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# BOUNDARY-INTEGRAL APPROACH FOR THE NUMERICAL SOLUTION OF THE CAUCHY PROBLEM FOR THE LAPLACE EQUATION

# МЕТОД ГРАНИЧНИХ ІНТЕГРАЛІВ ДЛЯ ЧИСЕЛЬНОГО РОЗВ'ЯЗУВАННЯ ЗАДАЧІ КОШІ ДЛЯ РІВНЯННЯ ЛАПЛАСА

We give a survey of a direct method of boundary integral equations for the numerical solution of the Cauchy problem for the Laplace equation in doubly connected domains. The domain of solution is located between two closed boundary surfaces (curves in the case of two-dimensional domains). This Cauchy problem is reduced to finding the values of a harmonic function and its normal derivative on one of the two closed parts of the boundary according to the information about these quantities on the other boundary surface. This is an ill-posed problem in which the presence of noise in the input data may completely destroy the procedure of finding the approximate solution. We describe and present a results for a procedure of regularization aimed at the stable determination of the required quantities based on the representation of the solution to the Cauchy problem in the form a single-layer potential. For the given data, this representation yields a system of boundary integral equations for two unknown densities. We establish the existence and uniqueness of these densities and propose a method for the numerical discretization in two- and three-dimensional domains. We also consider the cases of simply connected domains of solution and unbounded domains. Numerical examples are presented both for two- and three-dimensional domains. These numerical results demonstrate that the proposed method gives good accuracy with relatively small amount of computations.

Наведено огляд прямого методу граничних інтегральних рівнянь для чисельного розв'язування задачі Коші для рівняння Лапласа у двозв'язних областях; область розв'язування розміщена між двома замкненими граничними поверхнями (кривими у випадку двовимірних областей). Ця задача Коші полягає у знаходженні значень гармонічної функції та її нормальної похідної на одній із двох замкнених границь за інформацією про ці величини на іншій граничній поверхні. Це є некоректна задача, в якій шум у вхідних даних може призвести до непридатного обчисленого наближеного розв'язку. Ми описуємо і наводимо огляд регуляризуючого методу для стійкого визначення шуканих величин, ґрунтуючись на поданні розв'язку задачі Коші у формі потенціалу простого шару. Таке подання приводить до системи граничних інтегральних рівнянь відносно двох невідомих густин. Встановлено існування і єдиність густин та запропоновано спосіб чисельної дискретизації у дво- та тривимірних областях. Також дискутується випадок однозв'язних областей та випадок необмежених областей. Наведено чисельні приклади для дво- та тривимірних областей, які засвідчують, що запропонований підхід дає хорошу точність при економних обчислювальних затратах.

**1. Introduction.** Let $D_2 \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with boundary surface $\Gamma_2$. This surface is assumed to be simple (no self-intersections) closed (the surface has itself no boundary and is connected) and sufficiently smooth. In the case when $d = 2$, we have a boundary curve with the similar properties assumed; we shall not explicitly state each time the word "surface" appear that we also consider planar domains with boundary curves but ask the reader to keep in mind that the present work also covers the planar case when surfaces are replaced by curves.

Let then $\Gamma_1$ be a simple closed (smooth) surface lying wholly within $D_2$ with the interior of $\Gamma_1$ being denoted $D_1$. The solution domain $D$ is the region between the two surfaces $\Gamma_1$ and $\Gamma_2$, thus $D = D_2 \setminus \bar{D}_1$, see further Fig. 1 for examples of the configuration.

Let $u \in C^2(D) \cap C^1(\bar{D})$ be a harmonic function, that is a solution to the Laplace equation

$$\Delta u = 0 \quad \text{in} \quad D \tag{1.1}$$

and suppose additionally that $u$ satisfies the following boundary conditions on the outer surface $\Gamma_2$:

$$u = f \quad \text{on} \quad \Gamma_2 \qquad \text{and} \qquad \frac{\partial u}{\partial \nu} = g \quad \text{on} \quad \Gamma_2. \tag{1.2}$$

The linear inverse problem we study is: Given the function values and normal derivative on $\Gamma_2$, find a harmonic function $u$ in the domain $D$, matching this data. In particular, reconstruct the corresponding data $u$ and $\dfrac{\partial u}{\partial \nu}$ on the interior boundary $\Gamma_1$. Here, $\nu$ is the outward unit normal varying along the boundary surfaces.

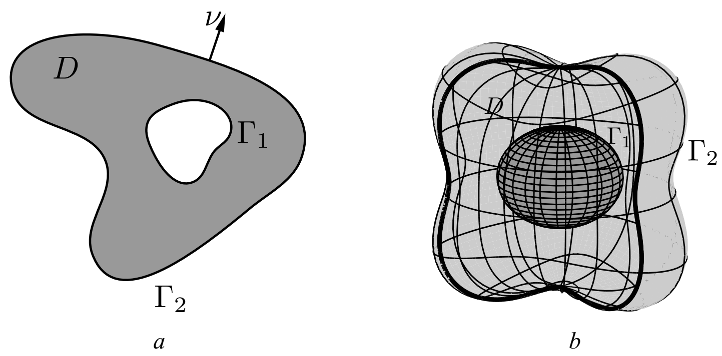


Fig. 1. Example of a two-dimensional (*a*) and three-dimensional (*b*) solution domain $D$ with the boundary part $\Gamma_1$ contained within the outer boundary $\Gamma_2$.

This type of problem is known as a Cauchy problem, and the given data on one boundary part is termed as Cauchy data. The problem is known to have a unique solution (a consequence of the Holmgren theorem), however, the continuous dependence on the data cannot be guaranteed making it fall into the category of ill-posed problems.

The Cauchy problem has a long history going back to Hadamard [17], and serves as a typical example of an ill-posed problem. The Cauchy problem has several important applications, for example, in cardiology, corrosion detection, electrostatics, geophysics, leak identification, nondestructive testing and plasma physics. Some works, where references to applications and methods for Cauchy problems can be found, are [7, 10, 19, 20]. We shall not go further into details or references on history or properties of Cauchy problems, but only state that we assume that data are compatible such that there exists a solution.

Two possible strategies for the numerical calculation of the solution to a Cauchy problem are the following. One can recast the problem as an equation for the missing boundary data, and for the obtained equation Tikhonov regularization is employed for the stable determination of the solution. The second strategy is to form a sequence of well-posed problems for the same equation and to prove that this sequence converges to the sought solution of the ill-posed problem. In either of the strategies, in terms of numerical calculations, approximations of harmonic functions, and in particular their boundary values, are needed. We shall outline a method belonging to the first category, a reference to a method in the second category is [23].

The solution domain of interest can be of different form to the one introduced above, in particular in some applications it can be unbounded or simply connected. The authors have been involved in developing methods based on boundary integral equations for Cauchy problems that are flexible in the sense that they can be adjusted to various solution domains, and numerically efficient since only boundary data are needed. In the present work, we shall give an overview of one such method and results obtained. This method corresponds to the first strategy mentioned above. The results presented are collected from work done by the authors in [1, 2, 4, 9 – 11].

The method is based on boundary integral techniques involving parameterisations of the boundary surfaces. The solution to (1.1), (1.2) in the region $D$ is represented as a sum of two single-layer potentials, one for each boundary surface, with unknown surface densities. Matching the given Cauchy data, a system of boundary integral equations is derived from which the densities over the two boundary surfaces can be obtained. We note that single-layer approaches have been used before for ill-posed problems, for example in inverse acoustic scattering, but seems somewhat overlooked for the Cauchy problem (for some properties of single-layer potentials and history, see [12]).

We point out that rather than using techniques based on parameterisations of the boundary surfaces, one can use the boundary element method (BEM) since only boundary data is needed in the Cauchy problem. However, in the BEM, the boundary surfaces are discretised into simpler ones, such as planes or quadratics, and this is a nontrivial task in itself for surfaces. If these boundary surfaces are instead known via given parameterisations, then it becomes advantageous not to use the BEM but instead make use of the parameterisations and to incorporate further transformations that can render faster and more accurate numerical results.

We mention that methods for Cauchy problems based on the BEM have been developed (mainly for bounded planar regions), see for example, [26, 27] (the authors of these works have plenty more results on techniques based on the BEM for ill-posed problems). Recently, meshless techniques have been advocated, see the survey [21]. Methods based on finite differences and finite elements have also been developed, see, for example, [3, 13]. Thus, most of the standard numerical methods for partial differential equations can be applied for solving the Cauchy problem but they tend to be cumbersome to adjust to, for example, unbounded domains or are not that efficient for three-dimensional regions.

For the outline of this work, in Section 2, we present the general approach for the Cauchy problem (1.1), (1.2) of representing the solution as a single-layer potential. The system obtained by matching the representation against the given data is derived together with properties in terms of uniqueness of a solution. In Section 3, we show how to discretise the obtained system for two-dimensional solution domains. In Section 4, we give the corresponding details for the discretisation in three-dimensional domains based on Weinert's method [29]. A note is included at the end of Section 4, discussing how to adjust the approach for various other types of domains such as unbounded as well as simply connected ones; references are given where further details can be found. In the final section, Section 5, we give some numerical results for two- and three-dimensional solution domains.

**2. A direct integral equation approach with Tikhonov regularization for the Cauchy problem (1.1), (1.2).** The solution to the Cauchy problem (1.1), (1.2) is sought in the form of a sum of single-layer potentials over the two boundary surfaces,

$$u(x) = \int\limits_{\Gamma_1} \phi_1(y)\Phi(x,y)\,ds(y) + \int\limits_{\Gamma_2} \phi_2(y)\Phi(x,y)\,ds(y), \quad x \in D, \tag{2.1}$$

where $\phi_1 \in C(\Gamma_1)$ and $\phi_2 \in C(\Gamma_2)$ are unknown densities (we enforce to have continuous densities for simplicity in terms of interpreting the boundary integrals), and

$$\Phi(x,y) = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{|x-y|}, & d=2, \\ \dfrac{1}{4\pi} \dfrac{1}{|x-y|}, & d=3, \end{cases} \tag{2.2}$$

is the fundamental solution of the Laplace equation in $\mathbb{R}^d$.

Using the classical jump properties of the single-layer potential and its normal derivative, the representation (2.1) satisfies (1.1), (1.2) provided that the two densities form a solution pair of the following system:

$$\begin{gathered} \int_{\Gamma_1} \phi_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_2} \phi_2(y)\Phi(x,y)\,ds(y) = f(x), \quad x \in \Gamma_2, \\ \frac{1}{2}\phi_2(x) + \int_{\Gamma_1} \phi_1(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) + \int_{\Gamma_2} \phi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) = g(x), \quad x \in \Gamma_2. \end{gathered} \tag{2.3}$$

For the moment, we take for granted that it exists a unique pair of densities $\phi_1$ and $\phi_2$ to this equation. Using these densities, the sought Cauchy data on the interior boundary $\Gamma_1$ can be found from

$$\begin{gathered} u(x) = \int_{\Gamma_1} \phi_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_2} \phi_2(y)\Phi(x,y)\,ds(y), \quad x \in \Gamma_1, \\ \frac{\partial u}{\partial\nu}(x) = -\frac{1}{2}\phi_1(x) + \int_{\Gamma_1} \phi_1(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) + \int_{\Gamma_2} \phi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y), \quad x \in \Gamma_1. \end{gathered} \tag{2.4}$$

Before continuing, we briefly mention that there exists other ways of representing the solution to the Cauchy problem, which also will render a system of boundary integral equations to solve for a pair of densities. For example, one can employ Green's representation formula for harmonic functions

$$u(x) = \int_{\Gamma_1} \left( \psi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} - \psi_1(y)\Phi(x,y) \right) ds(y) + Q(x), \quad x \in D,$$

with $\psi_1(x) = \dfrac{\partial u}{\partial\nu}(x)$ and $\psi_2(x) = u(x)$ for $x \in \Gamma_1$, and where

$$Q(x) = \int_{\Gamma_2} \left( g(y)\Phi(x,y) - f(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)} \right) ds(y).$$

Using jump properties of single- and double-layer potentials, we obtain the system

$$-\frac{1}{2}\psi_2(x) - \int_{\Gamma_1} \psi_1(y)\Phi(x,y)\,ds(y) + \int_{\Gamma_1} \psi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\,ds(y) = -Q(x), \quad x \in \Gamma_1,$$

$$-\frac{1}{2}\psi_1(x) - \int_{\Gamma_1} \psi_1(y)\frac{\partial\Phi(x,y)}{\partial\nu(x)}\,ds(y) + \frac{\partial}{\partial\nu(x)}\int_{\Gamma_1} \psi_2(y)\frac{\partial\Phi(x,y)}{\partial\nu(y)}\,ds(y) = -\frac{\partial Q}{\partial\nu}(x), \quad x \in \Gamma_1.$$

Clearly, the sought after Cauchy data on $\Gamma_1$ is the solution pair $(\psi_1, \psi_2)$. This approach however suffers from the drawback of generating a system with a hypersingular kernel and with a complicated right-hand side.

In the present work, we therefore only concentrate on the potential approach (2.1). We shall make use of the following boundary integral operators:

$$(S_{ij}\mu)(x) = \int\limits_{\Gamma_j} \mu(y)\Phi(x,y)\,ds(y), \quad x \in \Gamma_i, \tag{2.5}$$

and

$$(D_{ij}\mu)(x) = \int\limits_{\Gamma_j} \mu(y)\frac{\partial \Phi(x,y)}{\partial \nu(x)}\,ds(y), \quad x \in \Gamma_j, \tag{2.6}$$

with $i, j = 1, 2$.

We can rewrite the system (2.3) in the operator form

$$\begin{aligned} &S_{21}\phi_1 + S_{22}\phi_2 = f \quad \text{on} \quad \Gamma_2, \\ &\left(\frac{1}{2}I + D_{22}\right)\phi_2 + D_{21}\phi_1 = g \quad \text{on} \quad \Gamma_2. \end{aligned} \tag{2.7}$$

To investigate solutions to (2.7), we use $A : L^2(\Gamma_1) \times L^2(\Gamma_2) \to L^2(\Gamma_2) \times L^2(\Gamma_2)$, where

$$A = \begin{pmatrix} S_{21} & S_{22} \\ D_{21} & \frac{1}{2}I + D_{22} \end{pmatrix}. \tag{2.8}$$

The system (2.7) corresponds to the ill-posed Cauchy problem (1.1), (1.2), and therefore it will inherit the ill-posedness. Thus, rather than showing well-posedness it is important that the operator $A$ is such that Tikhonov regularization can be applied for the stable solution. Recalling the steps in [6] (Theorem 4.1) with a straightforward extension to the three-dimensional setting, the following result can be established:

**Theorem 1.** *The operator $A$ defined in* (2.8) *is injective and has a dense range.*

We can then write our inverse problem as an operator equation

$$A\phi = F \tag{2.9}$$

to be solved for $\phi = (\phi_1, \phi_2)$ given the data $F = (f, g)$. To restore stability, Tikhonov regularization shall be employed, that is we solve the regularized system

$$(A^*A + \alpha I)\phi_\alpha = A^*F,$$

where $A^*$ is the adjoint operator to $A$, and $\alpha > 0$ is a regularization parameter to be chosen appropriately.

We note that other spaces can be considered for the operator $A$ in (2.8). The $L^2$-setting is rather natural from a practical point of view, since data is typically contaminated with noise destroying any smoothness assumption on the data. Moreover, the element (2.1) with square integrable densities has traces in $H^1(\Gamma)$ and $L^2(\Gamma)$ for the function values and normal derivative, respectively. It is typically with such data that theoretical properties of the Cauchy problem has been derived; for example interior regularity and local estimates, see [18] (Theorem 3.3.1). It is possible though to instead consider properties of the operator $A$ having in mind the natural Sobolev trace spaces $H^{1/2}(\Gamma_2)$ and $H^{-1/2}(\Gamma_2)$ for the Cauchy data. An analysis in this direction for the Helmholtz equation is given in [5] for densities in $H^{-1/2}(\Gamma_j),\ j = 1, 2$. In this case, for noisy data, smoothing is in general required to have the given data belong to the required spaces.

In the case of noisy data, we solve

$$(A^*A + \alpha I)\phi_\alpha^\delta = A^* F^\delta.$$

Using the properties of the operator $A$ given in Theorem 1, it is known, see Theorem 16.13 [25], that one can devise a rule for choosing the regularizing parameter $\alpha$ such that $\phi_\alpha^\delta$ tends to the solution of (2.9), when the noise level $\delta$ tends to zero. Employing the densities constituting $\phi_\alpha^\delta$ in the representation (2.1), we obtain an element $u_\alpha^\delta$ in $H^1(D)$. Applying estimates for the single-layer operator in terms of the densities, see [28] (Theorem 7.1), we conclude that $u_\alpha^\delta$ tends to $u$, with $u$ obtained from the densities in (2.9) and the representation (2.1). This in turn via the trace theorem implies that we also obtain a sequence on $\Gamma_1$ converging with decreasing noise level to the sought after Cauchy data. In fact, since the difference $u - u_\alpha^\delta$ is a harmonic function, local estimates can be applied to conclude that $u - u_\alpha^\delta$ converges in $H^{\ell+1}(D')$, for $\ell = 1, 2, \ldots$, and $D'$ a sufficiently smooth domain with $\overline{D'} \subset D$.

**3. Full discretisation of (2.7) for two-dimensional domains.** In the case of a planar solution domain the two boundary parts $\Gamma_1$ and $\Gamma_2$ are simple smooth closed curves, which are assumed given by the parametric representation

$$\Gamma_i := \big\{p_i(t) = (x_{i1}(t), x_{i2}(t)),\ t \in [0, 2\pi]\big\},$$

where $p_i\colon \mathbb{R} \to \mathbb{R}^2$ is $2\pi$-periodic with $|p_i'(t)| > 0$ for all $t \in [0, 2\pi],\ p_i \in C^2([0, 2\pi] \times [0, 2\pi])$, $i = 1, 2$.

Using these parametric representations in (2.5) and (2.6), we have the parameterised integral operators

$$(\widetilde{S}_{ij}\psi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \psi(\tau) H_{ij}(t, \tau)\, d\tau$$

and

$$(\widetilde{D}_{ij}\psi)(t) = \frac{1}{2\pi}\int_0^{2\pi} \psi(\tau) K_{ij}(t, \tau)\, d\tau,$$

where $\psi(t) := \mu(p_i(t))\, |p_i'(t)|$, for $t \in [0, 2\pi]$ and $i, j = 1, 2$. Recalling the fundamental solution of the Laplace equation for planar domains, see (2.2), the kernels can be written

$$H_{ij}(t, \tau) = \ln \frac{1}{|p_i(t) - p_j(\tau)|}, \qquad t \neq \tau \quad \text{for} \quad i = j$$

and

$$K_{ij}(t, \tau) = \frac{(p_j(\tau) - p_i(t))\nu(p_i(t))}{|p_i(t) - p_j(\tau)|^2}, \quad t \neq \tau \quad \text{for} \quad i = j.$$

The diagonal values of the functions $K_{ij}$ when $i = j$ are

$$K_{ii}(t,t) = \frac{p_i''(t)\nu(p_i(t))}{2|p_i'(t)|^2}.$$

Elementary calculations reveal that the function $H_{ii}$ can be decomposed as

$$H_{ii}(t,\tau) = -\frac{1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) + \widetilde{H}_{ii}(t,\tau), \quad t \neq \tau,$$

with

$$\widetilde{H}_{ii}(t,\tau) = \frac{1}{2}\ln\left(\frac{(4/e)\sin^2((t-\tau)/2)}{|p_i(t)-p_i(\tau)|^2}\right), \quad t \neq \tau,$$

having the diagonal term

$$\widetilde{H}_{ii}(t,t) = \frac{1}{2}\ln\left(\frac{1}{e|p_i'(t)|^2}\right).$$

Using the above parameterisations of the boundary curves together with the derived expressions for the kernels in (2.3), we have a parameterised system of integral equations

$$\begin{gathered}\frac{1}{2\pi}\int\limits_0^{2\pi}\left\{H_{12}(t,\tau)\psi_1(\tau) + \left[-\frac{1}{2}\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right) + \tilde{H}_{22}(t,\tau)\right]\psi_2(\tau)\right\}d\tau = f(p_2(t)),\\ \frac{1}{2\pi}\int\limits_0^{2\pi}\Big\{K_{12}(t,\tau)\psi_1(\tau) + K_{22}(t,\tau)\psi_2(\tau)\Big\}d\tau + \frac{\psi_2(t)}{2|p_2'(t)|} = g(p_2(t))\end{gathered} \tag{3.1}$$

to be solved for $\psi_i(t) = \phi_i(p_i(t))|p_i'(t)|$, $i = 1, 2$, with $t \in [0, 2\pi]$. Finding these densities, we can use (2.4) to find the requested Cauchy data on $\Gamma_1$.

As explained at the end of the previous section, Tikhonov regularization is applied when solving (3.1) and this means that that the analogous transformations for the corresponding adjoint operators to (2.5) and (2.6) are needed. Going down that route will render a method requiring additional computational cost. This can be avoided by the simplistic but common approach of first discretising the parameterised integral equations (3.1) and then apply regularization. Numerically, the obtained results with this latter approach tend to match the more computationally demanding strategy. It can be made rigorous by showing an error estimate between the operator $A$ and the discretised one, and this can be done following ideas for a similar error estimate for the Symm's integral equation, see [22] (Section 3.4.2). One can then devise a parameter choice rule for Tikhonov regularization of the discretised operator based on the fineness of the discretisation and the error level.

For the discretisation of the involved integrals, we consider two quadrature rules both constructed via trigonometric interpolation with $2n$ equidistant nodal points

$$t_j := \frac{j\pi}{n}, \quad j = 0, \ldots, 2n-1. \tag{3.2}$$

The two quadrature rules are

$$\frac{1}{2\pi}\int\limits_0^{2\pi} f(\tau)\ln\left(\frac{4}{e}\sin^2\frac{t-\tau}{2}\right)d\tau \approx \sum_{k=0}^{2n-1} R_k(t)\, f(t_k) \tag{3.3}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(\tau)\,d\tau \approx \frac{1}{2n}\sum_{k=0}^{2n-1} f(t_k) \tag{3.4}$$

with explicit expressions for the weight functions given in [24].

Using the Nyström method with quadratures (3.3) and (3.4) in the integral equations (3.1), we obtain the following system of linear equations:

$$\begin{aligned} &\frac{1}{2n}\sum_{j=0}^{2n-1} \overline{\psi}_{1,j} H_{12}(t_i,t_j) + \sum_{j=0}^{2n-1} \overline{\psi}_{2,j}\left[\frac{1}{2n}\tilde{H}_{22}(t_i,t_j) - \frac{1}{2}R_j(t_i)\right] = \overline{f}_i,\\ &\frac{1}{2n}\sum_{j=0}^{2n-1} \overline{\psi}_{1,j} K_{12}(t_i,t_j) + \frac{1}{2n}\sum_{j=0}^{2n-1} \overline{\psi}_{2,j} K_{22}(t_i,t_j) + \frac{\overline{\psi}_{2,i}}{2|p_2'(t_i)|} = \overline{g}_i \end{aligned} \tag{3.5}$$

to be solved for $\overline{\psi}_{1,j} \approx \psi_1(t_j)$ and $\overline{\psi}_{2,j} \approx \psi_2(t_j)$ with the right-hand side $\overline{f}_i = f(p_2(t_i))$ and $\overline{g}_i = g(p_2(t_i))$, for $i = 0, \ldots, 2n-1$. Rearranging (3.5), we arrive at the following system of linear algebraic equations:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{3.6}$$

where the matrix $\mathbf{A} \in \mathbb{R}^{4n\times 4n}$, and $\mathbf{x} = [\overline{\psi}_1, \overline{\psi}_2]^\top$ and $\mathbf{b} = [\overline{f}, \overline{g}]^\top$. The matrix $\mathbf{A}$ will have a large condition number due to the ill-posedness of the Cauchy problem, and to obtain a stable smooth solution regularization of this system is necessary.

As explained at the end of the previous section, to solve (3.6) in a stable way, we employ Tikhonov regularization; the standard version of Tikhonov regularization amounts to solve the minimization problem

$$\min_x \left\{\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2 + \lambda\|\mathbf{x}\|_2^2\right\}, \tag{3.7}$$

where $\lambda \in \mathbb{R}$ is a regularization parameter that has to be appropriately chosen. The Tikhonov regularized solution $x_\lambda$ in (3.7) is equivalently given as the solution to the regularized normal equations

$$(\mathbf{A}^*\mathbf{A} + \lambda I)\mathbf{x}_\lambda = \mathbf{A}^*\mathbf{b},$$

where $\mathbf{A}^*$ is the transpose of the matrix $\mathbf{A}$. Although there are optimal choices for the regularization parameter (the discrepancy principle), it is often simpler and faster to use a heuristic choice such as the L-curve rule [14, 15].

Once the discrete (and regularized) densities $\overline{\psi}_1$ and $\overline{\psi}_2$ have been constructed, the corresponding discrete approximations for the Cauchy data on the interior boundary curve $\Gamma_1$ are obtained from (2.4) using the quadratures

$$u(p_1(t_i)) \approx \sum_{j=0}^{2n-1}\left\{\left[\frac{1}{2n}H_{11}(t_i,t_j) - \frac{1}{2}R_j(t_i)\right]\overline{\psi}_{1,j} + \frac{1}{2n}H_{21}(t_i,t_j)\overline{\psi}_{2,j}\right\}$$

and

$$\frac{\partial u}{\partial \nu}(p_1(t_i)) \approx -\frac{1}{2|p_1'(t_i)|}\overline{\psi}_{1,i} + \frac{1}{2n}\sum_{j=0}^{2n-1}\left\{K_{11}(t_i,t_j)\overline{\psi}_{1,j} + K_{21}(t_i,t_j)\overline{\psi}_{2,j}\right\}.$$

**4. Full discretisation of (2.7) for three-dimensional domains.** For planar domains certain parameterisations are assumed for the boundary parts. The analogous basic assumption for three-dimensional domains is that the closed boundary surfaces $\Gamma_1$ and $\Gamma_2$ can each be smoothly and bijectively mapped onto the unit sphere $\mathbb{S}^2$. This means that there shall exist one-to-one mappings

$$q_1 : \mathbb{S}^2 \to \Gamma_1 \qquad \text{and} \qquad q_2 : \mathbb{S}^2 \to \Gamma_2$$

with smoothly varying Jacobians $J_{q_1}$ and $J_{q_2}$, respectively.

We can then rewrite the integral equations (2.7) over the unit sphere and obtain

$$\begin{aligned} &\widetilde{S}_{21}\psi_1 + \widetilde{S}_{22}\psi_2 = \widetilde{f} \quad \text{on} \quad \mathbb{S}^2, \\ &\left(\frac{1}{2}I + \widetilde{D}_{22}\right)\psi_2 + \widetilde{D}_{21}\psi_1 = \widetilde{g} \quad \text{on} \quad \mathbb{S}^2 \end{aligned} \tag{4.1}$$

with the densities $\psi_\ell(\widehat{x}) = \phi_\ell(q_\ell(\widehat{x}))$, $\ell = 1, 2$, to be determined from the data $\widetilde{f}(\widehat{x}) = f(q_2(\widehat{x}))$, and $\widetilde{g}(\widehat{x}) = g(q_2(\widehat{x}))$ for $\widehat{x} \in \mathbb{S}^2$. The integral operators involved are parameterisations of (2.5), (2.6) over the unit sphere and given by

$$\left(\widetilde{S}_{\ell j}\phi\right)(\widehat{x}) = \int_{\mathbb{S}^2} \phi(\widehat{y}) L_{\ell j}(\widehat{x}, \widehat{y})\, ds(\hat{y}), \quad \widehat{x} \in \mathbb{S}^2, \tag{4.2}$$

and

$$\left(\widetilde{D}_{\ell j}\phi\right)(\widehat{x}) = \int_{\mathbb{S}^2} \phi(\widehat{y}) M_{\ell j}(\widehat{x}, \widehat{y})\, ds(\hat{y}), \quad \widehat{x} \in \mathbb{S}^2. \tag{4.3}$$

Recalling the fundamental solution to the Laplace equation in $\mathbb{R}^3$, see (2.2), the kernels are found to be

$$L_{\ell j}(\widehat{x}, \widehat{y}) = \begin{cases} \Phi(q_\ell(\widehat{x}), q_j(\widehat{y})) J_{q_j}(\widehat{y}), & \ell \neq j, \\ \dfrac{R_\ell(\widehat{x}, \widehat{y})}{|\widehat{x} - \widehat{y}|}, & \ell = j, \end{cases} \tag{4.4}$$

and

$$M_{\ell j}(\widehat{x}, \widehat{y}) = \begin{cases} -\dfrac{\langle q_\ell(\widehat{x}) - q_j(\widehat{y}), \nu(q_\ell(\widehat{x}))\rangle}{4\pi\, |q_\ell(\widehat{x}) - q_j(\widehat{y})|^3} J_{q_j}(\widehat{y}), & \ell \neq j, \\ \dfrac{\widetilde{R}_\ell(\widehat{x}, \widehat{y})}{|\widehat{x} - \widehat{y}|}, & \ell = j, \end{cases} \tag{4.5}$$

where

$$R_\ell(\widehat{x}, \widehat{y}) = J_{q_\ell}(\widehat{y}) \begin{cases} \dfrac{1}{4\pi} \dfrac{|\widehat{x} - \widehat{y}|}{|q_\ell(\widehat{x}) - q_\ell(\widehat{y})|}, & \widehat{x} \neq \widehat{y}, \\ \dfrac{1}{4\pi} \dfrac{1}{J_{q_\ell}(\widehat{x})}, & \widehat{x} = \widehat{y}, \end{cases} \tag{4.6}$$

and

$$\widetilde{R}_\ell(\widehat{x}, \widehat{y}) = -R_\ell(\widehat{x}, \widehat{y}) \begin{cases} \dfrac{(q_\ell(\widehat{x}) - q_\ell(\widehat{y}), \nu(q_\ell(\widehat{x})))}{|q_\ell(\widehat{x}) - q_\ell(\widehat{y})|^2}, & \widehat{x} \neq \widehat{y}, \\ -\dfrac{2\sum_{j=1}^3 q'_{\ell j}(\widehat{x})\nu_j(\widehat{x}) - \sum_{j=1}^3 q''_{\ell j}(\widehat{x})\nu_j(\widehat{x})}{2J_{q_\ell}^2(\widehat{x})}, & \widehat{x} = \widehat{y}. \end{cases} \tag{4.7}$$

We used that

$$\lim_{\widehat{y}\to\widehat{x}} \frac{|\widehat{x}-\widehat{y}|}{|q_\ell(\widehat{x})-q_\ell(\widehat{y})|} = \frac{1}{J_{q_\ell}(\widehat{x})}.$$

Points on the unit sphere are given using the standard spherical coordinates,

$$\widehat{x} = p(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \qquad \text{with} \quad \theta\in[0,\pi], \quad \varphi\in[0,2\pi]. \tag{4.8}$$

The integral operators $\widetilde{S}_{\ell\ell}$ and $\widetilde{D}_{\ell\ell}$, $\ell = 1,2$, are both weakly singular. For approximation and numerical implementation of these, we make the singularities explicit. In fact, we can make a transformation and move the singularities to appear at the north pole $\widehat{n} = (0,0,1)$ of the unit sphere. Define the orthogonal transformations for $\psi \in \mathbb{R}$ by

$$D_F(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad D_T(\psi) = \begin{pmatrix} \cos\psi & 0 & -\sin\psi \\ 0 & 1 & 0 \\ \sin\psi & 0 & \cos\psi \end{pmatrix}.$$

The linear orthogonal transformation

$$T_{\widehat{x}} = D_F(\varphi)D_T(\theta)D_F(-\varphi) \tag{4.9}$$

satisfies that $T_{\widehat{x}}\widehat{x} = \widehat{n}$ for every $\widehat{x}\in\mathbb{S}^2$. Moreover, $|\widehat{x}-\widehat{y}| = |T_{\widehat{x}}^{-1}(\widehat{n}-\widehat{\eta})| = |\widehat{n}-\widehat{\eta}|$, and $\widehat{\eta} = T_{\widehat{x}}\widehat{y}$.

Using this transformation in the operators $\widetilde{S}_{\ell\ell}$ and $\widetilde{D}_{\ell\ell}$, $\ell = 1,2$, defined in (4.2), (4.3), these operators are transformed into

$$(\widetilde{S}_{\ell\ell}\phi)(\widehat{x}) = \int\limits_{\mathbb{S}^2} \phi(T_{\widehat{x}}^{-1}\widehat{\eta}) \frac{R_\ell(\widehat{x}, T_{\widehat{x}}^{-1}\widehat{\eta})}{|\widehat{n}-\widehat{\eta}|}\, ds(\widehat{\eta}), \quad \widehat{x}\in\mathbb{S}^2,$$

and

$$(\widetilde{D}_{\ell\ell}\phi)(\widehat{x}) = \int\limits_{\mathbb{S}^2} \phi(T_{\widehat{x}}^{-1}\widehat{\eta}) \frac{\widetilde{R}_\ell(\widehat{x}, T_{\widehat{x}}^{-1}\widehat{\eta})}{|\widehat{n}-\widehat{\eta}|}\, ds(\widehat{\eta}), \quad \widehat{x}\in\mathbb{S}^2,$$

for $\ell = 1,2$.

The Cauchy data on the interior surface $\Gamma_1$ can, using the representation (2.4), be written over the unit sphere to get

$$\begin{aligned} u &= \widetilde{S}_{11}\psi_1 + \widetilde{S}_{12}\psi_2 \quad \text{on} \quad \mathbb{S}^2, \\ \frac{\partial u}{\partial\nu} &= \left(-\frac{1}{2}I + \widetilde{D}_{11}\right)\psi_1 + \widetilde{D}_{12}\psi_2 \quad \text{on} \quad \mathbb{S}^2 \end{aligned} \tag{4.10}$$

with the operators defined above.

To discretise (4.1), the following quadrature is used for integrals over the unit sphere having a continuous integrand

$$\int\limits_{\mathbb{S}^2} f(\widehat{y})\, ds(\widehat{y}) \approx \sum_{\rho'=0}^{2n'+1}\sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'}\tilde{a}_{s'} f(\widehat{y}_{s'\rho'}). \tag{4.11}$$

Here

$$\widehat{y}_{s'\rho'} = p(\theta_{s'}, \varphi_{\rho'}), \quad \text{with} \quad \varphi_{\rho'} = \rho'\pi/(n'+1) \qquad \text{and} \qquad \theta_{s'} = \arccos z_{s'} \tag{4.12}$$

and $z_{s'}$ being the zeros of the Legendre polynomials $P_{n'+1}$ and with $p$ as in (4.8) expressing an element on the unit sphere. For the coefficients,

$$\tilde{a}_{s'} = \frac{2(1 - z_{s'}^2)}{((n'+1)P_{n'}(z_{s'}))^2}$$

and

$$\tilde{\mu}_{\rho'} = \frac{\pi}{n'+1}.$$

In the case when the integrand has a weak singularity, we use the quadrature rule

$$\int\limits_{\mathbb{S}^2} \frac{f(\widehat{y})}{|\widehat{n} - \widehat{y}|}\, ds(\widehat{y}) \approx \sum_{\rho'=0}^{2n'+1} \sum_{s'=1}^{n'+1} \tilde{\mu}_{\rho'} \tilde{b}_{s'} f(\widehat{y}_{s'\rho'}) \tag{4.13}$$

with weights

$$\tilde{b}_{s'} = \tilde{a}_{s'} \sum_{i=0}^{n'} P_i(z_{s'}).$$

The above quadratures are both obtained from approximating the regular part of the integrand using spherical harmonics and then performing exact integration. According to [16, 29], the chosen quadrature rules have superalgebraic convergence order.

The ill-posed system of integral equations (4.1) is then discretised using a projection Galerkin method. In the previous section, trigonometric polynomials were used. To follow the similar idea, the analogue is to invoke spherical polynomials. We shall therefore search for the densities in terms of spherical polynomials of degree $n$. An orthonormal basis for the $(n+1)^2$-dimensional space of such spherical polynomials are given by the spherical harmonics.

Thus, we write the approximation of the densities as

$$\psi_\ell(\widehat{x}) \approx \tilde{\psi}_\ell(\hat{x}) = \sum_{k=0}^{n} \sum_{m=-k}^{k} \psi_{k,m}^\ell Y_{k,m}^R(\widehat{x}) \qquad \text{for} \quad \widehat{x} \in \mathbb{S}^2, \quad \ell = 1, 2, \tag{4.14}$$

where $\psi_{k,m}^\ell$ are unknown coefficients. Here, the real-valued spherical harmonics are

$$Y_{k,m}^R = \begin{cases} \operatorname{Im} Y_{k,|m|}, & 0 < m \le k, \\ \operatorname{Re} Y_{k,|m|}, & -k \le m \le 0, \end{cases} \tag{4.15}$$

with $Y_{k,m}(\theta, \varphi) = c_k^m P_k^{|m|}(\cos\theta) e^{im\varphi}$ the classical (complex-valued) spherical harmonic functions, $P_k^m$ the Legendre functions and

$$c_k^m = (-1)^{\frac{|m|-m}{2}} \sqrt{\frac{2k+1}{4\pi} \frac{(k-|m|)!}{(k+|m|)!}}, \qquad m = -k, \ldots, k, \quad k = 0, 1, \ldots.$$

Define a discrete inner product on the space of spherical polynomials of degree $n$ by

$$(v, w) = \sum_{\rho=0}^{2n+1} \sum_{s=1}^{n+1} \mu_\rho a_s v(\widehat{y}_{s\rho}) w(\widehat{y}_{s\rho}). \tag{4.16}$$

The coefficients $a_s$ and $\mu_\rho$ are generated as in (4.11) but with the integer $n'$ replaced by a possibly different integer $n$. The expression (4.16) is indeed an inner product on the space of spherical polynomials of degree $n$ and this is due to the fact that (4.11) is exact for spherical polynomials of degree $2n$.

We then employ the inner product (4.16) to the system of integral equations (4.1). This means first discretising (4.1) by replacing the densities by (4.14) and approximating the integrals via (4.11) and (4.13), and then employing the discrete inner product to identify the coefficients needed in (4.14) (multiplying with the basis elements $Y^R_{k,m}$). This strategy leads to the linear system

$$\begin{gathered}\sum_{k=0}^{n}\sum_{m=-k}^{k}\left(\psi^1_{k,m}A^{21}_{kk'mm'}+\psi^2_{k,m}A^{22}_{kk'mm'}\right)=\sum_{\rho=0}^{2n+1}\sum_{s=1}^{n+1}\mu_\rho a_s\widetilde{f}(\widehat{x}_{s\rho})Y^R_{k,m}(\widehat{x}_{s\rho}),\\ \sum_{k=0}^{n}\sum_{m=-k}^{k}\left(\psi^1_{k,m}\widetilde{A}^{21}_{kk'mm'}+\psi^2_{k,m}\widetilde{A}^{22}_{kk'mm'}\right)=\sum_{\rho=0}^{2n+1}\sum_{s=1}^{n+1}\mu_\rho a_s\widetilde{g}(\widehat{x}_{s\rho})Y^R_{k,m}(\widehat{x}_{s\rho}),\end{gathered}\tag{4.17}$$

where $k'=0,\ldots,n$, $m=-k',\ldots,k'$.

To give expressions for the coefficients in this linear system, let

$$V^j_{s,k,m}(\widehat{x}_{s\rho},\widehat{y}_{s'\rho'},\widehat{y}^{s'\rho'}_{s\rho})=\begin{cases}\widetilde{a}'_sL_{2j}(\widehat{x}_{s\rho},\widehat{y}_{s'\rho'})Y^R_{k,m}(\widehat{y}_{s'\rho'}), & j=1,\\ \widetilde{b}'_sR_2(\widehat{x}_{s\rho},\widehat{y}^{s'\rho'}_{s\rho})Y^R_{k,m}(\widehat{y}^{s'\rho'}_{s\rho}), & j=2,\end{cases}$$

and

$$W^j_{s,k,m}(\widehat{x}_{s\rho},\widehat{y}_{s'\rho'},\widehat{y}^{s'\rho'}_{s\rho})=\begin{cases}\widetilde{a}'_sL_{2j}(\widehat{x}_{s\rho},\widehat{y}_{s'\rho'})Y^R_{k,m}(\widehat{y}_{s'\rho'}), & j=1,\\ \widetilde{b}'_sR_2(\widehat{x}_{s\rho},\widehat{y}^{s'\rho'}_{s\rho})Y^R_{k,m}(\widehat{y}^{s'\rho'}_{s\rho})+\dfrac{1}{2}Y^R_{k,m}(\widehat{x}_{s\rho}), & j=2.\end{cases}$$

Here the kernels are given by (4.4)–(4.6), $\widehat{x}_{s\rho}$ and $\widehat{y}_{s'\rho'}$ are points on the unit sphere generated as in (4.12), and

$$\widehat{y}^{s'\rho'}_{sp}=T^{-1}_{\widehat{x}_{s\rho}}\widehat{y}_{s'\rho'}$$

with $T_{\widehat{x}_{s\rho}}$ given by (4.9).

Then the coefficients in (4.17) can be expressed as

$$A^{2j}_{kk'mm'}=\sum_{\rho=0}^{2n+1}\sum_{s=1}^{n+1}\sum_{\rho'=0}^{2n'+1}\sum_{s'=1}^{n'+1}\widetilde{\mu}_{\rho'}\mu_\rho a_sV^j_{s,k,m}(\widehat{x}_{s\rho},\widehat{y}_{s'\rho'},\widehat{y}^{s'\rho'}_{s\rho})Y^R_{k',m'}(\widehat{x}_{s\rho})$$

and

$$\widetilde{A}^{2j}_{kk'mm'}=\sum_{\rho=0}^{2n+1}\sum_{s=1}^{n+1}\sum_{\rho'=0}^{2n'+1}\sum_{s'=1}^{n'+1}\widetilde{\mu}_{\rho'}\mu_\rho a_sW^j_{s,k,m}(\widehat{x}_{s\rho},\widehat{y}_{s'\rho'},\widehat{y}^{s'\rho'}_{s\rho})Y^R_{k',m'}(\widehat{x}_{s\rho}),$$

where $j=1,2$.

Solving the linear system (4.17), using Tikhonov regularization, we obtain an approximation to the densities in (2.1) via the expression (4.14). Invoking the obtained approximation of the densities in the expression (4.10) together with quadrature, the sought values on $\Gamma_1$ are found to be

$$u_n(\widehat{x})=\sum_{s'=1}^{n'+1}\sum_{\rho'=0}^{2n'+1}\left(\tilde{b}_{s'}\tilde{\mu}_{\rho'}\tilde{\psi}_1(T^{-1}_{\widehat{x}}\widehat{y}_{s'\rho'})R_1(\widehat{x},T^{-1}_{\widehat{x}}\widehat{y}_{s'\rho'})+\tilde{a}_{s'}\tilde{\mu}_{\rho'}\tilde{\psi}_2(\widehat{y}_{s'\rho'})L_{12}(\widehat{x},\widehat{y}_{s'\rho'})\right)\tag{4.18}$$

and

$$\frac{\partial u_n}{\partial \nu}(\widehat{x}) = \sum_{s'=1}^{n'+1} \sum_{\rho'=0}^{2n'+1} \Big( \tilde{b}_{s'} \tilde{\mu}_{\rho'} \tilde{\psi}_1 (T_{\widehat{x}}^{-1} \widehat{y}_{s'\rho'}) \widetilde{R}_1 (\widehat{x}, T_{\widehat{x}}^{-1} \widehat{y}_{s'\rho'}) + \\ + \tilde{a}_{s'} \tilde{\mu}_{\rho'} \tilde{\psi}_2 (\widehat{y}_{s'\rho'}) M_{12} (\widehat{x}, \widehat{y}_{s'\rho'}) \Big) - \frac{1}{2} \tilde{\psi}_1 (\widehat{x}), \tag{4.19}$$

where $\widehat{x} \in \mathbb{S}^2$.

As was mentioned in Section 3 the proposed method is regularizing for planar domains. The same holds in the three-dimensional case; this follows by noting that also in the three-dimensional setting there are error estimates between the exact solution to (2.7) and its discretisation, and then using the similar arguments from Section 3. Rather than writing out and scrutinising the full details for showing that the method is regularizing, we give numerical evidence when performing the numerical experiments in Section 5 and present there a basic strategy for choosing the required parameters.

***4.1. Remark.*** The above introduced integral equation method for the elliptic Cauchy problem (1.1), (1.2) can be applied not only for doubly-connected domains. For example, the suggested approach was successfully applied in the following cases: a simply connected two-dimensional domain bounded by a simple closed curve, allowed to be nonsmooth in the sense of having corners, see [1], to a semiinfinite three-dimensional domain containing a cavity [10], and to a toroidal domain [2] (for such a domain the boundary surface is not simply connected).

In each of these cases, some adjustment is needed for the numerical implementation: in the case of a nonsmooth domain with corner points, we need to take into account the possible singularities that can be present at the corner points; for a semiinfinite domain Green's functions are incorporated to obtain integral equations over the cavity (which has a bounded boundary surface); for toroidal domains several transformations are used to take advantage of the symmetry of such a domain to obtain integral equations over a planar two-dimensional domain.

The proposed strategy can also be employed for well-posed problems for the Laplace equation such as mixed ones (which can be viewed as having incomplete Cauchy data). Having an efficient solver for mixed boundary value problems, it is possible to apply iterative regularizing methods for the Cauchy problem (1.1), (1.2), which at each iterative step solves such mixed problems. Methods in this direction, such as [23], which falls under the second category of regularizing methods mentioned in the introduction, and their numerical implementation for elliptic Cauchy problems in two- and three-dimensional regions can be found in for example [4, 8].

Based on the above research and results, we can conclude that our proposed approach is lightweight (in terms of computations) and flexible for elliptic Cauchy problems with boundary parts consisting of parameterised curves and surfaces isomorphic to the unit circle respectively the unit sphere.

We point out that the proposed approach can also be used for other Cauchy problems for elliptic equations or systems occurring in applications such as elasticity, fluid flow and wave propagation, for example, the Helmholtz equation, the Klein – Gordon equation and stationary Stokes system.

**5. Numerical experiments.** In this section, we illustrate by numerical examples the robustness of the proposed integral equation based method for the reconstruction of the harmonic function satisfying the Cauchy problem (1.1), (1.2), for both exact and noisy data. In the case of noisy data, random pointwise errors are added to the function values $f$ on the outer boundary with the percentage given in terms of the $L^2$-norm.

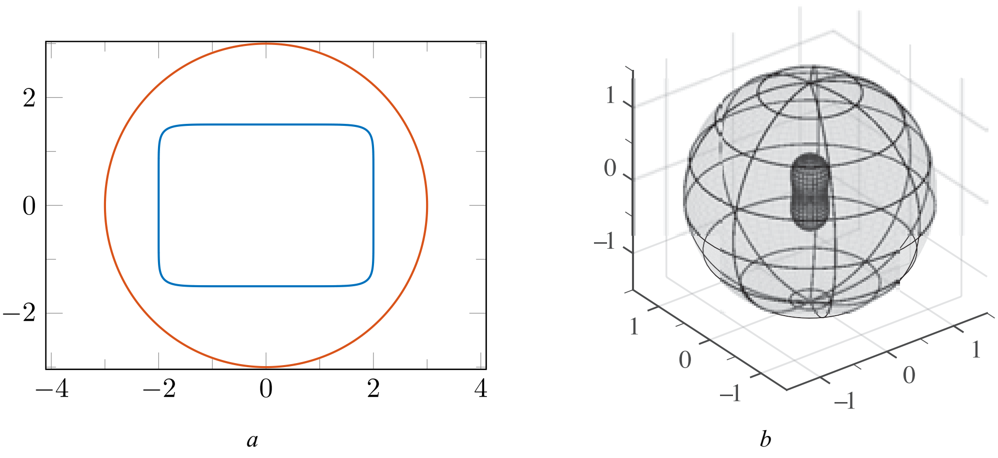


Fig. 2. The solution domain used in Example 1 (*a*) and Example 2 (*b*).

***Example* 1.** We use synthetic Cauchy data on the outer boundary part $_2$ constructed as follows: the Dirichlet boundary-value problem for the Laplace equation with boundary conditions $u = f_1$ on $_1$ and $u = f$ on $_2$ for given boundary functions $f_1$ and $f$ is numerically solved by the above boundary integral equation approach. Then, to generate the required trace of the normal derivative of the solution on $_2$ the following representation is employed:

$$\frac{u}{\phi}(x) = \frac{1}{2} \; _2(x) + \int_{1} \; _1(y) \frac{(x \; y)}{\phi(x)} \, ds(y) + \int_{2} \; _2(y) \frac{(x \; y)}{\phi(x)} \, ds(y) \qquad x \quad _2 \tag{5.1}$$

We consider the case when the outer boundary curve $_2$ is a circle of radius 3 and the interior boundary $_1$ is given by the parameterisation (see Fig. 2, *a*)

$$_1 = \quad p_1(t) = r(t)(\cos t \;\; \sin t) \qquad t \quad [0 \;\; 2\tau]$$

with the radial function

$$r(t) = \left\} \frac{1}{2} \cos t \right\{^{10} + \left\} \frac{2}{3} \sin t \right\{^{10} \quad ^{0\,1}$$

To generate the required synthetic Cauchy with the above explained strategy, the Dirichlet data functions are choosen as $f_1(x_1 \;\; x_2) = x_1^2$ on $_1$ and $f(x_1 \;\; x_2) = 1$ on $_2$ Then (5.1) is used to find the required normal derivative $g$

In the Cauchy problem (1.1), (1.2), it is data on the inner boundary $_1$ that has to be reconstructed from data on the outer boundary $_2$; the sought function value on $_1$ is thus the above chosen function $f_1$ restricted to $_1$ and this shall be compared with the one obtained numerically with the proposed procedure for the Cauchy problem.

The result of the reconstructions of the sought Cauchy data on the interior curve $_1$ are given in Fig. 3 and Fig. 4 for exact and 3% noisy data, respectively.

The discretization parameter (3.2) controlling the number of mesh points on each boundary curve was taken as $n = 64$ The value of the regularization parameter used, was chosen by trial and error; we calculated the numerical solutions for $= 10^{\;p}$ with $p = 1 \quad 15$ and use the value

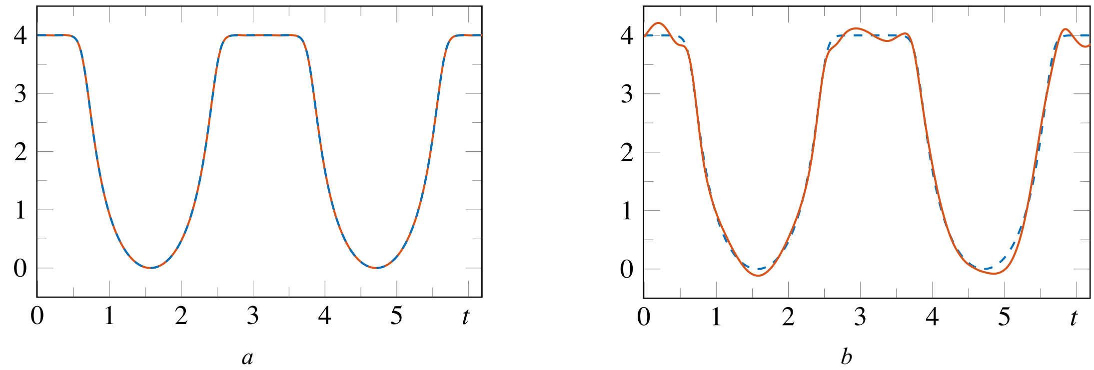


Fig. 3. Reconstruction (– – –) of the boundary function $u(p_1(t))$ (—–) on $\Gamma_1$ in Example 1 for exact data (*a*) and 3% noisy data (*b*).

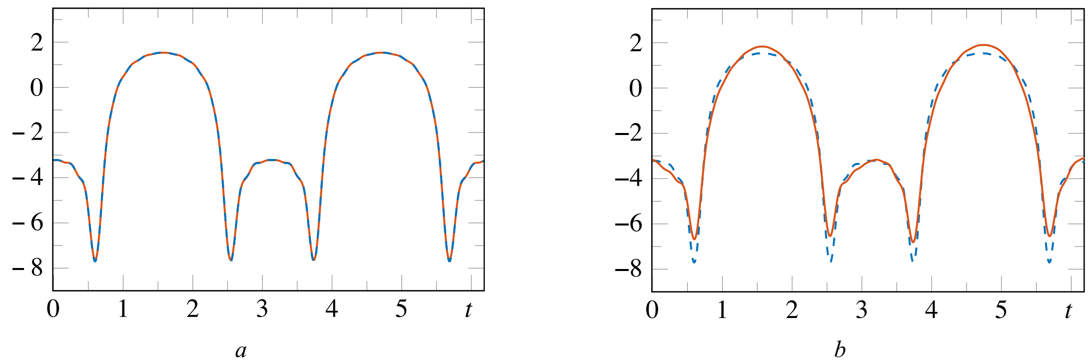


Fig. 4. Reconstruction (– – –) of the normal derivative $\dfrac{\partial u}{\partial \nu}(p_1(t))$ (—–) on $\Gamma_1$ in Example 1 for exact data (*a*) and 3% noisy data (*b*).

giving the most accurate result. Note here that we have compared $\alpha^*$ with the corresponding value for the regularization parameter obtained with the $L$-curve rule [15] and this value is near to $\alpha^*$.

It was mentioned at the end of Section 4 that the proposed method is regularizing. To exemplify this, in Table 1 are the discrete $L^2$-errors for the reconstructions of the solution and its normal derivative on the boundary $\Gamma_1$ as a function of the parameters $\alpha^*$, $\delta$ and $n$.

From Table 1 it can be seen that for a fixed error level the mesh size and regularizing parameter can be selected to give an approximation to the solution and its normal derivative, respectively, that decreases with the mesh size. However, for a fixed error a too fine mesh size will cause the error to start to increase since the linear system solved will be too ill-conditioned. The error in the reconstructions decrease as the error level decreases. One could elaborate further on this but we only note here that the basic strategy of, for a fixed error level, choosing a mesh size that does not render a too large condition number of the matrix corresponding to the linear system (4.17) and then choose $\alpha^*$ as above, will render approximations on $\Gamma_1$ having errors that decrease with $\alpha$ and which render pleasing numerical results.

As has been reported in the references mentioned in Section 4.1, one can change the solution domain and data, and as long as the distance between the boundary curves and the growth of the data are of the type as in the presented example, results of the similar kind are obtained. It is important to

Table 1. *Errors for Example* 1

| $\delta$ | $n$ | $\alpha^*$ | $e_2$ | $q_2$ |
|---|---|---|---|---|
| 0% | 16 | $10^{-4}$ | 0.07032 | 0.39314 |
| | 32 | $10^{-5}$ | 0.00281 | 0.02223 |
| | 64 | $10^{-8}$ | 0.00003 | 0.00029 |
| | 128 | $10^{-11}$ | $6E-7$ | 0.00002 |
| 3% | 16 | $10^{-3}$ | 0.15164 | 0.55168 |
| | 32 | $10^{-3}$ | 0.10286 | 0.33754 |
| | 64 | $10^{-4}$ | 0.07055 | 0.30161 |
| | 128 | $10^{-4}$ | 0.04409 | 0.20125 |
| 5% | 16 | $10^{-3}$ | 0.23139 | 0.75139 |
| | 32 | $10^{-3}$ | 0.18108 | 0.44474 |
| | 64 | $10^{-4}$ | 0.13227 | 0.48922 |
| | 128 | $10^{-4}$ | 0.07758 | 0.34147 |

have Cauchy data on a sufficiently large boundary part, and in general the derivative is reconstructed with less accuracy compared with the function values, as expected. Moreover, choosing a too fine mesh (a large number $n$) the numerical results will deteriorate since the condition number of the involved matrix of the linear system solved will have a too large condition number then reflecting the ill-posedness of the Cauchy problem. Thus, if the reader implements the procedure for a similar example, no surprises is to be expected but results of the same accuracy shall be obtained.

***Example* 2.** We also include an example in a three-dimensional domain. Let the solution domain $D$ be the region having the outer boundary surface being the sphere

$$\Gamma_2 = \left\{\xi_2(\theta,\varphi) = 1.5(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi\right\}$$

and the interior boundary surface being given by the parameterisation

$$\Gamma_1 = \left\{\xi_1(\theta,\varphi) = r(\theta,\varphi)(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\ 0 \le \theta \le \pi,\ 0 \le \varphi \le 2\pi\right\}$$

with the radial function

$$r(\theta,\varphi) = \frac{1}{2\sqrt{1+\sqrt{2}}}\sqrt{\cos 2\theta + \sqrt{2-\sin^2 2\theta}},$$

see Fig. 2, $b$. Both these surfaces satisfy, by construction, the assumption of the existence of a smooth one-to-one map to the unit sphere needed in the proposed method for the Cauchy problem (1.1), (1.2).

We choose as the exact solution of the Laplace equation the function

$$u_{ex}(x) = 2x_3^2 - 2x_2^2 + 3x_1$$

and this then generates the following Cauchy data:

$$f(x) = u_{ex}(x), \quad x \in \Gamma_2 \qquad \text{and} \qquad g(x) = \frac{\partial u_{ex}}{\partial \nu}(x), \quad x \in \Gamma_2.$$

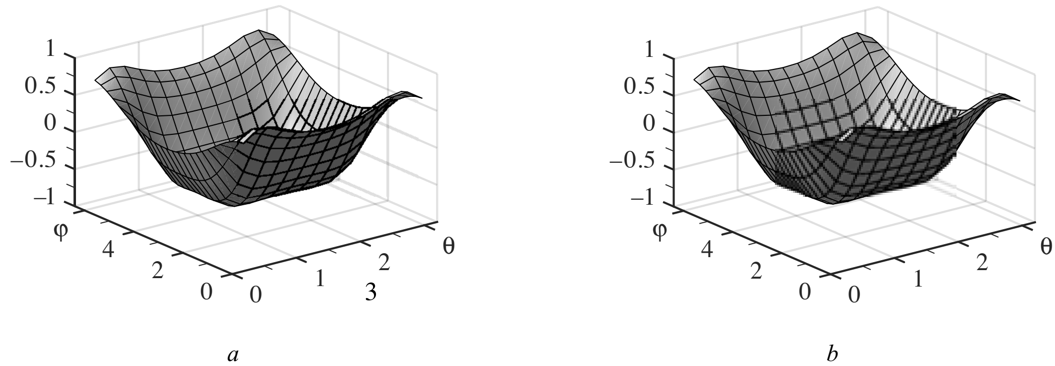


Fig. 5. The exact (*a*) values $u(\ _1(\quad))$ and numerical approximation (*b*) on the boundary surface $\ _1$ with 3% noise for Example 2.

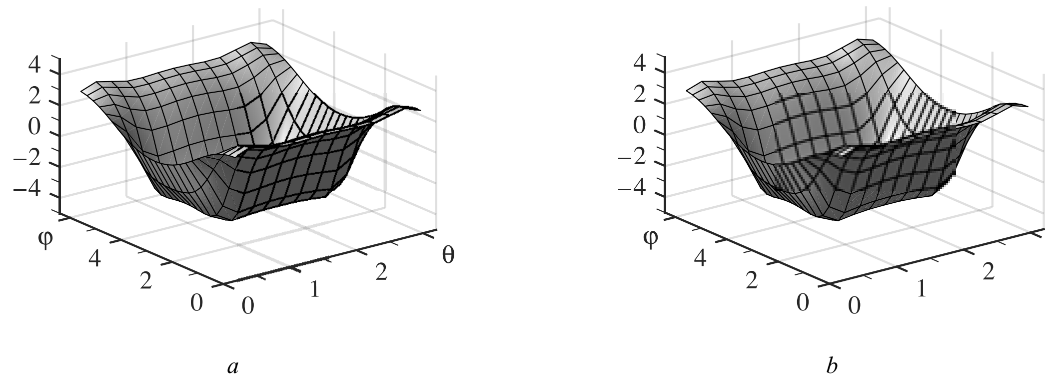


Fig. 6. The exact (*a*) normal derivative $\frac{u}{}(\ _1(\quad))$ and numerical approximation (*b*) on the boundary surface $\ _1$ with 3% noise for Example 2.

We recall that the integer $n$ is the degree of the spherical harmonic polynomials approximating the densities via (4.14), $n$ is the number of points chosen in the quadrature (cubature) rules (4.11) and (4.13); the numbers $n$ and $n$ enter into the approximation via (4.18) and (4.19). We give results when $n = n$ Given an integer $n$ the number of discretisation points on each surface is $(n + 1)^2$ Further improvements can possibly be made by other choices of $n$

The result of the reconstructions of the sought Cauchy data on the interior surface $\ _1$ are given in Fig. 5 and Fig. 6 for exact and 3% noisy data, respectively.

The similar conclusions as in the planar case can be drawn, see further the references for higher-dimensional domains given at the end of Section 4.1.

## References

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