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p-REGULARITY THEORY. TANGENT CONE DESCRIPTION IN SINGULAR CASE

ТЕОРІЯ p-РЕГУЛЯРНОСТІ. ОПИС ДОТИЧНОГО КОНУСА В СИНГУЛЯРНОМУ ВИПАДКУ

We present the new proof of the theorem which is one of the main results of the p-regularity theory. This gives a detailed description of the structure of the zero set of an singular nonlinear mapping. We say that $F\colon X \to Y$ is singular at some point x_0, where X and Y are Banach spaces if $\mathrm{Im}F'(x_0) \neq Y$. Otherwise, the mapping F is said to be regular.

Наведено нове доведення теореми, що є одним з основних результатів теорії p-регулярності. Дано детальний опис структури множини нулів сингулярного лінійного відображення. Кажуть, що $F\colon X \to Y$ є сингулярним у точці x_0, де X та Y — банахові простори, якщо $\mathrm{Im}F'(x_0) \neq Y$. В протилежному випадку відображення F називається регулярним.

A description of the solution set in regular case of the equation

$$F(x) = 0, \tag{1}$$

where X, Y are Banach spaces, we may obtain by means of the Lyusternik theorem, which says that the tangent cone of the solution set is equal to the kernel of the first derivative of F evaluated at some point $x_0 \in X$.

In this paper we consider the case when the regularity condition does not hold, i. e., $\mathrm{Im}\, F'(x_0) \neq Y$, but the mapping F is p-regular. Let us remind the definition of p-regularity and construction of p-factor operator [1, 2].

For $F\colon X \to Y$, p-times Fréchet differentiable mapping, we construct the p-factor operator under the assumption that Y is decomposed into a direct sum

$$Y = Y_1 \oplus \ldots \oplus Y_p, \tag{2}$$

where $Y_1 = \overline{\mathrm{Im}\, F'(x_0)}$ (the closure of the image of the first derivative of F evaluated at x_0), and the remaining spaces are defined as follows. Let $Z_1 = Y$, Z_2 be closed complementary subspace to Y_1 (we assume that such closed complement exists), and let $P_{Z_2}\colon Y \to Z_2$ be the projection operator onto Z_2 along Y_1. Let Y_2 be the closed linear span of the image of the quadratic map $P_{Z_2} F^{(2)}(x_0)[\cdot]^2$. More generally, define inductively,

$$Y_i = \overline{\mathrm{span}\, \mathrm{Im}\, P_{Z_i} F^{(i)}(x_0)[\cdot]^i} \subseteq Z_i, \quad i = 2, \ldots, p-1,$$

where Z_i is a choice of closed complementary subspace for $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$ and $P_{Z_i}\colon Y \to Z_i$ is the projection operator onto Z_i along $(Y_1 \oplus \ldots \oplus Y_{i-1})$ with respect to Y, $i = 2, \ldots, p$. Finally, $Y_p = Z_p$. The order p is chosen as the minimum number for which (2) holds. Now, define the following mappings (see [3, 7]):

$$f_i\colon U \to Y_i, \quad f_i(x) = P_{Y_i} F(x), \quad i = 1, \ldots, p,$$

where U is a neighborhood of x_0, $P_{Y_i}: Y \to Y_i$ is the projection operator onto Y_i along $(Y_1 \oplus \ldots \ldots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \ldots \oplus Y_p)$ with respect to Y, $i = 1, \ldots, p$.

If $F^{(i)}(x_0) = 0$, where $i = 1, \ldots, p-1$, then we say that F is *completely degenerate* at $x_0 \in X$ up to the order p.

Note that

$$f_k^{(i)}(x_0) = 0, \qquad i = 1, \ldots, k-1, \quad k = 1, \ldots, p. \tag{3}$$

Definition 1. *The linear operator $\Lambda_h \in \mathcal{L}(X, Y_1 \oplus \ldots \oplus Y_p)$ is defined for some $h \in X$ by*

$$\Lambda_h(x) = f_1'(x_0)[x] + f_2''(x_0)[h, x] + \ldots + \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h, \ldots, h, x], \quad x \in X,$$

and is called the p-factor operator.

We will also use more exact notation $\Lambda_h = (\Lambda_{h,1} + \Lambda_{h,2} + \ldots + \Lambda_{h,p})$, where

$$\Lambda_{h,k} = \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}, \quad k = 1, \ldots, p.$$

It is also convenient to use the following equivalent definition of p-factor operator $\widetilde{\Lambda}_h \in \mathcal{L}(X, Y_1 \times \ldots \ldots \times Y_p)$ for some fixed $h \in X$,

$$\widetilde{\Lambda}_h(x) = \left(f_1'(x_0)[x], f_2''(x_0)[h, x], \ldots, \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h, \ldots, h, x] \right), \quad x \in X.$$

Note that in completely degenerate case the p-factor operator has the form $F^{(p)}(x_0)[h]^{p-1}$.

In other words, we construct a decomposition of „non regular part" of the mapping F on partial mappings f_i in such a way that all of those mappings are completely degenerate up to the order $i-1$, where $i = 2, \ldots, p$.

For our further considerations we need the following generalization of the notion of regular mapping.

Definition 2. *We say that the mapping F is p-regular at x_0 along h if*

$$\operatorname{Im} \Lambda_h = Y.$$

Let us introduce a corresponding nonlinear operator

$$\Psi[x]^p = f_1'(x_0)[x] + f_2''(x_0)[x]^2 + \ldots + f_p^{(p)}(x_0)[x]^p$$

and k-kernel, $k = 1, \ldots, p$ of $F^{(k)}(x_0)$,

$$\operatorname{Ker}^k F^{(k)}(x_0) = \left\{ h \in X : F^{(k)}(x_0)[h]^k = 0 \right\}.$$

It is easy to see that $\Psi[h]^p = \Lambda_h(h)$.

Definition 3. *We say that the mapping F is p-regular at x_0 if it is p-regular along any h from the set*

$$H_p(x_0) \overset{\text{df}}{=} \left\{ \bigcap_{i=1}^{p} \operatorname{Ker}^i f_i^{(i)}(x_0) \right\} \setminus \{0\}.$$

Let us consider two examples of p-regular mappings.

***Example* 1.** Let us consider the equation (1) with

$$F(x) = \begin{pmatrix} x_1^2 - x_2^2 + x_3^2 + x_2^3 \\ x_1^2 - x_2^2 + x_3^2 + x_2x_3 + x_3^3 \end{pmatrix}, \quad x \in R^3,$$

$F\colon R^3 \to R^2$, $x_0 = (0,0)^T$. It is easy to verify $F'(x_0) = (0,0)^T$ and it is obvious that $F'(x_0)$ is not surjective. However, the mapping F is 2-regular at x_0. Indeed,

$$F''(0) = \begin{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \end{pmatrix}$$

and

$$\mathrm{Ker}^2 F''(0) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}, \qquad \bar{h} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \bar{\bar{h}} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

where $F''(0)\bar{h} = \begin{pmatrix} 2 & -2 & 0 \\ 2 & -2 & 1 \end{pmatrix}$, $F''(0)\bar{\bar{h}} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}$ are not degenerate, which means that the condition of 2-regularity of F at x_0 is valid.

***Example* 2.** Consider the type of (1) equation

$$\Delta u - (\varepsilon - 10)g(u) = 0$$

on $\Omega = [0,\pi] \times [0,\pi]$ in R^2 with $u = 0$ on $\partial\Omega$. If we denote $F(u,\varepsilon) = \Delta u - (\varepsilon - 10)g(u)$, then the mapping $F(u,\varepsilon)$ is 2-regular at the point $(u_0,\varepsilon_0) = (0,0)$ (see [7, p. 403], Example 1).

For a linear surjective operator $\Lambda\colon X \mapsto Y$ between Banach spaces we denote by Λ^{-1} its *right inverse*. Therefore $\Lambda^{-1}\colon Y \mapsto 2^X$ and we have

$$\Lambda^{-1}(y) = \{x \in X : \Lambda x = y\}.$$

We define the *norm* of Λ^{-1} via the formula

$$\|\Lambda^{-1}\| = \sup_{\|y\|=1} \inf\big\{\|x\| : x \in \Lambda^{-1}(y)\big\}.$$

We say that Λ^{-1} is *bounded* if $\|\Lambda^{-1}\| < \infty$.

Lemma 1 [4]**.** *Let X and Y be Banach spaces, and let $\Lambda \in \mathcal{L}(X,Y)$. We set*

$$C(\Lambda) = \sup_{y \in Y} \big(\|y\|^{-1} \inf\{\|x\| : x \in X,\ \Lambda x = y\}\big).$$

If $\mathrm{Im}\,\Lambda = Y$, *then* $C(\Lambda) < \infty$.

We shall give a „multivalued" generalization of the contraction mapping principle that is of independent interest. Let $\mathrm{dist}_{\mathrm{H}}(A_1, A_2)$ be the Hausdorff distance between sets A_1 and A_2.

Lemma 2 (contraction multimapping principle) [4]**.** *Let Z be a complete metric space with distance ρ. Assume that a multimapping*

$$\Phi\colon B(z_0, \varepsilon) \mapsto 2^Z,$$

on a ball $B(z_0, \varepsilon) = \{z\colon \rho(z, z_0) < \varepsilon\}$ $(\varepsilon > 0)$ where the sets $\Phi(z)$ are non-empty and closed for any $z \in B(z_0, \varepsilon)$. Further, assume that there exists a number θ, $0 < \theta < 1$, such that

1) $\mathrm{dist}_{\mathrm{H}}\big(\Phi(z_1), \Phi(z_2)\big) \le \theta\rho(z_1, z_2)$ *for any* $z_1, z_2 \in B(z_0, \varepsilon)$,

2) $\rho\big(z_0, \Phi(z_0)\big) < (1 - \theta)\varepsilon$.

Then, there exists an element $z \in B(z_0, \varepsilon)$ such that

$$z \in \Phi(z). \tag{4}$$

Moreover, among the points z satisfying (4) *there exists a point such that*

$$\|z - z_0\| \le \frac{2}{1-\theta}\rho\big(z_0, \Phi(z_0)\big). \tag{5}$$

Now we introduce an inverse multivalued operator for Λ_h as follows:

$$\Lambda_h^{-1}(y) = \left\{\xi \in X\colon \left(f_1'(x_0)[\xi], f_2''(x_0)[h, \xi], \ldots, \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h, \ldots, h, \xi]\right) = (y_1, \ldots, y_p)\right\},$$

where $y = (y_1, \ldots, y_p)$ and $y_i \in Y_i$, $i = 1, \ldots, p$.

Definition 4. *The mapping F is called strongly p-regular at the point x_0 if there exist $\gamma > 0$ such that*

$$\sup_{h \in H_\gamma} \|\Lambda_h^{-1}\| < \infty,$$

where

$$H_\gamma \overset{\mathrm{df}}{=} \left\{h \in X\colon \left\|f_k^{(k)}(x_0)[h]^k\right\|_{Y_k} \le \gamma,\ k = 1, \ldots, p,\ \|h\| = 1\right\}.$$

The following theorem is a generalization of the Banach open mapping theorem.

Theorem 1. *If $\|\Lambda_h^{-1}\| < \infty$, then there exists $C(h) \ge 0$ such that*

$$\|\Lambda_h^{-1} y\| \le C(h)\left(\|y_1\| + \frac{\|y_2\|}{\|h\|} + \ldots + (p-1)!\frac{\|y_p\|}{\|h\|^{p-1}}\right),$$

where $h \ne 0$, $y = (y_1, \ldots, y_p)$ and $y_i \in Y_i$, $i = 1, \ldots, p$.

The proof was given in [6].

The next theorem was first given and proved in [3, p. 158]. We give new improved proof of this theorem.

Theorem 2. *Let X and Y be Banach spaces, U a neighborhood of $x_0 \in X$ and let $F: U \to Y$ be a $(p+1)$-times continuously Fréchet differentiable mapping in U. Assume that F is strongly p-regular at x_0. Then there exists a neighborhood $U' \subseteq U$ of the point x_0, a mapping $\eta \to x(\eta)$: $U' \to X$ and constants $\beta_1 > 0$ and $\beta_2 > 0$, such that for $\eta \in U'$:*

$$F(\eta + x(\eta)) = F(x_0), \tag{6}$$

$$\|x(\eta)\|_X \leq \beta_1 \sum_{i=1}^{p} \frac{\|f_i(\eta) - f_i(x_0)\|_{Y_i}}{\|\eta - x_0\|^{i-1}} \tag{7}$$

and

$$\|x(\eta)\|_X \leq \beta_2 \sum_{i=1}^{p} \|f_i(\eta) - f_i(x_0)\|_{Y_i}^{1/i}. \tag{8}$$

Before stating the proof of the theorem, we prove the following lemma.

Lemma 3. *Let all the assumptions of Theorem 2 hold. Then for any $\varepsilon > 0$ there exist constants $\delta > 0$ and $R > 0$ such that for any $h \in X$, $\|h\| \leq \delta$ and for any $x_1,\ x_2 \in X$, $\|x_i\| \leq \|h\|/R$, $i = 1, 2$, the following estimation is satisfied*:

$$\|F(x_0 + h + x_1) - F(x_0 + h + x_2) - \Lambda_h(x_1 - x_2)\|_Y =$$

$$= \left\|f_1(x_0 + h + x_1) - f_1(x_0 + h + x_2) - f_1'(x_0)[h](x_1 - x_2)\right\|_{Y_1} + \ldots$$

$$\ldots + \left\|f_p(x_0 + h + x_1) - f_p(x_0 + h + x_2) - \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h]^{p-1}(x_1 - x_2)\right\|_{Y_p} \leq$$

$$\leq \varepsilon \sum_{i=1}^{p} \|h\|^{i-1} \|x_1 - x_2\|.$$

Proof. First we prove that for any $k = 1, \ldots, p$ inequality

$$\left\|f_k(x_0 + h + x_1) - f_k(x_0 + h + x_2) - \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}(x_1 - x_2)\right\|_{Y_k} \leq$$

$$\leq \sup_{\theta \in [0,1]} \left\|f_k(x_0 + h + x_1 + x_2 + \theta(x_1 - x_2)) - \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}\right\|_{Y_k} \|x_1 - x_2\|_X \tag{9}$$

is satisfied.

By Taylor's expansion,

$$f_k\big(x_0 + h + x_1 + x_2 + \theta(x_1 - x_2)\big) =$$

$$= f_k'(x_0) + \ldots + \frac{1}{(k-1)!} f_k^{(k)}(x_0)\big[h + x_2 + \theta(x_1 - x_2)\big]^{k-1} + \omega_k(x_0, h, x_1, x_2, \theta),$$

where $\big\|\omega_k(x_0, h, x_1, x_2, \theta)\big\| = o\big(\|x_0 + h + x_1 + x_2 + \theta\|\big)$.

Let $\varepsilon > 0$ be sufficiently small number. Define

$$R = \max\left\{1, \frac{2}{\varepsilon}\big\|f_1'(x_0)\big\|, \ldots, \frac{2}{\varepsilon}\frac{4^{p-1}}{(p-1)!}\big\|f_1^{(p)}(x_0)\big\|\right\}.$$

By an assumption, $x_i \le \dfrac{\|h\|}{R}$, $i = 1, 2$, so

$$\|x_0 + h + x_1 + x_2 + \theta\| \le 4\|h\|.$$

The last inequality yields

$$\big\|\omega_k(x_0, h, x_1, x_2, \theta)\big\| = o\big(\|h\|^{k-1}\big).$$

Then there exists $\delta > 0$ such that for $\|h\| \le \delta$ and $x_i \le \dfrac{\|h\|}{R}$, $i = 1, 2$, the inequality (9) holds and

$$\big\|\omega_k(x_0, h, x_1, x_2, \theta)\big\| \le \frac{2}{\varepsilon}\|h\|^{k-1}.$$

By (3), (9) and the last estimation we obtain

$$\left\|f_k(x_0 + h + x_1) - f_k(x_0 + h + x_2) - \frac{1}{(k-1)!}f_k^{(k)}(x_0)[h]^{k-1}(x_1 - x_2)\right\| \le$$

$$\le \sup_{\theta\in[0,1]}\left\|\frac{1}{(k-1)!}f_k^{(k)}(x_0)[h + x_2\theta(x_1 - x_2)]^{k-1} - \frac{1}{(k-1)!}f_k^{(k)}(x_0)[h]^{k-1}\right\|\|x_1 - x_2\| +$$

$$+\frac{2}{\varepsilon}\|h\|^{k-1}\|x_1 - x_2\|. \qquad (10)$$

Let us observe that for any $k = 1, \ldots, p-1$,

$$f_k^{(k)}(x_0)\big[h + x_2 + \theta(x_1 - x_2)\big]^{k-1} = \sum_{i=0}^{k-1} C_{k-1}^i f_k^{(k)}(x_0)[h]^{k-1-i}\big[x_2 + \theta(x_1 - x_2)\big]^i =$$

$$= f_k^{(k)}(x_0)[h]^{k-1} + \sum_{i=1}^{k-1} C_{k-1}^i f_k^{(k)}(x_0)[h]^{k-1-i}\big[x_2 + \theta(x_1 - x_2)\big]^i.$$

We may estimate the second term in the last equation. Since

$$\big\|x_2 + \theta(x_1 - x_2)\big\| \le \frac{3\|h\|}{R},$$

then

$$\left\|\sum_{i=1}^{k-1} C_{k-1}^i f_k^{(k)}(x_0)[h]^{k-1-i}[x_2 + \theta(x_1 - x_2)]^i\right\| \le$$

$$\le \left\|f_k^{(k)}(x_0)\right\| \sum_{i=1}^{k-1} C_{k-1}^i \|h\|^{k-1-i}\frac{3^i\|h\|^i}{R^i} \le \left\|f_k^{(k)}(x_0)\right\| \|h\|^{k-1}\frac{4^{k-1}}{R}.$$

Thus by definition of R,

$$\left\| \sum_{i=1}^{k-1} C_{k-1}^{i} f_k^{(k)}(x_0)[h]^{k-1-i}\big[x_2 + \theta(x_1 - x_2)\big]^i \right\| \le (k-1)!\frac{\varepsilon}{2}\|h\|^{k-1}.$$

The last inequalities and (10) yield for $k = 1, \ldots, p$

$$\left\| f_k(x_0 + h + x_1) - f_k(x_0 + h + x_2) - \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}(x_1 - x_2) \right\| \le$$

$$\le \left(\frac{\varepsilon}{2}\|h\|^{k-1} + \frac{\varepsilon}{2}\|h\|^{k-1} \right) \|x_1 - x_2\| = \varepsilon\|h\|^{k-1}\|x_1 - x_2\|.$$

Hence for $k = 1, \ldots, p$

$$\left\| f_k(x_0 + h + x_1) - f_k(x_0 + h + x_2) - \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}(x_1 - x_2) \right\| \le$$

$$\le \varepsilon\|h\|^{k-1}\|x_1 - x_2\|. \tag{11}$$

Adding up the inequalities in (11) for $k = 1, \ldots, p$ finishes proof of the lemma.

***Proof of Theorem* 2.** Let us consider U as a sufficiently small neighborhood of x_0 and divide our consideration into 2 cases.

Case 1. We consider $\eta \in U$ such that $h_\eta = \dfrac{\eta - x_0}{\|\eta - x_0\|} \notin H_\gamma$, that is there exists $k \le p$ such that

$$\left\| f_k^{(k)}(x_0)[h_\eta]^k \right\| > \gamma$$

or

$$\left\| f_k^{(k)}(x_0)[\eta - x_0]^k \right\| > \gamma\|\eta - x_0\|^k.$$

Taking $x(\eta) = \eta - x_0$ we get

$$\|f_k(\eta) - f_k(x_0)\| = \left\| \frac{1}{k!} f_k^{(k)}(x_0)[\eta - x_0]^k + \omega_k(\eta) \right\| \ge \frac{\gamma}{2k!}\|\eta - x_0\|^k,$$

where $\|\omega_k(\eta)\| = o\big(\|\eta - x_0\|^k\big)$. Hence,

$$\|x(\eta)\| = \|\eta - x_0\| \le \frac{2k!}{\gamma} \frac{\|f_k(\eta) - f_k(x_0)\|}{\|\eta - x_0\|^{k-1}} \le \frac{2k!}{\gamma} \sum_{i=1}^{p} \frac{\|f_i(\eta) - f_i(x_0)\|}{\|\eta - x_0\|^{i-1}},$$

which finishes the proof of the first case.

Case 2. We consider all $\eta \in U$ such that $h_\eta = \dfrac{\eta - x_0}{\|\eta - x_0\|} \in H_\gamma$, and hence for $i = 1, \ldots, p$

$$\left\| f_i^{(i)}(x_0)[\eta - x_0]^k \right\| \le \gamma\|\eta - x_0\|^i,$$

where γ is a sufficiently small number that does not depend on η. We also use notation $h = \eta - x_0$.

Let $\sup_{h\in H_\gamma} \|\Lambda_h^{-1}\| = C < \infty$. Note that $h_\eta = \eta - x_0$ and $h_\eta = \dfrac{h}{\|h\|} \in H_\gamma$, the statement of Theorem 1 holds with $C(h) = C$ for all $h = \eta - x_0$.

Let $\varepsilon > 0$ be a sufficiently small number such that $p!C\varepsilon < 1$. Then Lemma 3 and inequalities (11) imply that there exist numbers δ and R, such that $B_{2\delta}(0) \subset (U - x_0)$, $0 < \delta \le r$, and for $k = 1, \ldots, p$, and $h \in B_\delta(0)$ $x_1, x_2 \in B_{\|h/R\|(0)}$,

$$\frac{C(k-1)!}{\|h\|^{k-1}} \left\| f_k(x_0+h+x_1) - f_k(x_0+h+x_2) - \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}(x_1-x_2) \right\| \le$$

$$\le C(k-1)!\varepsilon\|x_1 - x_2\|. \tag{12}$$

Adding up the inequalities in (12) for $k = 1, \ldots, p$ we obtain

$$\sum_{k=1}^{p} \frac{C(k-1)!}{\|h\|^{k-1}} \left\| f_k(x_0+h+x_1) - f_k(x_0+h+x_2) - \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}(x_1-x_2) \right\| \le$$

$$\le \alpha\|x_1 - x_2\| \tag{13}$$

with $\alpha = p!C\varepsilon$. Note that $0 < \alpha < 1$.

Let us define $r(h) = \|h\|/R$ and a neighborhood $V \subset B_\delta(0)$ such that

$$\sum_{k=1}^{p} \frac{C(k-1)!}{\|h\|^{k-1}} \|f_k(x_0+h) - f_k(x_0)\| < (1-\alpha) r(h) \tag{14}$$

for all $h \in V$. Such a neighborhood exists by the definition of the set H_γ and property (3).

Now let us fix an element $h \in V$ and consider multivalued mapping

$$\Phi_h \colon B_{r(h)}(0) \to 2^X, \qquad \Phi_h(x) = x - \Lambda_h^{-1}\big(F(x_0+h+x) - F(x_0)\big).$$

Because of the choice of the number $r(h)$ and the neighborhood V for all $x \in B_{r(h)}(0)$ and $h \in V$ we have $x_0 + h + x \in U$.

For any $y \in Y$ the set $\Lambda_h^{-1} y$ is a linear manifold parallel to $\mathrm{Ker}\Lambda_h$. Thus the set $\Phi_h(x)$ is closed for any $x \in B_{r(h)}(0)$.

Next we verify that all assumptions of Lemma 2 are satisfied for $\Phi_h(x)$ with $B_{r(h)}(0)$ and $\omega_0 = 0$. First, it will be shown that assumption 1 of the mentioned lemma holds for all $x_1, x_2 \in B_{r(h)}(0)$. We have

$$\mathrm{dist}_{\mathrm{H}}(\Phi_h(x_1), \Phi_h(x_2)) = \inf\big\{\|z_1 - z_2\| \colon z_i \in \Phi_h(x_i), i = 1,2\big\} =$$

$$= \inf\big\{\|z_1 - z_2\| \colon \Lambda_h z_i = \Lambda_h x_i - F(x_0+h+x_i) + F(x_0),\ i = 1,2\big\} =$$

$$= \inf\Big\{\|z\| \colon f_1'(x_0)z = f_1'(x_0)(x_1-x_2) - f_1(x_0+h+x_1) + f_1(x_0+h+x_2), \ldots$$

$$\ldots, \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h]^{p-1} z =$$

$$= \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h]^{p-1}(x_1 - x_2) - f_p(x_0 + h + h_1) + f_p(x_0 + h + x_1)\Big\}. \qquad (15)$$

By Theorem 1 and (13) we get

$$\mathrm{dist}_{\mathrm{H}}(\Phi_h(x_1), \Phi_h(x_2)) =$$

$$= \left\| \Lambda_h^{-1} \begin{pmatrix} f_1'(x_0)(x_1 - x_2) - f_1(x_0 + h + x_1) + f_1(x_0 + h + x_2) \\ \vdots \\ \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h]^{p-1}(x_1 - x_2) - f_p(x_0 + h + h_1) + f_p(x_0 + h + x_1) \end{pmatrix} \right\| \leq$$

$$\leq \sum_{k=1}^{p} \frac{C(k-1)!}{\|h\|^{k-1}} \left\| f_k(x_0 + h + x_1) - f_k(x_0 + h + x_2) - \frac{1}{(k-1)!} f_k^{(k)}(x_0)[h]^{k-1}(x_1 - x_2) \right\| \leq$$

$$\leq \alpha \|x_1 - x_2\|$$

for all $x_1, x_2 \in B_{r(h)}(0)$.

Hence the assumption 1 of the lemma holds.

Next we verify that assumption 2 is also satisfied. Using the approach similar to above we get by (3), (14), and (15),

$$\mathrm{dist}_{\mathrm{H}}(0, \Phi_h(0)) = \inf \Big\{ \|z\| : f_1'(x_0)z = f_1(x_0 + h) - f_1(x_0), \ \ldots$$

$$\ldots, \frac{1}{(p-1)!} f_p^{(p)}(x_0)[h]^{k-1}[h]^{p-1} z = f_p(x_0 + h) - f_p(x_0) \Big\} \leq$$

$$\leq \sum_{k=1}^{p} \frac{C(k-1)!}{\|h\|^{k-1}} \|f_k(x_0 + h) - f_k(x_0)\| < (1 - \alpha) r(h).$$

Then for any $h \in V$ the mapping $\Phi_h(x)$ satisfies assumptions of the Lemma 2. Hence there exists $x(h) \in \Phi_h(x(h))$ which means that

$$0 \in \Lambda_h^{-1}\big(F(x_0 + h + x(h)) - F(x_0)\big).$$

Thus $F(x_0 + h + x(h)) = F(x_0)$ for all $h \in V$.

In addition, by (5),

$$\|x(h)\| \leq \frac{2}{1-\alpha} \mathrm{dist}_{\mathrm{H}}(0, \Phi_h(0)) \leq \frac{2}{1-\alpha} \sum_{k=1}^{p} \frac{C(k-1)!}{\|h\|^{k-1}} \|f_k(x_0 + h) - f_k(x_0)\| \leq$$

$$\leq \beta_1 \sum_{k=1}^{p} \frac{\|f_k(x_0 + h) - f_k(x_0)\|}{\|h\|^{k-1}},$$

for all $h \in V$, and $\beta_1 = \dfrac{Cp!}{1-\alpha}$.

Taking $x(\eta) = x(h)$ and $U' = x_0 + V$ we obtain that (6) and (7) hold for all $\eta \in U'$.

Note that

$$\frac{f_k(x_0 + h) - f_k(x_0)}{\|h\|^{k-1}} \leq C_1 \|f_k(x_0 + h) - f_k(x_0)\|^{1/k}$$

for all $k = 1, \ldots, p$ and $h \in V$, with some $C_1 \geq 0$. Then inequality (7) implies (8) with $\beta_2 = C_1 p \beta_1$ which ends the proof of the theorem.

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