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M. Eslamian*
(Univ. Sci. and Technology Mazandaran, Behshahr and School Math., Instit. Res. Fundam. Sci., Tehran, Iran),
J. Vahidi (Iran Univ. Sci. and Technology, Iran)

GENERAL PROXIMAL POINT ALGORITHM FOR MONOTONE OPERATORS

ЗАГАЛЬНИЙ АЛГОРИТМ НАЙБЛИЖЧОЇ ТОЧКИ ДЛЯ МОНОТОННИХ ОПЕРАТОРІВ

We introduce a new general proximal-point algorithm for an infinite family of monotone operators in a real Hilbert space. We establish strong convergence of the iterative process to a common zero point of the infinite family of monotone operators. Our result generalizes and improves numerous results in the available literature.

Введено новий загальний алгоритм найближчої точки для нескінченної сім'ї монотонних операторів в дійсному гільбертовому просторі. Встановлено сильну збіжність ітераційного процесу до спільної нульової точки нескінченної сім'ї монотонних операторів. Наш результат узагальнює та покращує численні результати, що відомі з літературних джерел.

1. Introduction. Let H be a real Hilbert space with scalar product $\langle .,. \rangle$ and $A\colon D(A) \subset H \to H$ be a set-valued operator. Recall that A is called monotone if $\langle u-v, x-y\rangle \geq 0$, for any $[x,u],[y,v] \in$ $\in G(A)$, where

$$G(A) = \big\{(x,u)\colon x \in D(A), u \in A(x)\big\}.$$

A monotone operator A is said to be maximal monotone if its graph $G(A)$ is not properly contained in the graph of any other monotone operator. Monotone operators have proven to be a key class of objects in modern Optimization and Analysis (see, e.g., the books [1 – 4] and the references therein). On the other hand, a variety of problems, including convex programming and variational inequalities, can be formulated as finding zeros of monotone operators. Consequently the problem of finding a solution $z \in H$ of $0 \in Az$ has been investigated by many researchers. A popular method used to solve iteratively $0 \in Az$ is the proximal point algorithm of Rockafellar [5], which is recognized as a powerful and successful algorithm in finding zeros of monotone operators. Starting from any initial guess $x_0 \in H$, this proximal point algorithm generates a sequence $\{x_n\}$ given by

$$x_{n+1} = J^A_{c_n}(x_n + e_n), \tag{1.1}$$

where $J^A_r = (I + rA)^{-1}$ for all $r > 0$ is the resolvent of A and $\{e_n\}$ is a sequence of errors. Rockafellar proved the weak convergence of the algorithm (1.1). However, as shown by Güler [6], the proximal point algorithm does not necessarily converge strongly. Since then, many authors have conducted worthwhile research on modifying the proximal point algorithm so that the strong convergence is guaranteed (see, for instance, [7 – 10]). In particular, Xu [11] introduced the following iterative scheme:

$$x_{n+1} = t_n x_0 + (1 - t_n) J^A_{r_n} x_n + e_n, \tag{1.2}$$

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where x_0 is the starting point and $\{e_n\}$ is the error sequence. For $\{e_n\}$ summable, it was proved that $\{x_n\}$ is strongly convergent if $r_n \to \infty$, and $\{t_n\} \subset (0,1)$ with $\lim_{n\to\infty} t_n = 0$, $\sum_{n=0}^{\infty} t_n = \infty$. Boikanyo and Morosanu [12] generalized this algorithm (1.2) with error sequences in l^p for $1 \leq p < 2$. Recently, Xu [13] proposed the following regularization for the proximal point algorithm:

$$x_{n+1} = J_{r_n}^A(t_n x_0 + (1-t_n)x_n + e_n) \tag{1.3}$$

which essentially includes the so called prox-Tikhonov algorithm introduced by Lehdili and Moudafi [14] as a special cases. Boikanyo and Morosanu [15] noted that the proximal point algorithm (1.3) is equivalent to algorithm (1.2). These algorithms have been further studied and analyzed by many authors (see [16–23]).

In this work we introduce a general proximal point algorithm for finding a common zero point for an infinite family of monotone operators. We establish strong convergence of the iterative process to a common zero of the family of monotone operators. Our result generalizes some results of Xu [11], Tian and Song [17], Boikanyo and Morosanu [16], Yao and Noor [23], and many others.

2. Preliminaries. Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and induced norm $\|.\|$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converge weakly to x, and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Let K be a nonempty, closed and convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in K, denoted by $P_K x$, such that

$$\|x - P_K x\| \leq \|x - y\| \quad \forall y \in K.$$

Operator P_K is called the metric projection of H onto K. We also know that for $x \in H$ and $z \in K$, $z = P_K x$ if and only if

$$\langle x - z, y - z \rangle \leq 0 \quad \forall y \in K.$$

It is known that H satisfies Opial's condition, i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$. We will use the following notions on $S\colon K \to H$.

(i) S is nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in K.$$

(ii) S is firmly nonexpansive if

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 - \|(x - Sx) - (y - Sy)\|^2 \quad \forall x, y \in K.$$

It is well known that P_K is a nonexpansive mapping.

The resolvent operator has the following properties:

Lemma 2.1 [1]. *For a* $\lambda > 0$,

(i) *A is monotone if and only if the resolvent J_λ^A of A is single valued and firmly nonexpansive.*

(ii) *A is maximal monotone if and only if J_λ^A of A is single valued and firmly nonexpansive and its domain is all of H.*

(iii) $0 \in A(x^\star) \iff x^\star \in Fix(J_\lambda^A)$, *where $Fix(J_\lambda^A)$ denotes the fixed point set of J_λ^A.*

Since the fixed point set of a nonexpansive operator is closed convex, the projection onto the solution set $Z = A^{-1}(0) = \{x \in D(A) \colon 0 \in Ax\}$ is well defined whenever $Z \neq \varnothing$. For more details, see [24].

Lemma 2.2 [1] (The resolvent identity). *For $\lambda, \mu > 0$, there holds the identity*

$$J_\lambda^A x = J_\mu^A \left(\frac{\mu}{\lambda} x + \left(1 - \frac{\mu}{\lambda}\right) J_\lambda^A x\right), \quad x \in H.$$

Let B be a strongly positive bounded linear operator on H, that is, there is a constant $\overline{\gamma} > 0$ such that

$$\langle Bx, x\rangle \geq \overline{\gamma}\|x\|^2 \qquad \forall x \in H.$$

A typical problem is to minimize a quadratic function over the set of fixed points of a nonexpansive mapping S :

$$\min_{x \in F(S)} \frac{1}{2}\langle Bx, x\rangle - \langle x, b\rangle.$$

Marino and Xu [25] introduced the following iterative process for finding a fixed point of a nonexpansive mapping based on the viscosity approximation method introduced by Moudafi [26]:

$$x_{n+1} = a_n \gamma f(x_n) + (I - a_n B)Sx_n \qquad \forall n \geq 0. \tag{2.1}$$

They proved that under some appropriate condition imposed on the parameters, the sequence $\{x_n\}$ generated by (2.1) converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^\star, x - x^\star\rangle \geq 0 \qquad \forall x \in F(S),$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(S)} \frac{1}{2}\langle Bx, x\rangle - h(x),$$

where h is a potential function for γf $\big($i.e., $h'(x) = \gamma f(x)\ \forall x \in H\big)$.

Lemma 2.3 [25]. *Assume that B is a strongly positive bounded linear operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|B\|^{-1}$. Then $\|I - \rho B\| \leq 1 - \rho\overline{\gamma}$.*

Lemma 2.4. *There holds the following inequality in a Hilbert space H*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

Lemma 2.5 [27]. *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H. Then for any given $\{\lambda_n\}_{n=1}^\infty \subset (0,1)$ with $\sum_{n=1}^\infty \lambda_n = 1$ and for any positive integer i, j with $i < j$,*

$$\left\|\sum_{n=1}^\infty \lambda_n x_n\right\|^2 \leq \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i\lambda_j\|x_i - x_j\|^2.$$

Lemma 2.6 [11]. *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\delta_n + \beta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ satisfy the conditions:

(i) $\gamma_n \subset [0,1]$, $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$,

(iii) $\beta_n \geq 0$ *for all* $n \geq 0$ *with* $\sum_{n=0}^{\infty} \beta_n < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2.7 [28]. *Let* $\{t_n\}$ *be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence* $\{t_{n_i}\}$ *of* $\{t_n\}$ *such that* $t_{n_i} \leq t_{n_i+1}$ *for all* $i \geq 0$. *For sufficiently large numbers* $n \in \mathbb{N}$, *define an integer sequence* $\{\tau(n)\}$ *as*

$$\tau(n) = \max\{k \leq n \colon t_k < t_{k+1}\}.$$

Then $\tau(n) \to \infty$ *as* $n \to \infty$ *and*

$$\max\{t_{\tau(n)}, t_n\} \leq t_{\tau(n)+1}.$$

3. Main result. Now, we state our main result.

Theorem 3.1. *Let* A_i, $\in \mathbb{N}$, *be an infinite family of monotone operators of a Hilbert space* H *with* $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(\{0\}) \neq \varnothing$. *Assume that* K *is a nonempty closed convex subset of* H *such that* $\bigcap_{i=1}^{\infty} \overline{D(A_i)} \subset K \subset \bigcap_{i=1}^{\infty} R(I + rA_i)$ *for all* $r > 0$. *Assume that* f *is a* b*-contraction of* K *into itself and* B *is a strongly positive bounded linear operator on* H *with coefficient* $\overline{\gamma}$ *and* $0 < \gamma < \dfrac{\overline{\gamma}}{b}$. *Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in H$ *and*

$$y_n = \alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_{r_n}^{A_i} x_n, \quad n \geq 0,$$

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n B) y_n \quad \forall n \geq 0,$$

where $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ *and* $\{\alpha_{n,i}\}$ *and* $\{\beta_n\}$ *satisfy the following conditions*:

(i) $\{\beta_n\} \subset (0,1)$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} \beta_n = \infty$,

(ii) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0$,

(iii) $\{\alpha_{n,i}\} \subset (0,1)$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \in \mathbb{N}$.

Then the sequence $\{x_n\}$ *converges strongly to* $z \in Z$, *which solves the variational inequality*;

$$\langle (B - \gamma f)z, x - z\rangle \geq 0 \quad \forall x \in Z.$$

Proof. Since $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(\{0\})$ is closed and convex, we have the projection P_Z is well defined. Since $\lim_{n\to\infty} \beta_n = 0$, we can assume that $\beta_n \in (0, \|B\|^{-1})$ for all $n \geq 0$. Applying Lemma 2.3 we have

$$\|I - \beta_n B\| \leq 1 - \beta_n \overline{\gamma}. \tag{3.1}$$

Next, we show that $\{x_n\}$ is bounded. By Lemma 2.1, the operators $J_{r_n}^{A_i}$ are nonexpansive and hence we get

$$\|y_n - z\| \leq \|\alpha_{n,0} x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_{r_n}^{A_i} x_n - z\| \leq$$

$$\leq \alpha_{n,0}\|x_n - z\| + \sum_{i=1}^{\infty}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - z\| \leq$$

$$\leq \alpha_{n,0}\|x_n - z\| + \sum_{i=1}^{\infty}\alpha_{n,i}\|x_n - z\| \leq \|x_n - z\|.$$

By using inequality (3.1) we obtain

$$\|x_{n+1} - z\| = \|\beta_n(\gamma f(x_n) - Bz) + ((I - \beta_n B)(y_n - z)\| \leq$$

$$\leq \beta_n\|\gamma f(x_n) - Bz\| + \|I - \beta_n B\|\|y_n - z\| \leq$$

$$\leq \beta_n\gamma\|f(x_n) - f(z)\| + \beta_n\|\gamma f(z) - Bz\| + (1 - \beta_n\overline{\gamma})\|x_n - z\| \leq$$

$$\leq \beta_n\gamma b\|x_n - z\| + \beta_n\|\gamma f(z) - Bz\| + (1 - \beta_n\overline{\gamma})\|x_n - z\| \leq$$

$$\leq (1 - \beta_n(\overline{\gamma} - \gamma b))\|x_n - z\| + \beta_n\|\gamma f(z) - Bz\| \leq$$

$$\leq \max\left\{\|x_n - z\|, \frac{1}{\overline{\gamma} - \gamma b}\|\gamma f(z) - Bz\|\right\}.$$

It follows by induction that

$$\|x_n - z\| \leq \max\left\{\|x_0 - z\|, \frac{1}{\overline{\gamma} - \gamma b}\|\gamma f(z) - Bz\|\right\} \quad \forall n \geq 0.$$

This shows that $\{x_n\}$ is bounded and so is $\{f(x_n)\}$. Next, we show that for each $i \in \mathbb{N}$,

$$\lim_{n\to\infty}\|x_n - J_{r_n}^{A_i}x_n\| = 0.$$

By using Lemma 2.5, for each $i \in \mathbb{N}$ we get

$$\|y_n - z\|^2 \leq \|\alpha_{n,0}x_n + \sum_{i=1}^{\infty}\alpha_{n,i}J_{r_n}^{A_i}x_n - z\|^2 \leq$$

$$\leq \alpha_{n,0}\|x_n - z\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - z\|^2 - \alpha_{n,0}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - x_n\|^2 \leq$$

$$\leq \alpha_{n,0}\|x_n - z\|^2 + \sum_{i=1}^{\infty}\alpha_{n,i}\|x_n - z\|^2 - \alpha_{n,0}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - x_n\|^2 \leq$$

$$\leq \|x_n - z\|^2 - \alpha_{n,0}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - x_n\|^2. \tag{3.2}$$

Consequently, we have

$$\|x_{n+1} - z\|^2 = \|\beta_n(\gamma f(x_n) - Bz) + (I - \beta_n B)(y_n - z)\|^2 \leq$$

$$\leq \|\beta_n(\gamma f(x_n) - Bz) + (I - \beta_n B)(y_n - z)\|^2 \leq$$

$$\leq \beta_n^2\|\gamma f(x_n) - Bz\|^2 + (1 - \beta_n\overline{\gamma})^2\|y_n - z\|^2 +$$

$$+2\beta_n(1 - \beta_n\overline{\gamma})\|\gamma f(x_n) - Bz\|\|y_n - z\| \leq$$

$$\leq \beta_n^2\|\gamma f(x_n) - Bz\|^2 + 2\beta_n(1-\beta_n\overline{\gamma})\|\gamma f(x_n) - Bz\|\|x_n - z\| +$$
$$+ (1-\beta_n\overline{\gamma})^2\|x_n - z\|^2 - (1-\beta_n\overline{\gamma})^2\alpha_{n,0}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - x_n\|^2. \tag{3.3}$$

So, we have for every $i \in \mathbb{N}$

$$(1-\beta_n\overline{\gamma})^2\alpha_{n,0}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - x_n\|^2 \leq$$
$$\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + 2\beta_n(1-\beta_n\overline{\gamma})\|\gamma f(x_n) - Bz\|\|x_n - z\| +$$
$$+ \beta_n^2\|\gamma f(x_n) - Bz\|^2. \tag{3.4}$$

We note that the Banach contraction mapping principle guarantees that $P_{\mathcal{Z}}(I - B + \gamma f)$ has a unique fixed point z which is the unique solution of the variational inequality

$$\langle (B - \gamma f)z, x - z\rangle \geq 0 \quad \forall x \in Z.$$

We finally analyze the inequality (3.4) by considering the following two cases.

Case 1. Assume that $\big\{\|x_n - z\|\big\}$ is a monotone sequence. In other words, for n_0 large enough, $\{\|x_n - z\|\}_{n\geq n_0}$ is either nondecreasing or non-increasing. Since $\|x_n - z\|$ is bounded, we have $\|x_n - z\|$ is convergent. Since $\lim_{n\to\infty}\beta_n = 0$, and $\{f(x_n)\}$ and $\{x_n\}$ are bounded, from (3.4) we get

$$\lim_{n\to\infty}(1-\beta_n\overline{\gamma})^2\alpha_{n,0}\alpha_{n,i}\|J_{r_n}^{A_i}x_n - x_n\|^2 = 0,$$

which implies that

$$\lim_{n\to\infty}\|J_{r_n}^{A_i}x_n - x_n\| = 0.$$

Using the resolvent identity (Lemma 2.2), for each $r > 0$ we have

$$\big\|x_n - J_r^{A_i}x_n\big\| \leq \big\|x_n - J_{r_n}^{A_i}x_n\big\| + \big\|J_{r_n}^{A_i}x_n - J_r^{A_i}x_n\big\| \leq$$
$$\leq \big\|x_n - J_{r_n}^{A_i}x_n\big\| + \left\|J_r^{A_i}\Big(\frac{r}{r_n}x_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}^{A_i}x_n\Big) - J_r^{A_i}x_n\right\| \leq$$
$$\leq \|x_n - J_{r_n}^{A_i}x_n\| + \left\|\frac{r}{r_n}x_n + \left(1 - \frac{r}{r_n}\right)J_{r_n}^{A_i}x_n - x_n\right\| \leq$$
$$\leq \big\|x_n - J_{r_n}^{A_i}x_n\big\| + \left|1 - \frac{r}{r_n}\right|\big\|J_{r_n}^{A_i}x_n - x_n\big\| \to 0, \quad n\to\infty.$$

Next we show that $\limsup_{n\to\infty}\langle (B-\gamma f)z, z - x_n\rangle \leq 0$. We can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{i\to\infty}(\langle B - \gamma f)z, z - x_{n_i}\rangle = \limsup_{n\to\infty}(\langle B - \gamma f)z, z - x_n\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to x^*. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup x^*$. We show that $x^* \in Z$. Indeed,

$$\|x_{n_i} - J_r^{A_i}x^*\| \leq \|x_{n_i} - J_r^{A_i}x_{n_i}\| + \|J_r^{A_i}x_{n_i} - J_r^{A_i}x^*\| \leq$$
$$\leq \|x_{n_i} - J_r^{A_i}x_{n_i}\| + \|x_{n_i} - x^*\|,$$

which implies that

$$\limsup_{i\to\infty} \|x_{n_i} - J_r^{A_i} x^*\| \leq \limsup_{i\to\infty} \|x_{n_i} - x^*\|.$$

By the Opial property of Hilbert space H we obtain $x^* = J_r^{A_i} x^*$, $i \in \mathbb{N}$. Hence $x^* \in Z$. Therefore, from $z = P_Z(I - B + \gamma f)z$ and $x^* \in \Psi$, it follows that

$$\limsup_{n\to\infty} \langle (B - \gamma f)z, z - x_n\rangle = \lim_{i\to\infty} (\langle B - \gamma f)z, z - x_{n_i}\rangle = (\langle B - \gamma f)z, z - x^*\rangle \leq 0.$$

Since

$$x_{n+1} - z = \beta_n(\gamma f(x_n) - Bz) + (I - \beta_n B)(y_n - z),$$

by using Lemma 2.4 and the inequality (3.2) we have

$$\begin{gathered}
\|x_{n+1} - z\|^2 \leq \|(I - \beta_n B)(y_n - z)\|^2 + 2\beta_n\langle \gamma f(x_n) - Bz, x_{n+1} - z\rangle \leq \\
\leq (1 - \beta_n\overline{\gamma})^2\|x_n - z\|^2 + \\
+2\beta_n\gamma\langle f(x_n) - f(z), x_{n+1} - z\rangle + 2\beta_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle \leq \\
\leq (1 - \beta_n\overline{\gamma})^2\|x_n - z\|^2 + 2\beta_n b\gamma\|x_n - z\|\|x_{n+1} - z\| + \\
+2\beta_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle \leq \\
\leq (1 - \beta_n\overline{\gamma})^2\|x_n - z\|^2 + \beta_n b\gamma(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \\
+2\beta_n\langle \gamma fz - Bz, x_{n+1} - z\rangle \leq \\
\leq \big((1 - \beta_n\overline{\gamma})^2 + \beta_n b\gamma\big)\|x_n - z\|^2 + \beta_n\gamma b\|x_{n+1} - z\|^2 + \\
+2\beta_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle.
\end{gathered}$$

This implies that

$$\begin{gathered}
\|x_{n+1} - z\|^2 \leq \frac{1 - 2\beta_n\overline{\gamma} + (\beta_n\overline{\gamma})^2 + \beta_n\gamma b}{1 - \beta_n\gamma b}\|x_n - z\|^2 + \\
+\frac{2\beta_n}{1 - \eta_n\gamma b}\langle \gamma fz - Bz, x_{n+1} - z\rangle = \\
= \left(1 - \frac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n\gamma b}\right)\|x_n - z\|^2 + \frac{(\beta_n\overline{\gamma})^2}{1 - \eta_n\gamma b}\|x_n - z\|^2 + \\
+\frac{2\beta_n}{1 - \beta_n\gamma b}\langle \gamma fz - Bz, x_{n+1} - z\rangle \leq \left(1 - \frac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n\gamma b}\right)\|x_n - z\|^2 + \\
+\frac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n\gamma b}\left(\frac{(\beta_n\overline{\gamma}^2)P}{2(\overline{\gamma} - \gamma b)} + \frac{1}{\overline{\gamma} - \gamma b}\right)\langle \gamma fz - Bz, x_{n+1} - z\rangle) = \\
= (1 - \gamma_n)\|x_n - z\|^2 + \gamma_n\delta_n
\end{gathered}$$

where $P = sup\{\|x_n - z\|^2 : n \geq 0\}$, $\gamma_n = \dfrac{2(\overline{\gamma} - \gamma b)\beta_n}{1 - \beta_n\gamma b}$ and

$$\delta_n = \frac{(\beta_n\overline{\gamma}^2)P}{2(\overline{\gamma} - \gamma b)} + \frac{1}{\overline{\gamma} - \gamma b}\langle \gamma fz - Bz, x_{n+1} - z\rangle).$$

It is easy to see that $\gamma_n \to 0$, $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$. Now by applying Lemma 2.6 we conclude that the sequence $\{x_n\}$ converges strongly to z.

Case 2. Assume that $\{\|x_n - z\|\}$ is not a monotone sequence. Then, we can define an integer sequence $\{\tau(n)\}$ for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) = \max\{k \in \mathbb{N}; k \leq n\colon \|x_k - z\| < \|x_{k+1} - z\|\}.$$

Clearly, $\tau(n)$ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and for all $n \geq n_0$,

$$\|x_{\tau(n)} - z\| < \|x_{\tau(n)+1} - z\|.$$

Now, it follows from (3.3) that

$$\|x_{n+1} - z\|^2 - \|x_n - z\|^2 \leq \beta_n^2 \|\gamma\mathfrak{f}(x_n) - Bz\|^2 + ((\beta_n\overline{\gamma})^2 - 2\beta_n\overline{\gamma})\|x_n - z\|^2 +$$
$$+ 2\beta_n(1 - \beta_n\overline{\gamma})\|\gamma\mathfrak{f}(x_n) - Bz\|\|x_n - z\|.$$

Since $\lim_{n\to\infty} \beta_n = 0$ and $\{f(x_n)\}$ and $\{x_n\}$ are bounded, we derive

$$\lim_{n\to\infty} \left(\|x_{\tau(n)+1} - z\|^2 - \|x_{\tau(n)} - z\|^2\right) = 0 \tag{3.5}$$

By the similar argument as in Case 1 we obtain

$$\lim_{n\to\infty} \left\|J_{r_n}^{A_i} x_{\tau(n)} - x_{\tau(n)}\right\| = 0,$$

and

$$\|x_{\tau(n)+1} - z\|^2 \leq (1 - \gamma_{\tau(n)})\|x_{\tau(n)} - z\|^2 + \gamma_{\tau(n)}\delta_{\tau(n)},$$

where $\limsup_{n\to\infty} \delta_{\tau(n)} \leq 0$. Since $\|x_{\tau(n)} - z\| \leq \|x_{\tau(n)+1} - z\|$, we have

$$\gamma_{\tau(n)}\|x_{\tau(n)} - z\|^2 \leq \gamma_{\tau(n)}\delta_{\tau(n)}.$$

Since $\gamma_{\tau(n)} > 0$ we deduce

$$\|x_{\tau(n)} - z\|^2 \leq \delta_{\tau(n)}.$$

From $\limsup_{n\to\infty} \delta_{\tau(n)} \leq 0$ we get $\lim_{n\to\infty} \|x_{\tau(n)} - z\| = 0$. This together with (3.5), implies that $\lim_{n\to\infty} \|x_{\tau(n)+1} - z\| = 0$. Thus by Lemma 2.7 we have

$$0 \leq \|x_n - z\| \leq \max\{\|x_{\tau(n)} - z\|, \|x_n - z\|\} \leq \|x_{\tau(n)+1} - z\|.$$

Therefore $\{x_n\}$ converges strongly to $z = P_{\mathcal{Z}}(I - B + \gamma f)z$.

Theorem 3.1 is proved.

Theorem 3.2. *Let A_i, $i \in \mathbb{N}$, be an infinite family of maximal monotone operators of a real Hilbert space H with $Z = \bigcap_{i=1}^{\infty} A_i^{-1}(\{0\}) \neq \varnothing$. Assume that f is a b-contraction of H into itself and A is a strongly positive bounded linear operator on H with coefficient $\overline{\gamma}$ and $0 < \gamma < \dfrac{\overline{\gamma}}{b}$. Let $\{x_n\}$ be a sequence generated by $x_0 \in H$ and*

$$y_n = \alpha_{n,0}x_n + \sum_{i=1}^{\infty} \alpha_{n,i} J_{r_n}^{A_i} x_n, \quad n \geq 0,$$

$$x_{n+1} = \beta_n \gamma f(x_n) + (I - \beta_n B)y_n \quad \forall n \geq 0,$$

where $\sum_{i=0}^{\infty} \alpha_{n,i} = 1$ and $\{\alpha_{n,i}\}$ and $\{\beta_n\}$ satisfy the following conditions:

(i) $\{\beta_n\} \subset (0,1),\ \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \beta_n = \infty,$

(ii) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0,$

(iii) $\{\alpha_{n,i}\} \subset (0,1)$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for all* $i \in \mathbb{N}$.

Then the sequence $\{x_n\}$ *converges strongly to* $z \in Z$, *which solves the variational inequality*

$$\langle (B - \gamma f)z, x - z\rangle \geq 0 \quad \forall x \in Z.$$

Proof. Since A_i are maximal monotones, then A_i are monotone and satisfy the condition $R(I + rA_i) = H$ for all $r > 0$. Putting $K = H$ in Theorem 3.1, the desired result follows.

Putting $B = I$ and $\gamma = 1$ in Theorem 3.1, for a finite family of monotone operators we obtain immediately the following result.

***Следствие* 3.1.** *Let* $A_i,\ i = 1, 2, \ldots, m$, *be a finite family of monotone operators of a Hilbert space* H *with* $Z = \bigcap_{i=1}^{m} A_i^{-1}(\{0\}) \neq \varnothing$. *Assume that* K *is a nonempty closed convex subset of* H *such that* $\bigcap_{i=1}^{m} \overline{D(A_i)} \subset K \subset \bigcap_{i=1}^{m} R(I + rA_i)$ *for all* $r > 0$. *Assume that* f *is a* b*-contraction of* K *into itself. Let* $\{x_n\}$ *be a sequence generated by* $x_0 \in H$ *and*

$$y_n = \alpha_{n,0}x_n + \sum_{i=1}^{m} \alpha_{n,i} J_{r_n}^{A_i} x_n, \quad n \geq 0,$$

$$x_{n+1} = \beta_n f(x_n) + (1 - \beta_n)y_n \quad \forall n \geq 0,$$

where $\sum_{i=0}^{m} \alpha_{n,i} = 1$ *and* $\{\alpha_{n,i}\},\ \{\beta_n\}$ *satisfy the following conditions*:

(i) $\{\beta_n\} \subset (0,1),\ \lim_{n\to\infty} \beta_n = 0,\ \sum_{n=1}^{\infty} \beta_n = \infty,$

(ii) $\{r_n\} \subset (0,\infty)$ *and* $\liminf_{n\to\infty} r_n > 0,$

(iii) $\{\alpha_{n,i}\} \subset (0,1)$ *and* $\liminf_{n\to\infty} \alpha_{n,0}\alpha_{n,i} > 0$ *for* $i = 1, 2, \ldots, m$.

Then the sequence $\{x_n\}$ *converges strongly to* $z \in Z$, *which solves the variational inequality*

$$\langle z - fz, x - z\rangle \geq 0 \quad \forall x \in Z.$$

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