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INTEGRAL FUNCTIONALS OF THE GASSER – MULLER REGRESSION FUNCTION *

ІНТЕГРАЛЬНІ ФУНКЦІОНАЛИ ФУНКЦІЇ РЕГРЕСІЇ ГАССЕРА – МЮЛЛЕРА

For integral functionals of the Gasser – Muller regression function and its derivatives, the plug-in estimator is considered. The consistency and asymptotic normality of the estimator are shown.

Для інтегральних функціоналів функції регресії Гассера – Мюллера та їх похідних розглядається оцінка, що підключається. Встановлено обгрунтованість та асимптотичну нормальність цієї оцінки.

1. Introduction. The study of functionals of a probability distribution density function or a regression function and its derivatives is an interesting task and attracts an active interest on the part of researchers (see, e.g., [4 – 9]). There are detailed studies functionals of a probability distribution density function and its derivatives (see [6 – 8] and the references therein). Investigations of functionals of a regression function and its derivatives are more modest [4, 5].

In the present paper we investigate the integral functional of a regression function and its derivatives. In our investigation we use the Gasser – Muller regression function introduced and studied in [1 – 3].

As it follows from these works consideration of these types of problems are important, particularly, while choosing asymptotical optimal bandwidth (see formula (5) in [5, p. 2584]). Our approach in this paper is based on the derivation of a representation theorem which we further use to obtain the results connected with asymptotic properties, in particular with consistency and the central limit theorem.

Suppose we have n measurements taken at the points

$$t_1, t_2, \ldots, t_n \ \ (0 \le t_1 \le t_2 \le \ldots \le t_n \le 1),$$

where the t_k, $k = 1, \ldots, n$, depend on n. The model considered is the following:

$$Y(t_k) = a(t_k) + \varepsilon_k, \quad k = 1, \ldots, n,$$

ε_k i.i.d. with $\mathbf{E}(\varepsilon_k) = 0$, $\mathbf{D}(\varepsilon_k) = \sigma^2 < \infty$.

The estimator of the unknown regression function $a(t)$ was introduced by Gasser and Muller [1] and defined by the expression

$$\widehat{a}_n(t) = \frac{1}{h_n} \sum_{i=1}^{n} \int\limits_{s_{i-1}}^{s_i} W\left(\frac{t-u}{h_n}\right) du \cdot Y(t_i),$$

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where $0 = s_1 \le s_2 \le \ldots \le s_n = 1$, $t_i \le s_i \le t_{i+1}$, $i = 1, 2, \ldots, n-1$, and $\max_i |s_i - s_{i-1}| = O\left(\frac{1}{n}\right)$; $\{h_n,\ n = 1, 2, \ldots\}$ is a sequence of positive numbers monotonically tending to zero. $W(u)$ is the function with probability density properties.

In the same paper [1], Gasser and Muller introduced the estimator of the k th derivative of the regression function $a^{(k)}(t)$

$$\widehat{a}_n^{(k)}(t) = \frac{1}{h_n^{k+1}} \sum_{i=1}^{n} \int\limits_{s_{i-1}}^{s_i} W^{(k)}\left(\frac{t-u}{h_n}\right) du \cdot Y(t_i) \tag{1}$$

for all $k = 0, 1, \ldots, m$. It was assumed that $\widehat{a}_n^{(0)}(t) \doteqdot \widehat{a}_n(t)$.

In the works, which we have referred to above, the theorems of consistency and asymptotic normality of estimators were obtained by imposing certain conditions.

Let $\varphi\colon R^{m+2} \to R$ be a continuous bounded smooth function. Consider an integral functional of the form

$$I(a) = \int\limits_{-\infty}^{\infty} \varphi\left(t, a(t), a'(t), \ldots, a^{(m)}(t)\right) dt.$$

We have the $(t_i, Y_i), i = 1, 2, \ldots, n$. This means that

$$Y_i = Y(t_i) = a(t_i) + \varepsilon_i.$$

To estimate $I(a)$, we use the plug-in estimator, i.e., consider the functional

$$I(\widehat{a}_n) = \int\limits_{-\infty}^{\infty} \varphi\left(t, \widehat{a}_n(t), \widehat{a}\,'_n(t), \ldots, \widehat{a}_n^{(m)}(t)\right) dt.$$

Here $\widehat{a}_n^{(k)}(t)$is defined from (1).

2. Representation theorem. Our consideration is based on the representation theorem which will lead to obtaining the results we are interested in. Let us list the conditions which the considered variables are supposed to satisfy:

***Conditions on $\boldsymbol{a}$*:**

(a_1) The function $a = a(t)$ is defined and continuous on $[0, 1]$ and takes its values in the interval $[-\mathbb{k}; \mathbb{k}]$.

(a_2) The function $a(t)$ has continuous derivatives up to order m inclusive.

***Conditions on $\boldsymbol{\varepsilon_k}$*:**

(ε_1) Random values ε_k, $k = 1, 2, \ldots$, are independent and equally distributed.

(ε_2) $\mathbf{E}\varepsilon_k = 0$, $\mathbf{D}\varepsilon_k^2 = \sigma^2 < \infty$.

(ε_3) The growth condition is $\mathbf{P}\{|\varepsilon_k| > n\} < e^{-n}$.

In the sequel, for brevity, we will use the notation

$$\frac{\partial \varphi}{\partial x_i} = \varphi_{(i)} \ \text{ for } \ i = 0, 1, \ldots, m$$

and

$$\frac{\partial^2\varphi}{\partial x_i \partial x_j} = \varphi_{(ij)} \quad \text{for} \quad i, j = 0, 1, \ldots, m.$$

Conditions on φ:

(φ_1) The function $\varphi\colon R^{m+2} \to R$ is continuous, bounded, integrable and has bounded continuous derivatives up to second order, inclusive, in some open convex domain A which contains the domain $R \times [-\Bbbk; \Bbbk]^{m+1}$.

(φ_2) All first and second derivatives of the function φ are uniformly bounded in the domain A by a constant $C_\varphi > 0$.

Therefore, by this condition, for the function φ we have for all $i, j = 0, 1, \ldots, m$:

$$\sup\Big\{|\varphi_{(ij)}|(s, s_0, s_1, \ldots, s_m)\colon\ (s, s_0, s_1, \ldots, s_m) \in A\Big\} \le C_\varphi. \tag{2}$$

Conditions on W:

(w_1) $\displaystyle\int_{-\infty}^{\infty} W(t)\,dt = 1.$

(w_2) The function $W(t)$ has continuous derivatives up to order m, $m \ge 1$.

(w_3) Function $W(t)$ has the compact support $[-\tau, \tau]$.

$$W(-\tau) = W(\tau) = 0;$$

(w_4) For any $i = 0, 1, \ldots, m$, $W^{(i)} \in L_1([-\tau, \tau])$.

Denote by $a_n(t)$ the mathematical expectation $\widehat{a}_n(t)$:

$$a_n(t) = \mathbf{E}\widehat{a}_n(t) = \mathbf{E}\frac{1}{h_n}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W\left(\frac{t-u}{h_n}\right) du \cdot Y(t_i) = \frac{1}{h_n}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W\left(\frac{t-u}{h_n}\right) du \cdot a(t_i).$$

Then we obtain

$$a_n^{(k)}(t) = \mathbf{E}\widehat{a}_n^{(k)}(t) = \frac{1}{h_n^{i+1}}\sum_{i=1}^{n}\int\limits_{s_{i-1}}^{s_i} W^{(k)}\left(\frac{t-u}{h_n}\right) du \cdot a(t_i).$$

Let us ascertain that there also exist expressions $I(a)$, $I(a_n)$ and $I(\widehat{a}_n)$ and they are finite. Using the Taylor formula, for any point $(s, s_0, s_1, \ldots, s_m) \in A$ and some $\widetilde{s}_m \in A$ we can write

$$|\varphi|(s, s_0, s_1, \ldots, s_m) = \left|\sum_{i=0}^{m}\varphi_{(i)}(s, 0, 0, \ldots, 0)s_i + \frac{1}{2}\sum_{i,j=0}^{m}\varphi_{(ij)}(s, \widetilde{s}_0, \widetilde{s}_1, \ldots, \widetilde{s}_m)s_i s_j\right|.$$

Accordingly, there exists a constant C such that

$$|\varphi|(s, s_0, s_1, \ldots, s_m) \le C\left(\sum_{i=0}^{m}|s_i| + \sum_{i=0}^{m}|s_i|^2\right).$$

Hence it follows that for any bounded measurable functions $f_0(t), f_1(t), \ldots, f_m(t)$ from $L_1(R)$ we have

$$\int_{-\infty}^{\infty} |\varphi|\,(t, f_0(t), \ldots, f_m(t))\; dt < \infty \tag{3}$$

and therefore $I(a)$ exists.

The conditions which are imposed on the function W ensure its boundedness and membership in $L_1(R)$. Then condition (w_4) and (2), (3) imply the finiteness of both variables $I(a_n)$ and $I(\widehat{a}_n)$ for any n.

By the Taylor formula we can write

$$I(\widehat{a}_n) - I(a_n) = S_n(h_n) + R_n, \tag{4}$$

where for any $h_n > 0$, $S_n(h)$ is the sum of independent random variables

$$S_n(h_n) = \sum_{i=0}^{m} \int_0^1 \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right)\left(\widehat{a}_n^{(i)}(t) - a_n^{(i)}(t)\right)\, dt. \tag{5}$$

A remainder R_n has the form

$$R_n = \frac{1}{2}\sum_{i,j=1}^{m} \int_0^1 \varphi_{(ij)}(\widetilde{b}_m(t))\left(\widehat{a}_n^{(i)}(t) - a_n^{(i)}(t)\right)\left(\widehat{a}_n^{(j)}(t) - a_n^{(j)}(t)\right)\, dt, \tag{6}$$

where $\widetilde{b}_m(t)$ is the straight line connecting the points

$$(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)) \quad \text{and} \quad (t, \widehat{a}_n(t), \widehat{a}\,'_n(t), \ldots, \widehat{a}_n^{(m)}(t)).$$

Let us estimate the remainder R_n. Applying the standard procedure, from (3) and (6) we obtain

$$|R_n| \le C_\varphi \int_0^1 \sum_{i=0}^{m} \left(\widehat{a}_n^{(i)}(t) - a_n^{(i)}(t)\right)^2 dt.$$

Let $C^m[0,1]$ denote the space of bounded real functions that are defined and continuous on $[0,1]$, having continuous derivatives of at least mth order. In this space we introduce the norm

$$\|f\|_m = \left(\sum_{i=0}^{m} \int_0^1 \left(\frac{d^i f}{dt^i}\right)^2 dt\right)^{1/2}, \quad f \in C^m[0,1].$$

The closure of $C^m[0,1]$ in this norm is defined by W_2^m and called the Sobolev space. This is completer separable Hilbert space with the scalar product

$$\langle f, g\rangle_m = \sum_{i=0}^{m} \int_0^1 \frac{d^i f}{dt^i}\,\frac{d^i g}{dt^i}\, dt, \quad f, g \in C^m[0,1].$$

Denote

$$r_n(m) = \|\widehat{a}_n - a_n\|_m^2,$$

then we can write

$$|R_n| \le C_\varphi r_n(m). \tag{7}$$

Assume

$$U_k = U_k(t) = \frac{1}{h_n}\int\limits_{s_{k-1}}^{s_k} W\left(\frac{t-u}{h_n}\right) du\big[Y(t_k) - a(t_k)\big], \quad k = 1, 2, \ldots, n,$$

where $a(t_k) = \mathbf{E}Y(t_k)$. Then

$$\sum_{k=1}^{n} U_k = \frac{1}{h_n}\sum_{k=1}^{n}\int\limits_{s_{k-1}}^{s_k} W\left(\frac{t-u}{h_n}\right) du\big[Y(t_k) - a(t_k)\big] = \widehat{a}_n(t) - a_n(t).$$

Therefore

$$r_n(m) = \left\| \sum_{k=1}^{n} U_k \right\|_m^2. \tag{8}$$

Let us estimate the norm of one of the summands U_k in (8) for each $k = 1, 2, \ldots, n$. We obtain

$$\|U_k\|_m = \left(\sum_{i=0}^{m} \int\limits_0^1 \left| \frac{1}{h_n^{i+1}} \int\limits_{s_{k-1}}^{s_k} W^{(i)}\left(\frac{t-u}{h_n}\right) du\big[Y(t_k) - a(t_k)\big] \right|^2 dt \right)^{1/2} =$$

$$= \left(\sum_{i=0}^{m} \frac{1}{h_n^{2i+2}} \int\limits_0^1 \left| h_n \int\limits_{\frac{t-s_k}{h_n}}^{\frac{t-s_{k-1}}{h_n}} W^{(i)}\left(\frac{t-u}{h_n}\right) d\left(\frac{t-u}{h_n}\right) \right|^2 \big|Y(t_k) - a(t_k)\big|^2 dt \right)^{1/2} \le$$

$$\le 2|\varepsilon_k| C_W \left(\sum_{i=0}^{m} \frac{1}{h_n^{2i}} \int\limits_0^1 \left| \frac{t-s_{k-1}}{h_n} - \frac{t-s_k}{h_n} \right|^2 dt \right)^{1/2} =$$

$$= 2|\varepsilon_k| C_W \frac{|s_k - s_{k-1}|\sqrt{1-h_n^{2m+2}}}{h_n^{m+1}\sqrt{1-h_n^2}} \le L\,\frac{1}{nh_n^{m+1}} =$$

$$= M_m \sim O\left(\frac{1}{nh_n^{m+1}}\right) \quad \text{for sufficiently large } L > 0. \tag{9}$$

To estimate $r_n(m)$, we use the McDiarmid's inequality which we give here for convenience (for details see [10]).

McDiarmid's inequality. Let $H(t_1, \ldots, t_k)$ be a real function such that for each $i = 1, \ldots, k$ and some c_i, the supremum in $t_1, \ldots, t_k, t$, of the difference

$$\Big|H(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_k)-H(t_1,\ldots,t_{i-1},t,t_{i+1},\ldots,t_k)\Big|\le c_i.$$

If $X_1,\ldots,X_k$ are independent random variables taking values in the domain of the function $H(t_1,\ldots,t_k)$, then for every $\varepsilon>0$,

$$\mathbf{P}\Big\{\big|H(X_1,\ldots,X_k)-\mathbf{E}H(X_1,\ldots,X_k)\big|>\varepsilon\Big\}\le 2\exp\left(-\frac{2\varepsilon^2}{\sum_{i=1}^{k}c_i^2}\right).$$

Let us apply McDiarmid's inequality for the functions

$$H(U_1,\ldots,U_m)=\left\|\sum_{k=1}^{n}U_k\right\|_m.$$

As c_k we take $c_k\equiv 2M_m$, $k=1,\ldots,n$. From (9), for any $\delta>0$ we obtain

$$\mathbf{P}\left\{\left|\left\|\sum_{k=1}^{n}U_k\right\|_m-\mathbf{E}\left\|\sum_{k=1}^{n}U_k\right\|_m\right|\ge\delta\right\}\le 2\exp\left\{-\frac{\delta^2 nh_n^{2m+2}}{2M_m^2}\right\}.$$

We substitute here

$$\delta=\frac{\sqrt{2\log n}}{\sqrt{n}\,h_n^{m+1}}$$

and, by the Borel – Cantelli lemma, write

$$\left\|\sum_{k=1}^{n}U_k\right\|_m=\mathbf{E}\left\|\sum_{k=1}^{n}U_k\right\|_m+O\left(\frac{\sqrt{\log n}}{\sqrt{n}\,h_n^{m+1}}\right). \tag{10}$$

Using the Jensen's inequality

$$\mathbf{E}\left\|\sum_{k=1}^{n}U_k\right\|_m^2\le\mathbf{E}\left\|\sum_{k=1}^{n}U_k\right\|_m^2=$$

$$=\sum_{k=1}^{n}\sum_{i=0}^{m}\int\limits_0^1\mathbf{E}\left|\frac{1}{h_n^{i+1}}\int\limits_{s_{k-1}}^{s_k}W^{(i)}\left(\frac{t-u}{h_n}\right)du\big[Y(t_k)-a(t_k)\big]\right|^2dt\le$$

$$\le 2C_W^2\sum_{k=1}^{n}\sum_{i=0}^{m}\int\limits_0^1\frac{1}{h_n^{2i+2}}\left|\int\limits_{s_{k-1}}^{s_k}du\right|^2\mathbf{E}\big[Y(t_k)-a(t_k)\big]^2dt=$$

$$=2C_W^2\sigma^2\frac{(1-h_n^{2m+2})}{(1-h_n^2)h_n^{2m+2}}\sum_{k=1}^{n}(s_k-s_{k-1})^2\le K\cdot\frac{1}{nh_n^{2m+2}}, \tag{11}$$

from (7), (8), (10) and (11) we conclude that

$$R_n=O\left(\frac{\log n}{nh_n^{2m+2}}\right).$$

Therefore the following statement is true.

Theorem 1. *Assume that conditions* (a_1) – (a_2), (ε_1) – (ε_3), (φ_1) – (φ_2) *and* (w_1) – (w_4) *are fulfilled. Then representation* (4), *where the remainder with probability* 1 *has the order*

$$R_n = O\left(\frac{\log n}{nh_n^{2m+2}}\right) \tag{12}$$

is valid.

3. Consistency. In this section of the paper we use Theorem 1 to prove that the estimator $I(\widehat{a}_n)$ is strictly consistent.

Theorem 2. *Let the conditions of Theorem* 1 *be fulfilled. If the positive sequence* $(h_n)_{n=1}^{\infty}$, $0 < h_n < 1$, *is chosen so that*

$$\frac{\log n}{nh_n^{2m+2}} \to 0,$$

then with probability 1 *we have*

$$I(\widehat{a}_n) \to I(a) \quad as \quad n \to \infty.$$

Proof. By Theorem 1 and formula (4)

$$I(\widehat{a}_n) - I(a_n) = S_n(h_n) + R_n, \quad R_n = o(1) \quad \text{a.e.},$$

where

$$S_n(h_n) = \sum_{i=0}^{m} \int\limits_0^1 \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right)\left(\widehat{a}_n^{(i)}(t) - a_n^{(i)}(t)\right) dt.$$

By condition (a_1),

$$\left\{\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}\right) : t \in [0,1]\right\} \subset [0,1] \times [-\Bbbk; \Bbbk]^{m+1}.$$

This and condition (φ_2) imply that there exists a constant $C_\varphi > 0$, such that

$$\sup\left\{|\varphi_{(i)}|(t, t_0, \ldots, t_m) \colon (t, t_0, \ldots, t_m) \in [0,1] \times [-\Bbbk; \Bbbk]^{m+1}\right\} \le C_\varphi.$$

Keeping this in mind, we can write

$$|S_n(h_n)| \le C_\varphi \sum_{i=0}^{m} \int\limits_0^1 \frac{1}{h_n^{i+1}} \sum_{k=1}^{n} \int\limits_{s_{k-1}}^{s_k} \left|W^{(i)}\left(\frac{t-u}{h_n}\right)\right| du \cdot \left[Y(t_k) - a(t_k)\right] dt \le$$

$$\le 2C_\varphi C_W \sum_{i=0}^{m} \int\limits_0^1 \frac{1}{h_n^{i+1}} \sum_{k=1}^{n} |\varepsilon_k|\, |s_k - s_{k-1}|\, dt \sim \quad \text{(by } (w_2)\text{)}$$

$$\sim M\, \frac{1}{nh_n^{m+1}} \quad \text{(for some } M\text{)}. \tag{13}$$

Hoeffding's inequality. Suppose $X_1, X_2, \ldots, X_n$ are independent real-valued random variables, such that for each i, X_i takes values from the interval $[r_i, p_i]$.

Let $Y \doteqdot \sum_{i=1}^{n} X_i$. Then for all $t > 0$,

$$\mathbf{P}\big\{|Y - \mathbf{E}|Y|| \geq t\big\} \leq 2\exp\left\{-\frac{2t^2}{\sum_{i=1}^{n}(p_i - r_i)^2}\right\}. \tag{14}$$

As X_k we take

$$X_k = \sum_{i=0}^{m} \int_0^1 \frac{1}{h_n^{i+1}} \int_{s_{k-1}}^{s_k} \left| W^{(i)}\left(\frac{t-u}{h_n}\right)\right| du \cdot \big[Y(t_k) - a(t_k)\big]\, dt.$$

Analogously to (13), it can be shown that X_k takes its values in the interval

$$\left[-M\,\frac{1}{n^2 h_n^{m+1}}, M\,\frac{1}{n^2 h_n^{m+1}}\right].$$

Therefore

$$\sum_{i=1}^{n}(p_i - r_i)^2 = \frac{2M}{n^3 h_n^{2m+2}}\,.$$

Besides, we take

$$t = \frac{2\sqrt{M\log n}}{n^{3/2} h_n^{m+1}}\,.$$

Then from (14) we obtain

$$\mathbf{P}\left\{\big|S_n(h_n)\big| > \frac{2\sqrt{M\log n}}{n^{3/2} h_n^{m+1}}\right\} \leq 2\exp\big\{-2\log n\big\}$$

by means of which, using the Borel – Cantelli lemma, we can conclude that

$$S_n(h_n) = O\left(\frac{\sqrt{\log n}}{n^{3/2} h_n^{m+1}}\right).$$

It is obvious that, for condition (12), $\dfrac{\sqrt{\log n}}{n^{3/2} h_n^{m+1}}$, too, tends to zero. Thus we conclude that $S_n(h_n) \to$ $\to 0$ as $n \to \infty$.

By formula (6) from [2] we can write

$$\mathbf{E}a_n^{(k)}(t) = \int_{-\tau}^{\tau} W(u) a^{(k)}(t - uh_n)\, du + O\left(\frac{1}{nh_n^k}\right).$$

Hence we make the following conclusions:

(i) for condition (12), $\dfrac{1}{nh_n^k}$, too, tends to zero for any $k = 0, 1, \ldots, m$;

(ii) $\mathbf{E}a_n^{(k)}(t) \to a^{(k)}(t)$ as $n \to \infty$.

Summarizing the above discussion, we ascertain that

$$I(a_n) = \int_0^1 \varphi\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right) dt \longrightarrow$$

$$\longrightarrow \int_0^1 \varphi\left(t, a(t), a'(t), \ldots, a^{(m)}(t)\right) dt = I(a) \text{ as } n \to \infty.$$

Since $I(\widehat{a}_n) - I(a_n) = o(1)$, we conclude that

$$I(\widehat{a}_n) - I(a) \longrightarrow 0 \quad \text{a.e.}$$

The theorem is proved.

4. Central limit theorem. Using our representation theorem we can obtain the limit distribution property for the integral functional

$$I(\widehat{a}_n) = \int_0^1 \varphi\left(t, \widehat{a}_n(t), \widehat{a}\,'_n(t), \ldots, \widehat{a}_n^{(m)}(t)\right) dt.$$

Consider the difference (4), where for any $h_n > 0$, $S_n(h)$ is the sum of independent random variables (5). R_n is a remainder having the form (6). Clearly,

$$\mathbf{E}S_n(h_n) = 0 \quad \text{and} \quad \mathbf{E}R_n \to 0 \quad \text{as} \quad n \to \infty. \tag{15}$$

Moreover,

$$\mathbf{E}(S_n(h_n))^2 = \sigma^2 \sum_{i=0}^{m} \left(\int_0^1 \varphi_{(i)}(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t))\, dt \right)^2$$
$$\text{and} \quad \mathbf{D}R_n \to 0 \quad \text{as} \quad n \to \infty. \tag{16}$$

Using appropriate conditions, we have to prove that the variable

$$\sqrt{n}\, \left(I(\widehat{a}_n) - I(a_n)\right)$$

is asymptotically normal and calculate the limiting variance. For this, according to the theorem and formulas (4), (15) and (16), we have to show the asymptotic normality of the variable $\sqrt{n}\, S_n(h_n)$. As follows from (6), in this case it suffices to study this property for the variables

$$d_k = Y(t_k) \sum_{i=0}^{m} \frac{1}{h_n^{i+1}} \int_0^1 \int_{s_{k-1}}^{s_k} W^{(i)}\left(\frac{t-u}{h_n}\right) \varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right) dt\, du.$$

It can be easily verified that

$$\mathbf{E}d_k = a(t_k)\sum_{i=0}^{m}\frac{1}{h_n^{i+1}}\int\limits_0^1\int\limits_{s_{k-1}}^{s_k} W^{(i)}\left(\frac{t-u}{h_n}\right)\varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right)dt\,du.$$

Thus we consider the sequence of independent random variables

$$f_k(n) = \alpha(n,k)(Y(t_k) - a(t_k)) = \alpha(n,k)\varepsilon_k,$$

where

$$\alpha(n,k) = \sum_{i=0}^{m}\frac{1}{h_n^{i+1}}\int\limits_0^1\int\limits_{s_{k-1}}^{s_k} W^{(i)}\left(\frac{t-u}{h_n}\right)\varphi_{(i)}\left(t, a_n(t), a_n'(t), \ldots, a_n^{(m)}(t)\right)dt\,du.$$

Let consider the sum

$$S_n(h_n) = \sum_{k=1}^{n}\alpha(n,k)\varepsilon_k.$$

Let $F_{k,n}$ be the probability distribution function of a random variable $\alpha(n,k)\varepsilon_k$, and F_ε be the distribution function of a random variable ε_k. The Lindeberg's condition is written in the form $\lim_{n\to\infty} L_n(\delta) = 0 \;\; \forall\,\delta > 0$, where

$$L_n(\delta) = \left(\sigma^2\sum_{k=1}^{n}\alpha^2(n,k)\right)^{-1}\sum_{j=1}^{n}\int x^2 J\left(|x| \geq \delta\sigma\left(\sum_{k=1}^{n}\alpha^2(n,k)\right)^{1/2}\right)dF_{k,n}(x),$$

where $J(A)$ is the indicator function of the set A.

It is easy to see that

$$L_n(\delta) \leq \frac{1}{\sigma^2}\max_{0\leq j\leq n}\int x^2 J\left(|x| \geq \delta\sigma v(n,j)\right)\,dF_\varepsilon,$$

where

$$v(n,j) = \frac{|\alpha(n,j)|}{\left(\sum_{j=1}^{n}\alpha^2(n,j)\right)^{1/2}}.$$

It remains to show that

$$\max_{1\leq j\leq n} v(n,j) \to 0 \quad \text{as} \quad n\to\infty.$$

But since

$$\max_{1\leq j\leq n}|\alpha(n,j)| = O\left(\frac{1}{nh_n^{m+1}}\right),$$

we have

$$\max_{1\leq j\leq n} v(n,j) = O\left(\frac{1}{\sqrt{nh_n^{m+1}}}\right).$$

Thus the Lindeberg's condition is fulfilled and we can conclude that the theorem is valid.

Theorem 3. *Let the conditions of Theorem* 1 *be fulfilled. Then if*

$$h_n \to 0 \ \text{ and } \ nh_n^{m+1} \to \infty \quad \text{as} \quad n \to \infty,$$

we have

$$\sqrt{n}\ (I(\widehat{a}_n) - I(a)) \xrightarrow{d} N(0, r^2),$$

where

$$r^2 = \sigma^2 \sum_{i=0}^{m} \left(\int_0^1 \varphi_{(i)} \left(t, a(t), a'(t), \ldots, a^{(m)}(t) \right) dt \right)^2 .$$

5. Example. As an example we consider the problem of estimation of total curvature (see [11, p. 22]) of a regression function a:

$$I = \int_0^1 (a''(t))^2 \, dt.$$

We obtain

$$\varphi(t, x_0, \ldots, x_m) = x_2^2.$$

Then we have

$$r^2 = 4\sigma^2 \left(a'(1) - a'(0) \right)^2 .$$

For $h_n \to 0$ and $\sqrt{n}\, h_n^6 \to \infty$, we have the convergence $\sqrt{n}(I_2(a) - I_2(\widehat{a}_n)) \xrightarrow{d} N(0, r^2)$.

These considerations can be used for checking hypothesis about total curvature of regression function.

6. Iterated logarithm law. Applying the well-known iterated logarithm law from Kuelbs paper [12], we ascertain that the following statement is true.

Theorem 4. *If the sequence h_n is chosen so that*

$$h_n = \left(\frac{\log n}{\sqrt{n \log \log n}} \right)^{\frac{1}{2m}},$$

then

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n}\,[I(\widehat{a}_n) - I(a_n)]}{\sqrt{2 \log \log n}} = r.$$

Proof. Note that for this h_n we have

$$R_n = o\left(\sqrt{\frac{\log \log n}{n}} \right).$$

It can be easily verified that

$$\limsup_{n\to\infty} \pm \frac{\sqrt{n}\,[I(\widehat{a}_n) - I(a_n)]}{\sqrt{2 \log \log n}} = \limsup_{n\to\infty} \pm \frac{\sqrt{n}\,\left[\alpha(n,k) Y(t_k) - \alpha(n,k) a(t_k)\right]}{\sqrt{2 \log \log n}} = r.$$

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