_{g*f} = W_f \circ W_g.$$

Proposition 3. *The space $\left(L^t(p,q)(G), \|\cdot\|_{p,q}^t\right)$ is*

(i) *a normed algebra under convolution,*

(ii) *a Banach $L^1(G)$-module under convolution,*

(iii) *strongly invariant under translation.*

Proposition 4. *The space $L^t(p,q)(G)$ is a Banach space under a new norm $|||\cdot|||_{p,q}^t = \|\cdot\|_{p,q} + \|\cdot\|_{p,q}^t$.*

Proof. For any $f \in L^t(p,q)(G) \subset L(p,q)(G)$, we know that $\|f\|_{p,q}^t < \infty$ and $\|f\|_{p,q} < \infty$. Since the function $|||\cdot|||_{p,q}^t = \|\cdot\|_{p,q} + \|\cdot\|_{p,q}^t$ is a sum of two known norms, the function $|||\cdot|||_{p,q}^t$ is a norm on $L^t(p,q)(G)$. We will prove the proposition in a classical way. Let $\{f_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $\left(L^t(p,q)(G), |||\cdot|||_{p,q}^t\right)$. Then, for each $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|||f_n - f_m|||_{p,q}^t < \varepsilon$ for all $m, n \ge N$. The inequality $|||\cdot|||_{p,q}^t \ge \|\cdot\|_{p,q}$ implies that $\|f_n - f_m\|_{p,q} < \varepsilon$ for all $m, n \ge N$ and so the sequence $\{f_n\}_{n\in\mathbb{N}}$ is also a Cauchy sequence in $\left(L(p,q)(G), \|\cdot\|_{p,q}\right)$. The completeness of $L(p,q)(G)$ shows that there exists a function $f \in L(p,q)(G)$ such that $f_n \to f$ in $L(p,q)(G)$. If we consider the sequence $\{W_{f_n}\}_{n\in\mathbb{N}}$ corresponding to $\{f_n\}_{n\in\mathbb{N}}$, then we see that $\{W_{f_n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $B_{p,q}$ and converges to an operator $W \in B_{p,q}$ such that

$$\lim_n \|f_n - f\|_{p,q} = 0 = \lim_n \|W_{f_n} - W\|.$$

Now, let's take any $g \in C_c(G)$ with $\|g\|_{p,q} \le 1$. Since $L(p,q)(G)$ is a Banach $L^1(G)$-module, we get $f_n * g \to f * g$ as $f_n \to f$ in $L(p,q)(G)$. Therefore, we have

$$\|g * f\|_{p,q} = \lim_n \|g * f_n\|_{p,q} \le \lim_n \|g\|_{p,q} \|f_n\|_{p,q} \le \lim_n \|f_n\|_{p,q} = \|f\|_{p,q}$$

and

$$\| f \|_{p,q}^t = \sup_{\|g\|_{p,q} \le 1} \|g * f\|_{p,q} \le \|f\|_{p,q} < \infty.$$

Also,

$$\lim_n |||f_n - f|||_{p,q}^t = \lim_n \left(\|f_n - f\|_{p,q} + \|f_n - f\|_{p,q}^t\right) =$$

$$= \lim_n \left(\|f_n - f\|_{p,q} + \sup_{\|g\|_{p,q} \le 1} \|g * (f_n - f)\|_{p,q} \right) \le$$

$$\le \lim_n \left(\|f_n - f\|_{p,q} + \sup_{\|g\|_{p,q} \le 1} \|g\|_{p,q} \|f_n - f\|_{p,q} \right) \le \lim_n 2 \|f_n - f\|_{p,q} = 0$$

and for any $h \in C_c(G)$

$$W(h) = \lim_n W_{f_n}(h) = \lim_n (h * f_n) = h * f = W_f(h)$$

can be found. Density of $C_c(G)$ in $L(p,q)(G)$ implies that $W = W_f$ and $\left(L^t(p,q)(G), \, |||\cdot|||^t_{p,q} \right)$ is a Banach space.

Proposition 4 is proved.

Proposition 5. *The space* $\left(L^t(p,q)(G), \, |||\cdot|||^t_{p,q} \right)$ *is a Banach algebra.*

Proof. Let's take any $f, g \in L^t(p,q)(G)$. Then

$$|||f * g|||^t_p = \|f * g\|_{p,q} + \|f * g\|^t_{p,q} \le \|W_f(g)\|_{p,q} + \|f\|^t_{p,q} \|g\|^t_{p,q} \le$$

$$\le \|W_f\| \, \|g\|_{p,q} + \|f\|^t_{p,q} \|g\|^t_{p,q} = \|f\|^t_{p,q} \|g\|_{p,q} + \|f\|^t_{p,q} \|g\|^t_{p,q} =$$

$$= \|f\|^t_{p,q} |||g|||^t_{p,q} \le |||f|||^t_{p,q} |||g|||^t_{p,q} \, .$$

Proposition 5 is proved.

Proposition 6. *The set*

$$\Lambda = \overline{\operatorname{span} \left\{ W_{f*g} \mid f \in L^t(p,q)(G), \, g \in C_c(G) \right\}}$$

is a complete subalgebra of $B_{p,q}$ *and it has a minimal approximate identity.*

Proof. By the definition of Λ, it is easy to see that Λ is a complete subalgebra of $B_{p,q}$. Let $\mathcal{F}$ be the family of all neighbourhoods of the identity of G and say $E_2 \prec E_1$ if $E_1 \subset E_2$ for $E_1, E_2 \in \mathcal{F}$. Then clearly $(\mathcal{F}, \prec)$ is a directed set. For every $E_i \in \mathcal{F}$, there exists a positive continuous function h_{E_i} on G such that $\int_G h_{E_i}(x) \, d\lambda(x) = 1$, the support of h_{E_i} is contained in E_i and $\|h_{E_i}\|_1 = 1$. This net $\{h_{E_i}\}_{i \in I} \subset C_c(G)$ is a minimal approximate identity for $L^1(G)$ [11, 16]. If $\{h_\gamma\}$ denotes the product net of $\{h_{E_i}\}_{i \in I}$ with itself, i.e., $h_\gamma = h_{E_i} * h_{E_j}$, then $\{h_\gamma\}$ is again a bounded approximate identity for $L^1(G)$. In other words, $\|h_\gamma\|_1 = \left\| h_{E_i} * h_{E_j} \right\|_1 \le \|h_{E_i}\|_1 \left\| h_{E_j} \right\|_1 \le 1$,

$$\lim_\gamma \|f - f * h_\gamma\|_1 = \lim_{i,j} \left\| f - f * \left(f_{E_i} * f_{E_j} \right) \right\| \le$$

$$\le \lim_i \|f - f * f_{E_i}\|_1 + \lim_j \left\| (f * f_{E_i}) - (f * f_{E_i}) * f_{E_j} \right\|_1 \le$$

$$\le \lim_i \|f - f * f_{E_i}\|_1 + \lim_j \left\| f - \left(f * f_{E_j} \right) \right\|_1 \|f_{E_i}\|_1 = 0$$

and

$$\|W_f\| = \sup_{\|h\|_{p,q}\le 1} \|W_f(h)\|_{p,q} = \sup_{\|h\|_{p,q}\le 1} \|h*f\|_{p,q} \le \|f\|_1 \tag{3}$$

for all $f \in L^1(G)$. Hence, it is seen that the net $\{W_{h_\gamma}\}$ is in Λ and $\lim_\gamma \|W_{h_\gamma}\| \le 1$. Since $C_c(G) \subset L^t(p,q)(G)$, we have

$$(W_f - W_g)(h) = h*f - h*g = h*(f-g) = W_{f-g}(h) \tag{4}$$

for any $f, g \in C_c(G)$ and $h \in L(p,q)(G)$. Therefore, by using (3), (4) and the equality $W_{f*g} = W_g \circ W_f$, we get

$$\overline{\lim_\gamma} \|W_{h_\gamma} \circ W_{f*g} - W_{f*g}\| = \overline{\lim_\gamma} \|W_{f*g*h_\gamma} - W_{f*g}\| =$$

$$= \overline{\lim_\gamma} \|W_{g*h_\gamma} \circ W_f - W_g \circ W_f\| =$$

$$= \overline{\lim_\gamma} \|(W_{g*h_\gamma} - W_g) \circ W_f\| \le \overline{\lim_\gamma} \|W_{g*h_\gamma} - W_g\| \, \|W_f\| =$$

$$= \overline{\lim_\gamma} \|W_{g*h_\gamma - g}\| \, \|W_f\| \le \overline{\lim_\gamma} \|g*h_\gamma - g\|_1 \|W_f\| = 0.$$

Therefore, the net $\{W_{h_\gamma}\}$ is a minimal approximate identity for the space Λ.

Proposition 6 is proved.

Proposition 7. *The space Λ is a complete subalgebra of* $\mathrm{Hom}_{L^1(G)}(L(p,q)(G))$.

Proof. Let $W \in \Lambda$. By the definition of the space Λ, there exists $\{W_{f_n}\}_{n\in\mathbb{N}} \subset \mathrm{span}\{W_{f*g} \mid f \in L^t(p,q)(G),\ g \in C_c(G)\}$ such that $W_{f_n} \to W$. Therefore for all $g \in L^1(G)$ and $h \in L(p,q)(G)$, we can write that

$$W(g*h) = \lim_n W_{f_n}(g*h) = \lim_n g*h*f_n = \lim_n g*W_{f_n}(h) = g*W(h). \tag{5}$$

From (5) we see that $W \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$. Since the space $\mathrm{Hom}_{L^1(G)}(L(p,q)(G))$ is a Banach algebra under usual operator norm, the space Λ is a complete subspace of $\mathrm{Hom}_{L^1(G)}(L(p,q)(G))$. Now, let $f_1, f_2 \in L^t(p,q)(G)$ with $W_{f_1}, W_{f_2} \in \Lambda$. Since $(f_1 * f_2) \in L^t(p,q)(G)$, we write that

$$(W_{f_1} \circ W_{f_2})(g*h) = W_{f_1}(W_{f_2}(g*h)) = W_{f_1}(g*h*f_2) =$$

$$= g*h*f_2*f_1 = g*W_{f_1}(h*f_2) =$$

$$= g*W_{f_1}(W_{f_2}(h)) = g*(W_{f_1} \circ W_{f_2})(h)$$

for all $g \in L^1(G)$ and $h \in L(p,q)(G)$. This shows that Λ is a complete subalgebra of the space $\mathrm{Hom}_{L^1(G)}(L(p,q)(G))$.

Proposition 7 is proved.

Proposition 8. *The space Λ is an essential Banach $L^1(G)$-module.*

Proof. Let's take any $W_f \in \Lambda$, $g \in L^1(G)$ and define the mapping $g \circ W_f : L(p,q)(G) \to L(p,q)(G)$ with $(g \circ W_f)(h) = W_f(g*h)$ for all $h \in L(p,q)(G)$. Then

$$\|g \circ W_f\| = \sup_{\|h\|_{p,q}\le 1} \|(g \circ W_f)(h)\|_{p,q} = \sup_{\|h\|_{p,q}\le 1} \|W_f(g*h)\|_{p,q} =$$

$$= \|W_f\| \sup_{\|h\|_{p,q}\leq 1} \|g * h\|_{p,q} \leq \|W_f\| \, \|g\|_1 = \|f\|_{p,q}^t \, \|g\|_1$$

is found. Thus, Λ is a Banach $L^1(G)$-module. Since any bounded approximate identity $\{e_\alpha\}_{\alpha\in I}$ of $L^1(G)$ is also an approximate identity for $L(p,q)(G)$, for any $W_f \in \Lambda$, we have

$$\|e_\alpha \circ W_f - W_f\| = \sup_{\|h\|_{p,q}\leq 1} \|(e_\alpha \circ W_f - W_f)(h)\|_{p,q} = \sup_{\|h\|_{p,q}\leq 1} \|W_f(h * e_\alpha) - W_f(h)\|_{p,q} =$$

$$= \sup_{\|h\|_{p,q}\leq 1} \|h * e_\alpha * f - h * f\|_{p,q} = \sup_{\|h\|_{p,q}\leq 1} \|h * (e_\alpha * f - f)\|_{p,q} =$$

$$= \sup_{\|h\|_{p,q}\leq 1} \|h\|_{p,q} \|e_\alpha * f - f\|_{p,q}$$

for all $h \in L(p,q)(G)$. As a result, Λ is an essential Banach $L^1(G)$-module by [3] (Corollary 15.3).

Proposition 8 is proved.

***Remark* 2.** If we consider any $f \in L^1(G)$ and the net $\{W_{e_\alpha}\}_{\alpha\in I} \in \Lambda$, then

$$\lim_\alpha \|f - f \circ W_{e_\alpha}\| = \lim_\alpha \left(\sup_{\|h\|_{p,q}\leq 1} \|(f - f \circ W_{e_\alpha})(h)\|_{p,q} \right) =$$

$$= \lim_\alpha \left(\sup_{\|h\|_{p,q}\leq 1} \|f * h - W_{e_\alpha}(f * h)\|_{p,q} \right) =$$

$$= \lim_\alpha \left(\sup_{\|h\|_{p,q}\leq 1} \|f * h - (e_\alpha * f) * h\|_{p,q} \right) \leq$$

$$\leq \lim_\alpha \left(\sup_{\|h\|_{p,q}\leq 1} \|f - e_\alpha * f\|_1 \|h\|_{p,q} \right) \leq$$

$$\leq \lim_\alpha \|f - e_\alpha * f\|_1 \sup_{\|h\|_{p,q}\leq 1} \|h\|_{p,q} = 0$$

is found. Therefore, $f \in \overline{L^1(G) \circ \Lambda} = \Lambda$. In other words, $L^1(G) \subset \Lambda$.

***Remark* 3.** If $p = q = 1$, then $L^t(p,q)(G) = L^t(1,1)(G) = L_1^t(G) = L_1(G)$ since $L_1(G)$ is a Banach algebra. If $p = q$, then $L^t(p,q)(G) = L^t(p,p)(G) = L_p^t(G)$ which is examined in [6,11,12].

2.2. Identification of multipliers space of $L^1(G)$-module with the multipliers space of certain normed algebra.

Proposition 9. *Let $f, g \in L(p,q)(G)$ and $T \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$.*

(i) *If $f \in L^t(p,q)(G)$, then $T(f) \in L^t(p,q)(G)$.*

(ii) *If $g \in L^t(p,q)(G)$, then $T(f * g) = f * T(g)$.*

Proof. (i) Let's take any $f \in L^t(p,q)(G)$. By the definition of $T \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$, we write

$$\|T(f)\|_{p,q}^t = \sup\left\{\|h * T(f)\|_{p,q} : h \in C_c(G),\ \|h\|_{p,q} \le 1\right\} =$$

$$= \sup\left\{\|T(h * f)\|_{p,q} : h \in C_c(G),\ \|h\|_{p,q} \le 1\right\} \le$$

$$\le \|T\| \sup\left\{\|h * f\|_{p,q} : h \in C_c(G),\ \|h\|_{p,q} \le 1\right\} \le \|T\|\,\|f\|_{p,q}^t < \infty$$

and the result follows.

(ii) Let $g \in L^t(p,q)(G)$. Since $C_c(G)$ is dense in $L(p,q)(G)$, we can find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_c(G)$ such that $\lim_n \|f_n - f\|_{p,q} = 0$ for all $f \in L(p,q)(G)$. Also, we can find a bounded linear mapping $W_g \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$ for all $g \in L^t(p,q)(G)$. Therefore,

$$\lim_n \|f_n * g - f * g\|_{p,q} = \lim_n \|W_g(f_n) - W_g(f)\|_{p,q} =$$

$$= \lim_n \|W_g(f_n - f)\|_{p,q} \le \|W_g\| \lim_n \|f_n - f\|_{p,q} = 0$$

is found. Since $T(g) \in L^t(p,q)(G)$ by (i), we get

$$\lim_n \|f_n * T(g) - f * T(g)\|_{p,q} = 0.$$

Therefore, the continuity of $T \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$ implies that

$$f * T(g) = \lim_n f_n * T(g) = \lim_n T(f_n * g) = T(f * g).$$

Proposition 9 is proved.

Definition 2. *For the space* Λ, *the space* Λ^* *is defined as follows:*

$$\Lambda^* = \left\{T \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G)) : T \circ W \in \Lambda \ \textit{for all}\ W \in \Lambda\right\}.$$

Proposition 10. $\Lambda^* = \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$.

Proof. Let $T \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$ and $W_{f*g} \in \Lambda$ such that $f \in L^t(p,q)(G)$ and $g \in C_c(G)$. By using Proposition 9, since $T(f) \in L^t(p,q)(G)$, we have

$$(T \circ W_{f*g})(h) = T(W_{f*g}(h)) = T(h * f * g) = h * T(f * g) = W_{T(f*g)}(h) =$$

$$= W_{g*T(f)}(h)$$

for all $h \in L(p,q)(G)$. This means that $T \in \Lambda^*$. On the other hand, the inclusion $\Lambda^* \subset \subset \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$ is obvious from the definition of Λ^*.

Proposition 10 is proved.

Proposition 11. $M(\Lambda,\Lambda)$, *the space of multipliers on Banach algebra* Λ, *is isometrically isometric to the space* Λ^*.

Proof. Define a mapping $F\colon \Lambda^* \to M(\Lambda,\Lambda)$ by letting $F(T) = \rho_T$ for each $T \in \Lambda^*$, where $\rho_T(S) = T \circ S$ for all $S \in \Lambda$. Note that, if $\rho_T(S \circ K) = T \circ S \circ K = \rho_T(S) \circ K$ for all $S, K \in \in \Lambda$, then we see that $F(T) = \rho_T \in M(\Lambda,\Lambda)$ and so F is well-defined. Also, it is obvious that the mapping F is linear.

For any $T_1, T_2 \in \Lambda^*$, if $F(T_1) = F(T_2)$, then $\rho_{T_1} = \rho_{T_2}$ and $\rho_{T_1}(S) = \rho_{T_2}(S)$ for all $S \in \Lambda$. This means that $T_1 \circ S = T_2 \circ S$ and $T_1 = T_2$. Hence, the mapping F is one-to-one.

Beside these, for any $T \in \Lambda^* = \operatorname{Hom}_{L^1(G)}(\Lambda,\Lambda)$ and for all $S \in \Lambda$, we get

$$\|T \circ S\| = \sup_{\|g\|_{p,q} \le 1} \|(T \circ S)(g)\|_{p,q} = \sup_{\|g\|_{p,q} \le 1} \|T(S(g))\|_{p,q} \le$$

$$\le \|T\| \sup_{\|g\|_{p,q} \le 1} \|S(g)\|_{p,q} = \|T\|\,\|S\|.$$

Therefore, we have

$$\|\rho_T\| = \sup_{S \in \Lambda} \frac{\|\rho_T(S)\|}{\|S\|} = \sup_{S \in \Lambda} \frac{\|T \circ S\|}{\|S\|} \le \sup_{S \in \Lambda} \frac{\|T\|\,\|S\|}{\|S\|} = \|T\|. \tag{6}$$

On the other hand, since the net $\{W_{h_\gamma}\}$ is a minimal approximate identity for Λ, we can write the following inequality:

$$\|\rho_T\| = \sup_{S \in \Lambda} \frac{\|T \circ S\|}{\|S\|} \ge \sup_{\gamma} \frac{\|T \circ W_{h_\gamma}\|}{\|W_{h_\gamma}\|} \ge \sup_{\gamma} \|T \circ W_{h_\gamma}\| \ge$$

$$\ge \lim_{\gamma} \|T \circ W_{h_\gamma}\| = \|T\|. \tag{7}$$

By (6) and (7), we see that $\|\rho_T\| = \|T\|$.

Lastly, we need to show that the mapping F is onto. Let's take any $\rho \in M(\Lambda,\Lambda)$ and the minimal approximate identity $\{e_\alpha\}_{\alpha \in I}$ of $L^1(G)$. Since, it is known that $\Lambda \subset \operatorname{Hom}_{L^1(G)}(L(p,q)(G))$ and $\rho e_\alpha \in \Lambda$, we get

$$\rho e_\alpha(f * g) = (f \circ (\rho e_\alpha))(g), \tag{8}$$

where $f \in L^1(G)$ and $g \in L(p,q)(G)$. If we use the property $M(\Lambda,\Lambda) \subset \operatorname{Hom}_{L^1(G)}(\Lambda,\Lambda)$, then we can write the equalities

$$\rho(f \circ e_\alpha)(g) = \rho(f * e_\alpha)(g) = (f \circ \rho e_\alpha)(g). \tag{9}$$

From (8) and (9), we have

$$\rho e_\alpha(f * g) = (f \circ (\rho e_\alpha))(g) = \rho(f * e_\alpha)(g).$$

Therefore, for all $f \in L^1(G)$ and $g \in L(p,q)(G)$

$$\lim_{\alpha} \|\rho(f * e_\alpha)(g) - \rho f(g)\|_{p,q} = \lim_{\alpha} \|(\rho(f * e_\alpha) - \rho f)(g)\|_{p,q} = \lim_{\alpha} \|\rho(f * e_\alpha - f)(g)\|_{p,q} \le$$

$$\le \lim_{\alpha} \|\rho(f * e_\alpha - f)\|\,\|g\|_{p,q} \le \lim_{\alpha} \|\rho\|\,\|(f * e_\alpha - f)\|\,\|g\|_{p,q} \le$$

$$\leq \lim_{\alpha} \|\rho\| \, \|(f * e_{\alpha} - f)\|_1 \, \|g\|_{p,q} = 0$$

is found and we can write that

$$\lim_{\alpha} \left(\rho e_{\alpha}\right)\left(f * g\right) = \lim_{\alpha} \left(f \circ \left(\rho e_{\alpha}\right)\right)\left(g\right) = \lim_{\alpha} \rho \left(f * e_{\alpha}\right)\left(g\right) = \rho f\left(g\right).$$

Since $L\left(p,q\right)\left(G\right)$ is an essential Banach $L^1\left(G\right)$-module, the following expression:

$$\left(\rho e_{\alpha}\right)\left(f * g\right) = \left(f \circ \rho e_{\alpha}\right)\left(g\right) = f * \left(\rho e_{\alpha}\right)\left(g\right)$$

converges to such a $f * T\left(g\right) \in L\left(p,q\right)\left(G\right)$ where $T \in \mathrm{Hom}_{L^1(G)}\left(L\left(p,q\right)\left(G\right)\right)$. Since $\lim_{\alpha}\left(\rho e_{\alpha}\right)\left(f * g\right) = \lim_{\alpha}\left(f \circ \left(\rho e_{\alpha}\right)\right)\left(g\right) = \rho f\left(g\right)$, we write $f \circ T = \rho f$ for all $f \in L^1\left(G\right)$. Thus, for any $W \in \Lambda$, we have

$$e_{\alpha} \circ T \circ W = \left(\rho e_{\alpha}\right) \circ W = \rho\left(e_{\alpha} \circ W\right).$$

Since Λ is an essential Banach $L^1\left(G\right)$-module, we have $T \circ W = \rho\left(W\right)$ and $\rho\left(W\right) = \rho_T\left(W\right)$ for all $W \in \Lambda$. This means that $\rho = \rho_T$.

Proposition 11 is proved.

Conclusion 1. *The space* $M\left(\Lambda,\Lambda\right)$ *is isometrically isomorphic to* $\mathrm{Hom}_{L^1(G)}\left(L\left(p,q\right)\left(G\right)\right)$.

2.3. Multipliers on group $\mathbf{L}\left(\mathbf{p},\mathbf{q}\right)\left(\mathbf{G}\right)$-algebras. If an operator $T \in B_{p,q}$ satisfies the condition $T\left(f * g\right) = f * T\left(g\right)$ for all $f, g \in L^t\left(p,q\right)\left(G\right)$, then we will call the operator T as a multiplier on the space $L^t\left(p,q\right)\left(G\right)$. The space of all multipliers on $L^t\left(p,q\right)\left(G\right)$ will be showed by the symbol $m_{p,q}$. Since, one can show by using usual techniques that the space $m_{p,q}$ is a Banach algebra under operator norm with composition, we will omit its proof.

Lemma 1. *For any* $f \in L\left(p,q\right)\left(G\right)$ *with* $f \neq 0$, *there exists a* $g \in C_c\left(G\right)$ *such that* $f * g \neq 0$.

Proof. Let's take a function $f \in L\left(p,q\right)\left(G\right)$ with $f \neq 0$ and assume that $f * g = 0$ for all $g \in C_c\left(G\right)$. We know by [5] that

$$\|f\|_{p,q} \leq p' \|f\|_{p,q}^{*} \leq p' C \sup \left\{ \left| \int_G f\left(x\right) g\left(-x\right) d\mu\left(x\right) \right| : \, g \in C_c\left(G\right), \, \|g\|_{p',q'}^{*} \leq 1 \right\} =$$

$$= p' C \sup \left\{ \left|\left(f * g\right)\left(0\right)\right| : \, g \in C_c\left(G\right), \, \|g\|_{p',q'}^{*} \leq 1 \right\}.$$

Therefore $\|f\|_{p,q} = 0$ and so $f = 0$ (a.e.). This contradiction proves the lemma.

Lemma 2. *For any* $T \in m_{p,q}$ *and* $V \in \Lambda$, *we have*

$$\sup \left\{ \|T \circ V\left(h\right)\|_{p,q} : \, h \in L^t\left(p,q\right)\left(G\right), \, \|h\|_{p,q} \leq 1 \right\} \leq \|T\| \, \|V\|.$$

Proof. Let's define a set $D = \left\{W_f : f \in L^t\left(p,q\right)\left(G\right), \, W_f \in \Lambda\right\}$ and take any $W \in \Lambda$. By the definiton of Λ, for all $\varepsilon > 0$, we can find $f \in L^t\left(p,q\right)\left(G\right)$ and $g \in C_c\left(G\right)$ such that $\|W - W_{f*g}\| < \varepsilon$. Since $f * g \in L^t\left(p,q\right)\left(G\right)$, we can conclude that $W_{f*g} \in D$ and so $\bar{D} = \Lambda$. Now, define a map $\varphi' \colon D \to B_{p,q}$ such that $\varphi'\left(W_f\right) = W_{T(f)}$ for any $T \in m_{p,q}$. Since it is easy to show that φ' is bounded, it has a continuous extension $\varphi \colon \Lambda \to B_{p,q}$ with $\|\varphi'\| = \|\varphi\|$.

Now, we will show the operators $\varphi\left(V\right)$ and $\left(T \circ V\right)$ coinciding on $L^t\left(p,q\right)\left(G\right)$. Let's take any $h \in L^t\left(p,q\right)\left(G\right)$, $\|h\|_{p,q} \leq 1$ and $V \in \Lambda$. Since $\bar{D} = \Lambda$, we can write

$$\lim_n \|W_{f_n} - V\| = 0, \tag{10}$$

where $\{W_{f_n}\}_{n\in\mathbb{N}} \in D$ and $\{f_n\}_{n\in\mathbb{N}} \in L^t(p,q)(G)$ for all $n \in \mathbb{N}$. Also, since

$$W_g(L_x f) = g * L_x f = L_x(f * g) = L_x W_g(f)$$

for all $f \in L(p,q)(G)$ and $W_g \in \Lambda$, we say that $\Lambda \subset M_{p,q}$ and so $V \in M_{p,q}$. By Proposition 9, we have

$$(V \circ W_h)(g) = V(W_h(g)) = V(g * h) = g * V(h) = W_{V(h)}(g)$$

for all $g \in L(p,q)(G)$ and so $V \circ W_h = W_{V(h)}$. Again, the equality $W_{W_{f_n}(h)} = W_{f_n} \circ W_h$ implies that

$$\|W_{f_n}(h) - V(h)\|_{p,q}^t = \left\|W_{W_{f_n}(h)} - W_{V(h)}\right\| = \|W_{f_n} \circ W_h - V \circ W_h\| =$$

$$= \|(W_{f_n} - V) \circ W_h\|$$

for all $n \in \mathbb{N}$. Therefore

$$\overline{\lim_n} \|W_{f_n}(h) - V(h)\|_{p,q}^t = \overline{\lim_n} \|(W_{f_n} - V) \circ W_h\| \leq \overline{\lim_n} \|W_{f_n} - V\| \|W_h\| = 0$$

is found. As a result, for $T \in m_{p,q}$,

$$\lim_n \|T(W_{f_n}(h)) - T(V(h))\|_{p,q}^t = 0 \tag{11}$$

can be written. For any $g \in L^t(p,q)(G)$ and $n \in \mathbb{N}$, we have

$$W_{T(f_n)}(g) = g * T(f_n) = T(g * f_n) = T(W_{f_n}(g)) = (T \circ W_{f_n})(g)$$

or simply $W_{T(f_n)} = T \circ W_{f_n}$. This says that $\varphi(W_{f_n}) = \varphi'(W_{f_n}) = W_{T(f_n)} = T \circ W_{f_n}$. Therefore, we get

$$\overline{\lim_n} \|T \circ W_{f_n} - \varphi(V)\| = \lim_n \|\varphi(W_{f_n}) - \varphi(V)\| \leq$$

$$\leq \|\varphi\| \lim_n \|W_{f_n} - V\| = 0$$

by (10). Since $\lim_n \|(T \circ W_{f_n})(h) - \varphi(V)(h)\|_{p,q} = 0$, we can easily obtain that $\overline{\lim}_n \|g * ((T \circ \circ W_{f_n})(h)) - g * (\varphi(V)(h))\|_{p,q} = 0$ and $g * \varphi(V)(h) = g * T(V(h))$ by (11) for all $g \in C_c(G)$. By Lemma 1, we get $\varphi(V)(h) = (T \circ V)(h)$. Since $L^t(p,q)(G)$ is a normed algebra, it is easy to see that

$$\|(T \circ V)(h)\|_{p,q} = \|\varphi(V)(h)\|_{p,q} = \lim_n \|\varphi(W_{f_n})(h)\|_{p,q} = \lim_n \left\|W_{T(f_n)}(h)\right\|_{p,q} \leq$$

$$\leq \lim_n \left\|W_{T(f_n)}\right\| \|h\|_{p,q} = \|h\|_{p,q} \lim_n \|T(f_n)\|_{p,q}^t.$$

Therefore, by using equalities $\|W_f\| = \|f\|_{p,q}^t$, $\|h\|_{p,q} \leq 1$ and (10), we see that

$$\|(T \circ V)(h)\|_{p,q} \leq \|T\| \lim_n \|f_n\|_{p,q}^t = \|T\| \lim_n \|W_{f_n}\| \leq \|T\| \|V\|.$$

Lemma 2 is proved.

Proposition 12. *For any $T \in m_{p,q}$, $V \in \Lambda$ and $f \in L^t(p,q)(G)$, we get*

$$\|(T \circ V)(f)\|_{p,q} \leq \|T\| \, \|V(f)\|_{p,q}.$$

Proof. Let $\varepsilon > 0$. Since Λ is a Banach algebra with an approximate identity, for any $V \in \Lambda$, we can find $P, S \in \Lambda$ such that $\|P\| = 1$, $\|S - V\| < \varepsilon$ and $V = P \circ S$ by Cohen factorization theorem in [8]. Therefore, we get

$$\|S(f)\|_{p,q} = \|(S - V + V)(f)\|_{p,q} \leq \|(S - V)(f)\|_{p,q} + \|V(f)\|_{p,q} \leq$$

$$\leq \varepsilon \|f\|_{p,q} + \|V(f)\|_{p,q}$$

and

$$\|(T \circ V)(f)\|_{p,q} = \|(T \circ (P \circ S))(f)\|_{p,q} = \|T(P(S(f)))\|_{p,q} \leq$$

$$\leq \|T\| \, \|P\| \, \|S(f)\|_{p,q} \leq \|T\| \left(\varepsilon \|f\|_{p,q} + \|V(f)\|_{p,q}\right)$$

by Lemma 2. Since ε is arbitrary, we have $\|(T \circ V)(f)\|_{p,q} \leq \|T\| \, \|V(f)\|_{p,q}$.

Proposition 12 is proved.

Lemma 3. *The set $\wp = \left\{V(f) : f \in L^t(p,q)(G), \ V \in \Lambda\right\}$ is dense in $L(p,q)(G)$.*

Proof. Let's take any $g \in L(p,q)(G)$ and $\varepsilon > 0$. Since $\overline{C_c(G)} = L(p,q)(G)$, there exists a function $f \in C_c(G)$ such that $\|g - f\|_{p,q} < \dfrac{\varepsilon}{2}$. If we take the approximate identity $\left\{W_{h_\gamma}\right\}$ of Λ into consideration, then $W_{h_\gamma}(f) \in L^t(p,q)(G)$ for each γ by Proposition 9. Since the net $\{h_\gamma\}$ is an approximate identity for $L(p,q)(G)$, we have $\lim_\gamma \left\|W_{h_\gamma}(f) - f\right\|_{p,q} = \lim_\gamma \|h_\gamma * f - f\|_{p,q} = 0$ and so $\left\{W_{h_\gamma}(f)\right\} \subset \wp$. As a result, we get $\left\|W_{h_\gamma}(f) - g\right\|_{p,q} \leq \left\|W_{h_\gamma}(f) - f\right\|_{p,q} + \|f - g\|_{p,q} < \varepsilon$ and $\overline{\wp} = L(p,q)(G)$.

Lemma 3 is proved.

Lemma 4. *Let $\Im$ be a dense subspace of $L(p,q)(G)$ and $V \in B_{p,q}$. If $V(h * f) = h * V(f)$ for all $h \in C_c(G)$ and $f \in \Im$, then $V \in M_{p,q}$.*

Proof. Let's take any $x \in G$. Then by [7], we can find a net $\{u_\alpha\}_{\alpha \in I} \subset C_c(G)$ such that $\lim_\alpha \|L_x h - u_\alpha * h\|_{p,q} = 0$ for all $h \in \Im$. Since $V \in B_{p,q}$ is bounded, $\lim_\alpha \|V(L_x h) - V(u_\alpha * h)\|_{p,q} = 0$ can be written. Also by the hypothesis, we get $\lim_\alpha \|L_x V(h) - u_\alpha * V(h)\|_{p,q} = 0$ as $V(h) \in L(p,q)(G)$. Therefore, we have

$$\|V(L_x h) - L_x V(h)\|_{p,q} = \|V(L_x h) - V(u_\alpha * h) + V(u_\alpha * h) - L_x V(h)\|_{p,q} \leq$$

$$\leq \|V(L_x h) - V(u_\alpha * h)\|_{p,q} + \|V(u_\alpha * h) - L_x V(h)\|_{p,q} \leq$$

$$\leq \|V(L_x h) - V(u_\alpha * h)\|_{p,q} + \|u_\alpha * V(h) - L_x V(h)\|_{p,q}$$

for all $h \in \Im$. It means that $V(L_x h) = L_x V(h)$ and $V \in M_{p,q}$ as $\overline{\Im} = L(p,q)(G)$.

Lemma 4 is proved.

Lemma 5. *Let $f, g \in L(p,q)(G)$ and $T \in M_{p,q}$.*

(i) *If $f \in L^1(G)$, then $T(f * g) = f * T(g)$.*

(ii) *If $f \in L^t(p,q)(G)$, then $T(f) \in L^t(p,q)(G)$.*

(iii) *If $g \in L^t(p,q)(G)$, then $T(f * g) = f * T(g)$.*

Proof. (i) is proved in [4] (Lemma 2.1).

(ii) Let $f \in L^t(p,q)(G)$ and $T \in M_{p,q}$. Then by (i), we get

$$\|T(f)\|_{p,q}^t = \sup\left\{\|h * T(f)\|_{p,q} : h \in C_c(G),\ \|h\|_{p,q} \le 1\right\} =$$

$$= \sup\left\{\|T(h * f)\|_{p,q} \ : h \in C_c(G),\ \|h\|_{p,q} \le 1\right\} \le$$

$$\le \|T\| \sup\left\{\|h * f\|_{p,q} : h \in C_c(G),\ \|h\|_{p,q} \le 1\right\} \le \|T\|\, \|f\|_{p,q}^t < \infty.$$

As a result, $T(f) \in L^t(p,q)(G)$.

(iii) Let $g \in L^t(p,q)(G)$. Since $C_c(G)$ is dense in $L(p,q)(G)$, we can find a sequence $\{f_n\}_{n\in\mathbb{N}} \subset C_c(G)$ such that $\lim_n \|f_n - f\|_{p,q} = 0$. If we remember Proposition 7, then it says that $W_g \in \mathrm{Hom}_{L^1(G)}(L(p,q)(G))$ and so

$$\lim_n \|f_n * g - f * g\|_{p,q} = \lim_n \|W_g(f_n) - W_g(f)\|_{p,q} =$$

$$= \lim_n \|W_g(f_n - f)\|_{p,q} \le \|W_g\| \lim_n \|f_n - f\|_{p,q} = 0.$$

Since $T(g) \in L^t(p,q)(G)$ by (ii), we get

$$\lim_n \|f_n * T(g) - f * T(g)\|_{p,q} = 0.$$

Therefore,

$$f * T(g) = \lim_n f_n * T(g) = \lim_n T(f_n * g) = T(f * g)$$

by the continuity of T and (i).

Lemma 5 is proved.

Proposition 13. *Let ω be a map such that $\omega\colon M_{p,q} \to m_{p,q}$, $\omega(T) = \omega_T$ and $\omega(T)(f) = \omega_T(f) = T(f)$ for all $T \in M_{p,q}$ and $f \in L^t(p,q)(G)$. Then ω is an isometric isomorphism.*

Also, for $T \in m_{p,q}$, there exists at least one $S \in M_{p,q}$ such that $\omega_S(V(f)) = (T \circ V)(f)$ for all $V \in \Lambda$ and $f \in L^t(p,q)(G)$.

Proof. We showed that $T(f) \in L^t(p,q)(G)$ for $T \in M_{p,q}$ and $f \in L^t(p,q)(G)$ in Lemma 5. If we use $\Lambda \subset M_{p,q}$, then

$$\omega(T)(f * g) = \omega_T(f * g) = T(f * g) = T(W_f(g)) = W_f(T(g)) =$$

$$= f * T(g) = f * \omega_T(g)$$

for all $f, g \in L^t(p,q)(G)$ and $T \in M_{p,q}$ by Lemma 5. This means that $\omega(T) \in m_{p,q}$ and ω is well-defined.

On the other hand, we see the linearity of ω by the following equality:

$$\omega(\alpha T_1 + \beta T_2)(f) = \omega_{\alpha T_1 + \beta T_2}(f) = (\alpha T_1 + \beta T_2)(f) = \alpha T_1(f) + \beta T_2(f) =$$

$$= \alpha \omega_{T_1}(f) + \beta \omega_{T_2}(f) = \alpha \omega(T_1)(f) + \beta \omega(T_2)(f),$$

where α, β numbers, $f \in L^t(p,q)(G)$ and $T_1, T_2 \in M_{p,q}$.

Now, let's take any $T_1, T_2 \in M_{p,q}$ and assume that $\omega(T_1) = \omega(T_2)$. Then, by definition of ω, we get $T_1(f) = T_2(f)$ for all $f \in L^t(p,q)(G)$. Since $\overline{L^t(p,q)(G)} = L(p,q)(G)$, we get $T_1 = T_2$ on $L(p,q)(G)$ and ω is injective.

For any $T \in M_{p,q}$, we now know that $\omega_T \in m_{p,q}$. From this point, since $\overline{L^t(p,q)(G)} = = L(p,q)(G)$, we have

$$\|\omega_T\| = \sup \frac{\|\omega_T(f)\|^t_{p,q}}{\|f\|^t_{p,q}} = \sup \frac{\|T(f)\|^t_{p,q}}{\|f\|^t_{p,q}} = \|T\|^t_{p,q} = \left\|\tilde{T}\right\|_{p,q},$$

where $\tilde{T}$ is extension of T. Therefore, $\|\omega\| = 1$ and ω is continuous. Let's take any $T \in m_{p,q}$. By Proposition 12 and Lemma 3, there exists a $S \in B_{p,q}$ such that $S(V(f)) = T(V(f))$ for all $V \in \Lambda$ and $f \in L^t(p,q)(G)$. Therefore, we get

$$S(h * V(f)) = S(V(h * f)) = T(V(h * f)) = T(h * V(f)) =$$

$$= h * T(V(f)) = h * S(V(f))$$

for all $h \in C_c(G)$, $V \in \Lambda$ and $f \in L^t(p,q)(G)$ by Proposition 9 and Lemma 5. Besides these, $S \in M_{p,q}$ by Lemmas 3 and 4. As a result, we obtain

$$\omega(S)(V(h)) = \omega_S(V(h)) = S(V(h)) = T(V(h)) \tag{12}$$

for all $h \in L^t(p,q)(G)$ and $V \in \Lambda$ as $S \in M_{p,q}$.

Now, let's assume that there is a function $h \in L^t(p,q)(G)$ such that $\omega_S(h) \neq T(h)$. Then $\omega_S(h) - T(h) \neq 0$ and $g * (\omega_S(h) - T(h)) \neq 0$ for all $g \in C_c(G)$ by Lemma 1. Since we can find a sequence $\{h_n\}_{n\in\mathbb{N}} \subset C_c(G)$ with $\lim_n \|h_n - h\|_{p,q} = 0$ for $h \in L^t(p,q)(G) \subset L(p,q)(G)$, we write the following:

$$\|g * (\omega_S(h) - T(h))\|_{p,q} = \|g * (\omega_S - T)(h)\|_{p,q} = \|(\omega_S - T)(g * h)\|_{p,q} =$$

$$= \|h * (\omega_S - T)(g)\|_{p,q} = \lim_n \|h_n * (\omega_S - T)(g)\|_{p,q} =$$

$$= \lim_n \|(\omega_S - T)(h_n * g)\|_{p,q} = \lim_n \|(\omega_S - T)(W_{h_n}(g))\|_{p,q}.$$

Therefore, $W_{h_n}(g) \in \wp$ and $\overline{\wp} = L(p,q)(G)$ imply that $\|g * (\omega_S(h) - T(h))\|_{p,q} = 0$ by (12). This contradiction shows that $\omega(S) = \omega_S = T$, i.e., ω is surjective.

For any $T \in M_{p,q}$, $f \in L^t(p,q)(G)$ and $\varepsilon > 0$, there exists a $g \in L(p,q)(G)$ with $\|g\|_{p,q} \leq 1$ such that $\|\omega_T(f)\|^t_{p,q} < \|g * \omega_T(f)\|_{p,q} + \varepsilon$ by the property of $\|\cdot\|^t_{p,q}$ norm. Since Lemma 5 says that

$$g * \omega_T(f) = g * T(f) = T(g * f) = T(W_f(g)) = (T \circ W_f)(g),$$

we get

$$\|\omega_T(f)\|^t_{p,q} \leq \|g * \omega_T(f)\|_{p,q} + \varepsilon \leq$$

$$\leq \|T\| \|W_f\| \|g\|_{p,q} + \varepsilon \leq \|T\| \|f\|^t_{p,q} + \varepsilon.$$

Therefore, $\|\omega\,(T)\| = \|\omega_T\| \,\le \|T\|$. Conversely, with Proposition 12 and Lemma 3, we have

$$\|T\| = \sup\left\{\|T\,(V\,(h))\|_{p,q} : V\,(h) \in \wp,\ \|V\,(h)\|_{p,q} \le 1\right\} =$$

$$= \sup\left\{\|\omega_T\,(V\,(h))\|_{p,q} : V\,(h) \in \wp,\ \|V\,(h)\|_{p,q} \le 1\right\} \le \|\omega_T\|$$

for $T \in M_{p,q}$. Lastly, we have $\|\omega\,(T)\| = \|T\|$ and $M_{p,q} \cong m_{p,q}$.

Proposition 13 is proved.

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