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ON FUNDAMENTAL THEOREMS FOR HOLOMORPHIC CURVES ON ANNULI *

ФУНДАМЕНТАЛЬНІ ТЕОРЕМИ ДЛЯ ГОЛОМОРФНИХ КРИВИХ НА КІЛЬЦЯХ

We prove some fundamental theorems for holomorphic curves on annuli intersecting a finite set of fixed hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ with ramification.

Доведено деякі фундаментальні теореми для голоморфних кривих на кільцях, що перетинають скінченну множину фіксованих гіперплощин загального положення в $\mathbb{P}^n(\mathbb{C})$ з розгалудженням.

1. Introduction and main results. In 1933, H. Cartan (see [3]) proved the second main theorem for holomorphic curves with targets being hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. Since that times, the problem which studies the characteristics of holomorphic maps has been attracted by many authors. For example, in 1983, E. I. Nochka (see [6]) proved the second main theorem in the case of the hyperplanes is in N-subgeneral position in $\mathbb{P}(\mathbb{C})$ with ramification. In 2003, M. Ru and T. Y. Wang (see [7]) proved an inequality of the second main theorem type, with ramification for a holomorphic curve intersecting a finite set of moving or fixed hyperplanes. In 2004, M. Ru (see [9]) showed the second main theorem for holomorphic curves with targets being hypersurfaces in general position in $\mathbb{P}(\mathbb{C})$ without ramification. In 2009, T. T. H. An and H. T. Phuong (see [1]) gave the result on the second main theorem for holomorphic curves from $\mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$ intersecting hypersurfaces with ramification.

Recently, there exist the results about of the characteristics of meromorphic functions on annuli in complex plane $\mathbb{C}$. In 2005, A. Y. Khrystiyanyn and A. A. Kondratyuk (see [4, 5]) showed some results about of the fundamental theorems and defect relation, which were considered again by T. B. Cao and Z. S. Deng in [2] and by Y. Tan and Q. Zhang in [11]. Our idea is to prove some fundamental theorems for holomorphic mappings from annuli $\Delta \subset \mathbb{C}$ to $\mathbb{P}^n(\mathbb{C})$ intersecting a finite set of hyperplanes. To state our results, we first introduce some notations.

Let $R_0 > 1$ be a fixed positive real number or $+\infty$, set

$$\Delta = \left\{ z \in \mathbb{C} \colon \frac{1}{R_0} < |z| < R_0 \right\},$$

be a annuli in $\mathbb{C}$, and for any real number r such that $1 < r < R_0$ we denote

$$\Delta_r = \left\{ z \in \mathbb{C} \colon \frac{1}{r} < |z| < r \right\}, \qquad \Delta_{1,r} = \left\{ z \in \mathbb{C} \colon \frac{1}{r} < |z| \le 1 \right\},$$

$$\Delta_{2,r} = \left\{ z \in \mathbb{C} \colon 1 < |z| < r \right\}.$$

Let $f = (f_0 \colon \ldots \colon f_n) \colon \Delta \to \mathbb{P}^n(\mathbb{C})$ be a holomorphic map where $f_0, \ldots, f_n$ are holormorphic functions and without common zeros in Δ. For $1 < r < R_0$, characteristic function $T_f(r)$ of f is

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defined by

$$T_f(r) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi}\int\limits_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and

$$L(z_0, \ldots, z_n) = \sum_{j=0}^{n} a_j z_j$$

be linear form defined H, where $a_j \in \mathbb{C}$, $j = 0, \ldots, n$, be constants. Denote by $a = (a_0, \ldots, a_n)$ the non-zero associated vector with H. And denote

$$(H, f) = (a, f) = \sum_{j=0}^{n} a_j f_j.$$

Under the assumption that $(a, f) \not\equiv 0$, for $1 < r < R_0$, the proximity function of f with respect to H is defined as

$$m_f(r, H) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log \frac{\|f(re^{i\theta})\|}{|(a, f)(re^{i\theta})|} d\theta + \frac{1}{2\pi}\int\limits_0^{2\pi} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a, f)(r^{-1}e^{i\theta})|} d\theta,$$

where the above definition is independent, up to an additive constant, of the choice of the reduced representation of f.

Next, we denote by $n_{1,f}(r, H)$ the number of zeros of (a, f) in $\Delta_{1,r}$, counting multiplicity and by $n_{2,f}(r, H)$ the number of zeros of (a, f) in $\Delta_{2,r}$, counting multiplicity too. Set

$$N_{1,f}(r, H) = N_{1,f}(r, L) = \int\limits_{r^{-1}}^{1} \frac{n_{1,f}(t, H)}{t} dt,$$

$$N_{2,f}(r, H) = N_{2,f}(r, L) = \int\limits_{1}^{r} \frac{n_{2,f}(t, H)}{t} dt.$$

The counting function function of f is defined by

$$N_f(r, H) = N_{1,f}(r, H) + N_{2,f}(r, H).$$

Now let δ be a positive integer, we denote by $n^{\delta}_{1,f}(r, H)$ and $n^{\delta}_{2,f}(r, H)$ be the numbers of zeros of (a, f) in $\Delta_{1,r}$ and $\Delta_{2,r}$ respectively, where any zero of multiplicity greater than δ is “truncated" and counted as if it only had multiplicity δ. Set

$$N^{\delta}_{1,f}(r, H) = N^{\delta}_{1,f}(r, L) = \int\limits_{r^{-1}}^{1} \frac{n^{\delta}_{1,f}(t, H)}{t} dt,$$

$$N_{2,f}^{\delta}(r,H)=N_{2,f}^{\delta}(r,L)=\int\limits_{1}^{r}\frac{n_{2,f}^{\delta}(t,H)}{t}dt.$$

The truncated counting function of f is defined by

$$N_{f}^{\delta}(r,H)=N_{1,f}^{\delta}(r,H)+N_{2,f}^{\delta}(r,H).$$

Recall that hyperplanes $H_1,\ldots,H_q,\ q>n$, in $\mathbb{P}^n(\mathbb{C})$ are said to be in general position if for any distinct $i_1,\ldots,i_{n+1}\in\{1,\ldots,q\}$,

$$\bigcap_{k=1}^{n+1}\operatorname{supp}(H_{i_k})=\varnothing,$$

this is equivalence to the $H_{i_1},\ldots,H_{i_{n+1}}$ being linearly independent.

In this paper, a notation "$\|$" in the inequality is mean that for $R_0=+\infty$, the inequality holds for $r\in(1,+\infty)$ outside a set Δ'_r satisfying $\int_{\Delta'_r} r^{\lambda-1}dr<+\infty$, and for $R_0<+\infty$, the inequality holds for $r\in(1,R_0)$ outside a set Δ'_r satisfying $\int_{\Delta'_r}\frac{1}{(R_0-r)^{\lambda+1}}dr<+\infty$, where $\lambda\geq 0$.

Our main results are:

Theorem 1.1. *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and $f=(f_0\colon\ldots\colon f_n)\colon\Delta\to\mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained H. Then we have for any $1<r<R_0$,*

$$T_f(r)=m_f(r,H)+N_f(r,H)+O(1).$$

Theorem 1.2. *Let $f=(f_0\colon\ldots\colon f_n)\colon\Delta\to\mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve and $H_1,\ldots,H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have*

$$\Bigg\|\quad (q-n-1)T_f(r)\leq\sum_{j=1}^{q}N_f^{n}(r,H_j)+O_f(r)\ ,$$

where

$$O_f(r)=\begin{cases}O(\log r+\log T_f(r)) & \textit{if}\quad R_0=+\infty,\\ O\left(\log\dfrac{1}{R_0-r}+\log T_f(r)\right) & \textit{if}\quad R_0<+\infty.\end{cases}$$

Theorem 1.1 is first main theorem, and Theorem 1.2 is second main theorem for holomorphic curves from annuli Δ to $\mathbb{P}^n(\mathbb{C})$ intersecting a collection of fixed hyperplanes in general position with truncated counting functions. When one applies inequalities of second main theorem type, it is often crucial to the application to have the inequality with truncated counting functions. For example, all existing constructions of unique range sets depend on a second main theorem with truncated counting functions.

2. Some preliminaries in Nevanlinna theory for meromorphic functions. In order to prove theorems, we need the following lemmas. Let f be a meromorphic function on Δ, we recall that

$$m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta,$$

$$m(r, f) = m(r, \infty) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta,$$

where $\log^+ x = \max\{0, \log x\}, a \in \mathbb{C}$ and $r \in (R_0^{-1}; R_0)$. For $r \in (1, R_0)$, we denote

$$m_0\left(r, \frac{1}{f-a}\right) = m\left(r, \frac{1}{f-a}\right) + m\left(\frac{1}{r}, \frac{1}{f-a}\right),$$

$$m_0(r, f) = m(r, f) + m(r^{-1}, f).$$

Denote by $n_1\left(t, \dfrac{1}{f-a}\right)$ the number of zeros of $f - a$ in $\{z \in \mathbb{C}\colon t < |z| \le 1\}$, $n_2\left(t, \dfrac{1}{f-a}\right)$ the number of zeros of $f-a$ in $\{z \in \mathbb{C}\colon 1 < |z| < t\}$, $n_1(t, \infty)$ the number of poles in $\{z \in \mathbb{C}\colon t < |z| \le 1\}$ and $n_2(t, \infty)$ the number of poles in $\{z \in \mathbb{C}\colon 1 < |z| < t\}$. For any $r\colon\ 1 < r < R_0$, put

$$N_1\left(r, \frac{1}{f-a}\right) = \int\limits_{1/r}^{1} \frac{n_1\left(t, \dfrac{1}{f-a}\right)}{t} dt, \quad N_2\left(r, \frac{1}{f-a}\right) = \int\limits_{1}^{r} \frac{n_2\left(t, \dfrac{1}{f-a}\right)}{t} dt,$$

and

$$N_1(r, f) = N_1(r, \infty) = \int\limits_{1/r}^{1} \frac{n_1(t, \infty)}{t} dt, \quad N_2(r, f) = N_2(r, \infty) = \int\limits_{1}^{r} \frac{n_2(t, \infty)}{t} dt.$$

Let

$$N_0\left(r, \frac{1}{f-a}\right) = N_1\left(r, \frac{1}{f-a}\right) + N_2\left(r, \frac{1}{f-a}\right),$$

$$N_0(r, f) = N_1(r, f) + N_2(r, f).$$

Denote the Nevanlinna characteristic of f by

$$T_0(r, f) = m_0(r, f) - 2m(1, f) + N_0(r, f).$$

Lemma 2.1 [4]. *Let f be a nonconstant meromorphic function on Δ. Then for any $r \in (1, R_0)$, we have*

$$N_0\left(r,\frac{1}{f}\right)-N_0(r,f)=\frac{1}{2\pi}\int\limits_0^{2\pi}\log|f(re^{i\theta})|d\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}\log|f(r^{-1}e^{i\theta})|d\theta-$$

$$-\frac{1}{\pi}\int\limits_0^{2\pi}\log|f(e^{i\theta})|d\theta.$$

Lemma 2.2 [5]. *Let f be a nonconstant meromorphic function on Δ and $\lambda\geq 0$. Then for any* $r\in(1,R_0)$

(i) *if* $R_0=+\infty$,

$$\left\|\; m_0\left(r,\frac{f'}{f}\right)=O(\log r+\log T_0(r,f))\;;\right.$$

(ii) *if* $R_0<+\infty$,

$$\left\|\; m_0\left(r,\frac{f'}{f}\right)=O\left(\log\frac{1}{R_0-r}+\log T_0(r,f)\right).\right.$$

Lemma 2.3 [4]. *Let f be a nonconstant meromorphic function on Δ. Then we have for any* $r\in(1,R_0)$

$$T_0(r,f_1+f_2)\leq T_0(r,f_1)+T_0(r,f_2)+O(1),$$

$$T_0\left(r,\frac{f_1}{f_2}\right)\leq T_0(r,f_1)+T_0(r,f_2)+O(1).$$

3. Proofs of Theorems 1.1 and 1.2. ***Proof of Theorem* 1.1.** Let $a=(a_0,\ldots,a_n)$ is the associated vector with H. First we note that $N_0(r,(H,f))=0$. By the definitions of $T_f(r)$, $N_f(r,H)$, $m_f(r,H)$ and apply to Lemma 2.1 for (H,f), we have

$$N_f(r,H)=N_0\left(r,\frac{1}{(H,f)}\right)=$$

$$=\frac{1}{2\pi}\int\limits_0^{2\pi}\log|(a,f)(re^{i\theta})|d\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}\log|(a,f)(r^{-1}e^{i\theta})|d\theta+O(1).$$

Hence, we get

$$N_f(r,H)+m_f(r,H)=$$

$$=\frac{1}{2\pi}\int\limits_0^{2\pi}\log\frac{\|f(re^{i\theta})\|}{|(a,f)(re^{i\theta})|}d\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}\log\frac{\|f(r^{-1}e^{i\theta})\|}{|(a,f)(r^{-1}e^{i\theta})|}d\theta+$$

$$+\frac{1}{2\pi}\int\limits_0^{2\pi}\log|(a,f)(re^{i\theta})|d\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}\log|(a,f)(r^{-1}e^{i\theta})|d\theta+O(1)=$$

$$= \frac{1}{2\pi}\int\limits_0^{2\pi} \log\|f(re^{i\theta})\|d\theta + \frac{1}{2\pi}\int\limits_0^{2\pi} \log\|f(r^{-1}e^{i\theta})\|d\theta + O(1) =$$

$$= T_f(r) + O(1).$$

Theorem 1.1 is proved.

To prove Theorem 1.2, we need some lemmas. First we recall the property of Wronskian. Let $f = (f_0 \colon \ldots \colon f_n) \colon \Delta \to \mathbb{P}^n(\mathbb{C})$ be holomorphic curves, the determining of Wronskian of f is defined by

$$W = W(f) = W(f_0, \ldots, f_n) = \begin{vmatrix} f_0(z) & f_1(z) & \ldots & f_n(z) \\ f_0'(z) & f_1'(z) & \ldots & f_n'(z) \\ \ldots & \ldots & \ldots & \ldots \\ f_0^{(n)}(z) & f_1^{(n)}(z) & \ldots & f_n^{(n)}(z) \end{vmatrix}.$$

We denote by $N_W(r, 0)$ the counting function of zeros of $W(f_0, \ldots, f_n)$ in Δ_r, namely

$$N_W(r, 0) = N_0\left(r, \frac{1}{W}\right) + O(1).$$

Let $L_0, \ldots, L_n$ are linearly independent forms of $z_0, \ldots, z_n$. For $j = 0, \ldots, n$, set

$$F_j(z) := L_j(f(z)).$$

By the property of Wronskian there exists a constant $C \neq 0$ such that

$$W(F_0, \ldots, F_n) = CW(f_0, \ldots, f_n).$$

Lemma 3.1. *Let $f = (f_0 \colon \ldots \colon f_n) \colon \Delta \to \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve and $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let a_j is the associated vector with H_j for $j = 1, \ldots, q$. Then we have*

$$\left\| \int\limits_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|}\frac{d\theta}{2\pi} + \int\limits_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi} \le \right.$$

$$\le (n+1)T_f(r) - N_W(r, 0) + O_f(r),$$

where

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)) & \textit{if} \quad R_0 = +\infty, \\ O\left(\log\dfrac{1}{R_0 - r} + \log T_f(r)\right) & \textit{if} \quad R_0 < +\infty. \end{cases}$$

Here the maximum is taken over all subsets K of $\{1, \ldots, q\}$ such that a_j, $j \in K$, are linearly independent.

Proof. We prove the case $R_0 = +\infty$, case $R_0 < +\infty$ can be proved similarly. First, we prove

$$\Bigg\| \int\limits_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \frac{1}{2\pi} \int\limits_0^{2\pi} \log |W(f)(re^{i\theta})| d\theta \le$$

$$\le (n+1) \frac{1}{2\pi} \int\limits_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + O(\log r + \log T_f(r)), \tag{3.1}$$

holds for any $r \in (1, R_0)$. Let $K \subset \{1, \ldots, q\}$ such that $a_j, j \in K$, are linearly independent. Without loss of generality, we may assume that $q \ge n+1$ and $\# K = n+1$. Let $\mathcal{T}$ is the set of all injective maps $\mu \colon \{0, 1, \ldots, n\} \to \{1, \ldots, q\}$. Noting that $\# \mathcal{T} < +\infty$, then we have

$$\int\limits_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} =$$

$$= \int\limits_0^{2\pi} \max_{\mu\in\mathcal{T}} \sum_{j=0}^{n} \log \frac{\|f(re^{i\theta})\|}{|(a_{\mu(j)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} =$$

$$= \int\limits_0^{2\pi} \max_{\mu\in\mathcal{T}} \left\{ \log \prod_{j=0}^{n} \frac{\|f(re^{i\theta})\|}{|(a_{\mu(j)}, f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} =$$

$$= \int\limits_0^{2\pi} \max_{\mu\in\mathcal{T}} \left\{ \log \frac{\|f(re^{i\theta})\|^{n+1}}{\prod_{j=0}^{n} |(a_{\mu(j)}, f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} =$$

$$= \int\limits_0^{2\pi} \log \left\{ \max_{\mu\in\mathcal{T}} \frac{\|f(re^{i\theta})\|^{n+1}}{\prod_{j=0}^{n} |(a_{\mu(j)}, f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1) \le$$

$$\le \int\limits_0^{2\pi} \log \sum_{\mu\in\mathcal{T}} \frac{\|f(re^{i\theta})\|^{n+1}}{\prod_{j=0}^{n} |(a_{\mu(j)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1) =$$

$$= \int\limits_0^{2\pi} \log \sum_{\mu\in\mathcal{T}} \frac{|W((a_{\mu(0)}, f), \ldots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{j=0}^{n} |(a_{\mu(j)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} +$$

$$+ \int\limits_0^{2\pi} \log \sum_{\mu\in\mathcal{T}} \frac{\|f(re^{i\theta})\|^{n+1}}{|W((a_{\mu(0)}, f), \ldots, (a_{\mu(n)}, f))(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1).$$

By the property of Wronskian, we see that

$$|W((a_{\mu(0)},f),\ldots,(a_{\mu(n)},f))| = |C|\,|W(f_0,\ldots,f_n)|,$$

where $C \neq 0$ is constant. So we obtain

$$\int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j,f)(re^{i\theta})|}\frac{d\theta}{2\pi} \leq$$

$$\leq \int_0^{2\pi} \log \sum_{\mu\in T} \frac{|W((a_{\mu(0)},f),\ldots,(a_{\mu(n)},f))(re^{i\theta})|}{\prod_{j=0}^{n}|(a_{\mu(j)},f)(re^{i\theta})|}\frac{d\theta}{2\pi}+$$

$$+\int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^{n+1}}{|W(f_0,\ldots,f_n)(re^{i\theta})|}\frac{d\theta}{2\pi} + O(1). \tag{3.2}$$

We have

$$\frac{W((a_{\mu(0)},f),\ldots,(a_{\mu(n)},f))(re^{i\theta})}{\prod_{j=0}^{n}(a_{\mu(j)},f)(re^{i\theta})} =$$

$$= \begin{vmatrix} 1 & 1 & \ldots & 1 \\ \dfrac{(a_{\mu(0)},f)'}{(a_{\mu(0)},f)} & \dfrac{(a_{\mu(1)},f)'}{(a_{\mu(1)},f)} & \ldots & \dfrac{(a_{\mu(n)},f)'}{(a_{\mu(n)},f)} \\ \ldots & \ldots & \ldots & \ldots \\ \dfrac{(a_{\mu(0)},f)^{(n)}}{(a_{\mu(0)},f)} & \dfrac{(a_{\mu(1)},f)^{(n)}}{(a_{\mu(1)},f)} & \ldots & \dfrac{(a_{\mu(n)},f)^{(n)}}{(a_{\mu(n)},f)} \end{vmatrix} (re^{i\theta}). \tag{3.3}$$

We see that

$$\Big\| \; m\left(r, \frac{(a_{\mu(j)},f)^{(k)}}{(a_{\mu(j)},f)}\right) \leq m_0\left(r, \frac{(a_{\mu(j)},f)^{(k)}}{(a_{\mu(j)},f)}\right) =$$

$$= m_0\left(r, \frac{(a_{\mu(j)},f)^{(k)}}{(a_{\mu(j)},f)^{(k-1)}}\frac{(a_{\mu(j)},f)^{(k-1)}}{(a_{\mu(j)},f)^{(k-2)}}\ldots\frac{(a_{\mu(j)},f)'}{(a_{\mu(j)},f)}\right) \leq$$

$$\leq \sum_{l=1}^{k} m_0\left(r, \frac{(a_{\mu(j)},f)^{(l)}}{(a_{\mu(j)},f)^{(l-1)}}\right). \tag{3.4}$$

By Lemma 2.2, we get

$$m_0\left(r, \frac{(a_{\mu(j)},f)'}{(a_{\mu(j)},f)}\right) = O(\log r + \log T_0(r,(a_{\mu(j)},f)). \tag{3.5}$$

From the definition of $T_0(r,(a_{\mu(j)},f)')$, $N((a_{\mu(j)},f)') = 0$ and (3.5), we obtain

$$T_0(r,(a_{\mu(j)},f)')=m_0(r,(a_{\mu(j)},f)')=$$

$$=m_0\left(r,\frac{(a_{\mu(j)},f)'}{(a_{\mu(j)},f)}(a_{\mu(j)},f)\right)\leq$$

$$\leq m_0(r,(a_{\mu(j)},f))+O(\log r+\log T_0(r,(a_{\mu(j)},f))=$$

$$=T_0(r,(a_{\mu(j)},f))+O(\log r+\log T_0(r,(a_{\mu(j)},f)). \tag{3.6}$$

Similarly, again using Lemma 2.2 and (3.6), we have

$$T_0(r,(a_{\mu(j)},f)'')=m_0(r,(a_{\mu(j)},f)'')=$$

$$=m_0\left(r,\frac{(a_{\mu(j)},f)''}{(a_{\mu(j)},f)'}(a_{\mu(j)},f)'\right)\leq$$

$$\leq m_0(r,(a_{\mu(j)},f)')+O(\log r+\log T_0(r,(a_{\mu(j)},f)')=$$

$$=T_0(r,(a_{\mu(j)},f))+O(\log r+\log T_0(r,(a_{\mu(j)},f)). \tag{3.7}$$

By argument as (3.7) and using inductive method, we obtain that the inequality

$$T_0(r,(a_{\mu(j)},f)^{(l)})\leq T_0(r,(a_{\mu(j)},f))+O(\log r+\log T_0(r,(a_{\mu(j)},f)) \tag{3.8}$$

holds for all $l\in\mathbb{N}^*$. Furthemore, by Lemma 2.2, we also have the equality

$$m_0\left(r,\frac{(a_{\mu(j)},f)^{(l+1)}}{(a_{\mu(j)},f)^{(l)}}\right)=O(\log r+\log T_0(r,(a_{\mu(j)},f)^{(l)})), \tag{3.9}$$

holds for all $l\in\mathbb{N}$. Combining (3.4), (3.8) and (3.9), we get for any $k\in\{1,\ldots,n\}$ and $j\in$ $\in\{0,\ldots,n\}$,

$$\left\|\ m\left(r,\frac{(a_{\mu(j)},f)^{(k)}}{(a_{\mu(j)},f)}\right)\leq O(\log r+\log T_0(r,(a_{\mu(j)},f))\ .\right. \tag{3.10}$$

By the definition of $T_0(r,f_t),T_f(r)$, we have for any $t\in\{0,\ldots,n\}$,

$$T_0(r,f_t)+O(1)=m_0(r,f_t)=m(r,f_t)+m\left(\frac{1}{r},f_t\right)\leq$$

$$\leq O\left(\frac{1}{2\pi}\int\limits_0^{2\pi}\log\|f(re^{i\theta})\|d\theta+\frac{1}{2\pi}\int\limits_0^{2\pi}\log\|f(r^{-1}e^{i\theta})\|d\theta\right)=O(T_f(r))$$

and

$$T_0(r,(a_{\mu(j)},f))\leq\sum_{t=0}^{n}T_0(r,f_t)+O(1).$$

Then we have from (3.10),

$$\left\| \; m\left(r, \frac{(a_{\mu(j)}, f)^{(k)}}{(a_{\mu(j)}, f)}\right) \le O(\log r + \log T_f(r)) \; .\right.$$

Hence for any $\mu \in \mathcal{T}$, we have from (3.3)

$$\left\| \; \int\limits_0^{2\pi} \log^+ \frac{|W((a_{\mu(0)}, f), \ldots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{j=0}^n |(a_{\mu(j)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \le O(\log r + \log T_f(r)) \; .\right.$$

This implies that

$$\left\| \; \int\limits_0^{2\pi} \log \sum_{\mu \in \mathcal{T}} \frac{|W((a_{\mu(0)}, f), \ldots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{j=0}^n |(a_{\mu(j)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \le \right.$$

$$\le \int\limits_0^{2\pi} \log^+ \sum_{\mu \in \mathcal{T}} \frac{|W((a_{\mu(0)}, f), \ldots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{j=0}^n |(a_{\mu(j)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \le$$

$$\le \sum_{\mu \in \mathcal{T}} \int\limits_0^{2\pi} \log^+ \frac{|W((a_{\mu(0)}, f), \ldots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{j=0}^n |(a_{\mu(j)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1) \le$$

$$\le O(\log r + \log T_f(r)). \tag{3.11}$$

We may obtain the inequality (3.1) from (3.4) and (3.11). Similarly, we get

$$\left\| \; \int\limits_0^{2\pi} \max_K \sum_{j \in K} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} + \frac{1}{2\pi} \int\limits_0^{2\pi} \log |W(f)(r^{-1}e^{i\theta})| d\theta \le \right.$$

$$\le (n+1) \frac{1}{2\pi} \int\limits_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta + O(\log r + \log T_f(r)) \tag{3.12}$$

holds for any $r \in (1, R_0)$. Combining (3.1) and (3.12) we obtain

$$\left\| \; \int\limits_0^{2\pi} \max_K \sum_{j \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \int\limits_0^{2\pi} \max_K \sum_{j \in K} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} \le \right.$$

$$\le (n+1) \left(\frac{1}{2\pi} \int\limits_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi} \int\limits_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta \right) -$$

$$- \frac{1}{2\pi} \left(\int\limits_0^{2\pi} \log |W(f)(re^{i\theta})| d\theta + \int\limits_0^{2\pi} \log |W(f)(r^{-1}e^{i\theta})| d\theta \right) +$$

$$+O(\log r + \log T_f(r)).$$

Since

$$N_W(r,0) = \frac{1}{2\pi}\int\limits_0^{2\pi} \log|W(f)(re^{i\theta})|d\theta + \frac{1}{2\pi}\int\limits_0^{2\pi} \log|W(f)(r^{-1}e^{i\theta})|d\theta + O(1).$$

Lemma 3.1 is proved.

Lemma 3.2. *Let $f = (f_0 : \ldots : f_n) : \Delta \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve and $H_1, \ldots, H_q$ be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position and let a_j is the associated vector with H_j for $j = 1, \ldots, q$. Then we have*

$$\sum_{j=1}^{q} m_f(r, H_j) \leq \int\limits_0^{2\pi} \max_K \sum_{j\in K} \log\frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|}\frac{d\theta}{2\pi} +$$

$$+ \int\limits_0^{2\pi} \max_K \sum_{j\in K} \log\frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|}\frac{d\theta}{2\pi} + O(1).$$

Proof. Let $a_j = (a_{j,0}, \ldots, a_{j,n})$ is the associated vector of H_j, $1 \leq j \leq q$, and let $\mathcal{T}$ is the set of all injective maps $\mu\colon \{0, 1, \ldots, n\} \longrightarrow \{1, \ldots, q\}$. By hypothesis that $H_1, \ldots, H_q$ are in general position that for any $\mu \in \mathcal{T}$, the vectors $a_{\mu(0)}, \ldots, a_{\mu(n)}$ are linearly independent.

Let $\mu \in \mathcal{T}$, we have

$$(f, a_{\mu(t)}) = a_{\mu(t),0} f_0 + \ldots + a_{\mu(t),n} f_n, \quad t = 0, 1, \ldots, n. \tag{3.13}$$

Solve the system of linear equations (3.13), we get

$$f_t = b_{\mu(t),0}(a_{\mu(0)}, f) + \ldots + b_{\mu(t),n}(a_{\mu(n)}, f), \quad t = 0, 1, \ldots, n,$$

where $\left(b_{\mu(t),j}\right)_{t,j=0}^{n}$ is the inverse matrix of $\left(a_{\mu(t),j}\right)_{t,j=0}^{n}$. So there is a constant C_μ satisfying

$$\|f(z)\| \leq C_\mu \max_{0\leq t\leq n} |(a_{\mu(t)}, f)(z)|.$$

Set $C = \max_{\mu\in\mathcal{T}} C_\mu$. Then for any $\mu \in \mathcal{T}$, we have

$$\|f(z)\| \leq C \max_{0\leq t\leq n} |(a_{\mu(t)}, f)(z)|.$$

For any $z \in \Delta_r$, there exists the mapping $\mu \in \mathcal{T}$ such that

$$0 < |(a_{\mu(0)}, f)(z)| \leq |(a_{\mu(1)}, f)(z)| \leq \ldots. \leq |(a_{\mu(n)}, f)(z)| \leq |(a_j, f)(z)|,$$

for $j \notin \{\mu(0), \ldots, \mu(n)\}$. Hence

$$\prod_{j=1}^{q} \frac{\|f(z)\|}{|(a_j, f)(z)|} \leq C^{q-n-1} \max_{\mu\in\mathcal{T}} \prod_{t=0}^{n} \frac{\|f(z)\|}{|(a_{\mu(t)}, f)(z)|}.$$

We have

$$\sum_{j=1}^{q} m_f(r, H_j) =$$

$$= \sum_{j=1}^{q} \frac{1}{2\pi} \int_{0}^{2\pi} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \sum_{j=1}^{q} \frac{1}{2\pi} \int_{0}^{2\pi} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} =$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \log \prod_{j=1}^{q} \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \frac{1}{2\pi} \int_{0}^{2\pi} \log \prod_{j=1}^{q} \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} \leq$$

$$\leq \frac{1}{2\pi} \int_{0}^{2\pi} \log \max_{\mu \in \mathcal{T}} \prod_{t=0}^{n} \frac{\|f(re^{i\theta})\|}{|(a_{\mu(t)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} +$$

$$+ \frac{1}{2\pi} \int_{0}^{2\pi} \log \max_{\mu \in \mathcal{T}} \prod_{t=0}^{n} \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_{\mu(t)}, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1) =$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \max_{\mu \in T} \log \prod_{t=0}^{n} \frac{\|f(re^{i\theta})\|}{|(a_{\mu(t)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} +$$

$$+ \frac{1}{2\pi} \int_{0}^{2\pi} \max_{\mu \in T} \log \prod_{t=0}^{n} \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_{\mu(t)}, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1) =$$

$$= \frac{1}{2\pi} \int_{0}^{2\pi} \max_{\mu \in T} \sum_{t=0}^{n} \log \frac{\|f(re^{i\theta})\|}{|(a_{\mu(t)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} +$$

$$+ \frac{1}{2\pi} \int_{0}^{2\pi} \max_{\mu \in T} \sum_{t=0}^{n} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_{\mu(t)}, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1).$$

So we obtain

$$\sum_{j=1}^{q} m_f(r, H_j) \leq \int_{0}^{2\pi} \max_{K} \sum_{j \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} +$$

$$+ \int_{0}^{2\pi} \max_{K} \sum_{j \in K} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1).$$

Lemma 3.2 is proved.

***Proof of Theorem* 1.2.** We prove for $R_0 = +\infty$, the case $R_0 < +\infty$ can be proved similarly. By Lemmas 3.1 and 3.2, we obtain

$$\left\| \sum_{j=1}^{q} m_f(r, H_j) \le \int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \right.$$

$$+ \int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\|f(r^{-1}e^{i\theta})\|}{|(a_j, f)(r^{-1}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1) \le$$

$$\le (n+1)T_f(r) - N_W(r, 0) + O(\log r + \log T_f(r)). \tag{3.14}$$

By Theorem 1.1, we get that

$$T_f(r) = N_f(r, H_j) + m_f(r, H_j) + O(1)$$

for any $j \in \{1, \ldots, q\}$. So from (3.14), we have

$$\left\| \quad (q - n - 1)T_f(r) \le \sum_{j=1}^{q} N_f(r, H_j) - N_W(r, 0) + O(\log r + \log T_f(r)) \right. . \tag{3.15}$$

For $z_0 \in \Delta_r$, we may assume that (a_j, f) vanishes at z_0 for $1 \le j \le q_1$, (a_j, f) does not vanish at z_0 for $j > q_1$. Hence, there exists a nonnegative integer k_j and nowhere vanishing holomorphic function g_j in neighborhood U of z such that

$$(a_j, f)(z) = (z - z_0)^{k_j} g_j(z), \qquad \text{for} \quad j = 1, \ldots, q,$$

here $k_j = 0$ for $q_1 < j \le q$. We may assume that $k_j \ge n$ for $1 \le j \le q_0$, and $1 \le k_j < n$ for $q_0 < j \le q_1$. By property of the Wronskian, we have

$$W(f) = CW((a_0, f), \ldots, (a_n, f)) = \prod_{j=1}^{q_0} (z - z_0)^{k_j - n} h(z),$$

where $h(z)$ is holomorphic function on U. Then $W(f)$ is vanishes at z_0 with order at least

$$\sum_{j=1}^{q_0} (k_j - n) = \sum_{j=1}^{q_0} k_j - q_0 n.$$

By the definition of $N_f(r, H)$, $N_W(r, 0)$ and $N_f^n(r, H)$, we get

$$\sum_{j=1}^{q} N_f(r, H_j) - N_W(r, 0) =$$

$$= \left(\sum_{j=1}^{q} N_{1,f}(r, H_j) - N_1\left(r, \frac{1}{W}\right) \right) + \left(\sum_{j=1}^{q} N_{2,f}(r, H_j) - N_2\left(r, \frac{1}{W}\right) \right) + O(1) \le$$

$$\leq \sum_{j=1}^{q} N_{1,f}^{n}(r,H_j) + \sum_{j=1}^{q} N_{2,f}^{n}(r,H_j) + O(1) = \sum_{j=1}^{q} N_{f}^{n}(r,H_j) + O(1).$$

So from (3.15), we obtain

$$\Bigg\| \ (q-n-1)T_f(r) \leq \sum_{j=1}^{q} N_f^n(r,H_j) + O(\log r + \log T_f(r)) \ .$$

Theorem 1.2 is proved.

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