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APPLICATION OF FABER POLYNOMIALS TO APPROXIMATE SOLUTION OF THE RIEMANN PROBLEM

ЗАСТОСУВАННЯ ПОЛІНОМІВ ФАБЕРА ДО НАБЛИЖЕНОГО РОЗВ'ЯЗАННЯ ПРОБЛЕМИ РІМАНА

In the paper, Faber polynomials are used to derive an approximate solution of the Riemann problem on a Lyapunov curve. Moreover, an estimation of the error of the approximated solution is presented and proved.

Поліноми Фабера застосовано для отримання наближеного розв'язку проблеми Рімана на кривій Ляпунова. Наведено і обгрунтовано оцінку похибки цього наближеного розв'язку.

1. Introduction. Let L be a closed Lyapunov curve on the complex plane and $G(t) \neq 0$ and $g(t)$ be given functions of Hölder continuous class $H(\mu)$, $0 < \mu \leq 1$, defined on L. The Riemann boundary-value problem for analytic functions consists in finding a pair of functions $F^+(z)$, $z \in D^+$, and $F^-(z)$, $z \in D^-$, analytic on the inside (D^+) and outside (D^-) of the curve L, respectively, such that the following condition is fulfilled

$$F^+(t) = G(t)F^-(t) + g(t), \qquad F^-(\infty) = 0, \quad t \in L. \tag{1}$$

Let us recall that a simple continuous curve is called Lyapunov curve if it satisfies the following conditions:

(i) at every point of L there exists a well-defined tangent,

(ii) the angle $\theta(s)$ between OX axis and the tangent to L at the point M whose distance from a fixed point, measured along the curve L, is equal to s, satisfies

$$|\theta(s_2) - \theta(s_1)| \leq k|s_1 - s_2|^\alpha, \quad 0 < \alpha \leq 1.$$

The Riemann problem (1) has numerous applications [8, 22, 6, 21]. The main arise in the theory of singular integral equations. The homogeneous Riemann problem $\big(g(t) \equiv 0\big)$ was first considered by Hilbert [11], and the nonhomogeneous problem (1) by Privalov [23]. They reduced it to the problem of solving integral equations. Next, Gakhov in the monograph [7] presented an effective solution of (1) in terms of Cauchy type integrals.

We will recall this solution. Let $\varkappa = \operatorname{Ind} G(t) \geq 0$, then the solution has the following form [8, 22]:

$$F^\pm(z) = X^\pm(z)\big(\Psi^\pm(z) + P_{\varkappa-1}(z)\big), \tag{2}$$

where

$$X^+(z) = \exp\Gamma^+(z), \quad z \in D^+, \qquad X^-(z) = z^{-\varkappa}\exp\Gamma^-(z), \quad z \in D^-, \tag{3}$$

$P_{\varkappa-1}(z) = \gamma_0 + \gamma_1 z + \ldots + \gamma_{\varkappa-1}z^{\varkappa-1}$, and $\gamma_0, \gamma_1, \ldots, \gamma_{\varkappa-1}$ are arbitrary constants. Here

$$\Gamma^{\pm}(z) = \frac{1}{2\pi i} \int\limits_L \frac{\ln\left(\tau^{-\varkappa} G(\tau)\right)}{\tau - z}\, d\tau, \quad z \in D^+, \tag{4}$$

and

$$\Psi^{\pm}(z) = \frac{1}{2\pi i} \int\limits_L \frac{g(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - z}, \quad z \in D^+. \tag{5}$$

If $\varkappa < 0$ then the solution $F^{\pm}(z)$, given by the formula (2) with $P_{\varkappa-1}(z) \equiv 0$, exists if and only if the following conditions hold:

$$\int\limits_L \frac{g(\tau)}{X^+(\tau)} \tau^{j-1}\, d\tau = 0, \quad j = 1, 2, \ldots, |\varkappa|. \tag{6}$$

Over the last decades the Riemann problem has been intensively investigated. Many generalizations and modifications can be found in the literature [6, 20]. One of the most famous problem is the nonlinear conjugation problem of power type [3, 21]. Many research has been done under various assumptions about the curve and coefficients [13, 14]. However, even in the classical Riemann problem the Cauchy-type integrals occurring in (4), (5) have very complicated forms. Their exact values can be calculated only in special cases. Therefore, to solve the problem (1), we apply the approximate methods [2]. Well-known numerical solutions of the problem (1) have been constructed for the case of zero index and the case when the contour is a circle or a segment of a real line [24, 25].

In this paper Faber polynomials are used to derive an approximate solution of the boundary problem (1) on any Lyapunov curve in the case of an arbitrary index. Moreover, the convergence of the approximate solution is proved and the rate of convergence is established.

Faber polynomials and their numerous modifications are a very useful tool in modern investigation of analytic functions [18, 25] and approximation theory. One can find their applications to a numerical solution of the Dirichlet problem in the plane [4], approximate solutions of singular integral equations [17, 26] and many other numerical methods for analytic functions (see monograph by P. K. Suetin [25] and papers [1, 5, 9, 10, 12, 15, 16, 19]).

2. Approximate solution. If $\varkappa > 0$ then the right-hand side of the formula (2) contains $\varkappa$ arbitrary constants. Therefore, we have to find the conditions for the uniqueness of the solution. In our opinion, the most convenient conditions are the following:

$$-\operatorname*{Res}_{z=\infty} \left(F^-(z) z^{j-1}\right) = A_j, \quad j = 1, 2, \ldots, \varkappa, \tag{7}$$

where $A_j, j = 1, 2, \ldots, \varkappa$, are given numbers.

By (2), taking into account the Laurent expansions of $X^-(z)$ and $\Psi^-(z)$ about the point $z = \infty$, i.e.,

$$X^-(z) = \frac{1}{z^{\varkappa}} \left(1 + \frac{p_1}{z} + \frac{p_1}{z^2} + \ldots\right),$$

$$\Psi^-(z) = \frac{h_1}{z} + \frac{h_2}{z^2} + \ldots,$$

from (7) we derive the system of linear equations

$$A_1 = \gamma_{\varkappa-1},$$

$$A_2 = p_1\gamma_{\varkappa-1} + \gamma_{\varkappa-2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$A_\varkappa = p_{\varkappa-1}\gamma_{\varkappa-1} + p_1\gamma_1 + \ldots + \gamma_0.$$

It enables us to compute the unknown coefficients $\gamma_0, \gamma_1, \ldots, \gamma_{\varkappa-1}$.

In what follows, it will be convenient to have an expansion in Faber series of the function $F^{\pm}(z)$ defined by (2). For this purpose, we first find the expansion of the function $\Gamma^{\pm}(z)$. By Theorem 6 from [25, p. 192] we obtain

$$\Gamma^+(z) = \sum_{k=0}^{\infty} a_k \Phi_k(z), \quad z \in \overline{D^+}, \tag{8}$$

where

$$a_k = \frac{1}{2\pi i}\int\limits_L \frac{\ln\left(\tau^{-\varkappa}G(\tau)\right)}{\Phi^{k+1}(\tau)}\Phi'(\tau)\,d\tau = \frac{1}{2\pi i}\int\limits_{|t|=1} \frac{\ln\left[(\varphi(t))^{-\varkappa}G(\varphi(t))\right]}{t^{k+1}}\,d\tau,$$

and

$$\Gamma^-(z) = \sum_{k=1}^{\infty} \frac{b_k}{\Phi^k(z)}, \quad z \in \overline{D^-}, \tag{9}$$

where

$$b_k = \frac{1}{2\pi i}\int\limits_L \ln\left(\tau^{-\varkappa}G(\tau)\right)\Phi^{k-1}(\tau)\Phi'(\tau)\,d\tau = \frac{1}{2\pi i}\int\limits_{|t|=1} \ln\left[\varphi^{-\varkappa}(t)G(\varphi(t))\right]t^{k-1}\,d\tau.$$

The function $w = \Phi(z)$ is a Riemann mapping, i.e., a conformal and univalent mapping from D^- of $\mathbb{C}_z$ onto the exterior of the unit circle $|w| > 1$ of $\mathbb{C}_w$, while $\varphi(t)$ is a limit value of the inverse function $z = \varphi(w)$, and $\Phi_k(z),\ k = 0, 1, \ldots,$ are the Faber polynomials on the area D^+.

Similarly as in the previous case, we have the following expansions:

$$\Psi^+(z) = \sum_{k=0}^{\infty} c_k \Phi_k(z), \quad z \in \overline{D^+}, \tag{10}$$

$$\Psi^-(z) = \sum_{k=1}^{\infty} \frac{d_k}{\Phi^k(z)}, \quad z \in \overline{D^-}, \tag{11}$$

where

$$c_k = \frac{1}{2\pi i}\int\limits_{|t|=1} \frac{g\big(\varphi(t)\big)}{X^+\big(\varphi(t)\big)}\,\frac{d\tau}{t^{k+1}},$$

$$d_k = \frac{1}{2\pi i}\int\limits_{|t|=1} \frac{g\big(\varphi(t)\big)}{X^+\big(\varphi(t)\big)}\,t^{k-1}d\tau.$$

Thus, in the case of nonnegative index $\varkappa \geq 0$, the desired functions $F^{\pm}(z)$ can be expressed as

$$F^{+}(z) = \exp\left(\sum_{k=0}^{\infty} a_k \Phi_k(z)\right)\left(\sum_{k=0}^{\infty} c_k \Phi_k(z) + P_{\varkappa-1}(z)\right), \quad z \in \overline{D^{+}},$$

$$F^{-}(z) = z^{-\varkappa} \exp\left(\sum_{k=1}^{\infty} \frac{b_k}{\Phi^k(z)}\right)\left(\sum_{k=1}^{\infty} \frac{d_k}{\Phi^k(z)} + P_{\varkappa-1}(z)\right), \quad z \in \overline{D^{-}}.$$

As the approximate solution of the problem (1), (7), we take the function

$$F_n^{\pm}(z) = X_n^{\pm}(z)\big(\Psi_n^{\pm} + P_{\varkappa-1}(z)\big), \quad z \in D^{\pm}, \tag{12}$$

where

$$\begin{gathered} X_n^{+}(z) = \exp \Gamma_n^{+}(z), \qquad X_n^{-}(z) = z^{-\varkappa} \exp \Gamma_n^{-}(z), \\ \Gamma_n^{+}(z) = \sum_{k=0}^{n} a_k \Phi_k(z), \qquad \Gamma_n^{-}(z) = \sum_{k=1}^{n} \frac{b_k}{\Phi^k(z)}, \end{gathered} \tag{13}$$

and

$$\Psi_n^{+}(z) = \sum_{k=0}^{n} c_k^{*} \Phi_k(z), \qquad \Psi_n^{-}(z) = \sum_{k=1}^{n} \frac{d_k^{*}}{\Phi^k(z)},$$

$$c_k^{*} = \frac{1}{2\pi i} \int\limits_{|t|=1} \frac{g(\varphi(t))}{X_n^{+}(\varphi(t))} \frac{d\tau}{t^{k+1}}, \qquad d_k^{*} = \frac{1}{2\pi i} \int\limits_{|t|=1} \frac{g(\varphi(t))}{X_n^{+}(\varphi(t))} t^{k-1} d\tau.$$

If $\varkappa < 0$ then an approximate solution of the problem (1) can be found from the boundary condition

$$F_n^{+}(t) = G_n(t) F_n^{-}(t) + g(t) + X_n^{+}(t)\left(\frac{q_1}{t} + \frac{q_2}{t^2} + \ldots + \frac{q_{|\varkappa|}}{t^{|\varkappa|}}\right), \quad t \in L, \tag{14}$$

where

$$G_n(t) = X_n^{+}(t)\big(X_n^{-}(t)\big)^{-1},$$

and the functions $X_n^{\pm}(t)$ are defined by (13). The coefficients $q_1, q_2, \ldots, q_{|\varkappa|}$ should be chosen in such a way that the solvability conditions for Eq. (14) are satisfied. These conditions, by (6), have the form

$$\frac{1}{2\pi i} \int\limits_{L} \left(\frac{q_1}{\tau} + \frac{q_2}{\tau^2} + \ldots + \frac{q_{|\varkappa|}}{\tau^{|\varkappa|}}\right) \tau^{j-1}\, d\tau = -\frac{1}{2\pi i} \int\limits_{L} \frac{g(\tau)}{X_n^{+}(\tau)} \tau^{j-1}\, d\tau, \quad j = 1, 2, \ldots, |\varkappa|\,.$$

Hence, we obtain

$$q_j = -\frac{1}{2\pi i} \int\limits_{L} \frac{g(\tau)}{X_n^{+}(\tau)} \tau^{j-1}\, d\tau, \quad j = 1, 2, \ldots, |\varkappa|.$$

In accordance with (2), the solution of the problem (14) has the following structure:

$$F_n^{\pm}(z) = X_n^{\pm}(z) \Psi_n^{\pm}(z), \quad z \in D^{\pm}, \tag{15}$$

where $X_n^{\pm}(t)$ are defined by (13), and

$$\Psi_n^+(z) = \sum_{k=0}^{n} \alpha_k \Phi_k(z), \qquad \Psi_n^-(z) = \sum_{k=1}^{n} \frac{\beta_k}{\Phi^k(z)},$$

$$\alpha_k = \frac{1}{2\pi i} \int\limits_{|t|=1} H_n(t) \frac{dt}{t^{k+1}}, \qquad \beta_k = \frac{1}{2\pi i} \int\limits_{|t|=1} H_n(t) t^{k-1}\, dt,$$

and

$$H_n(t) = \frac{g\big(\varphi(t)\big)}{X_n^+\big(\varphi(t)\big)} + \frac{q_1}{\varphi(t)} + \frac{q_2}{\big(\varphi(t)\big)^2} + \ldots + \frac{q_{|\varkappa|}}{\big(\varphi(t)\big)^{|\varkappa|}}.$$

3. Estimation of errors. Now we will provide the error estimations of the approximate solutions obtained above. Let the function $G^*(t) = \ln(t^{-\varkappa} G(t)), t \in L$, be continuously differentiable up to the order r and the rth derivative fulfill the Hölder inequality with the constant $0 < \mu < 1$. Then we say that the function $G^*(t)$ belongs to the class $W^r H^\mu$. According to [25, p. 262] we have

$$\left| \Gamma^+(z) - \sum_{k=0}^{n} a_k \Phi_k(z) \right| \le K_1 \frac{\ln(n)}{n^{r+\mu}}, \quad z \in \overline{D^+}, \tag{16}$$

$$\left| \Gamma^-(z) - \sum_{k=1}^{n} \frac{b_k}{\Phi^k(z)} \right| \le K_2 \frac{\ln(n)}{n^{r+\mu}}, \quad z \in \overline{D^-}, \tag{17}$$

where K_j are constants independent of n. Taking into account the inequality

$$|1 - e^z| \le (e-1)\,|z|\,, \quad |z| \le 1,$$

and using the maximum modulus principle, we obtain

$$\big|X^{\pm}(z) - X_n^{\pm}(z)\big| \le \max_{t \in L} \big|X^{\pm}(z) - X_n^{\pm}(z)\big| \le K_3 \frac{\ln n}{n^{r+\mu}}, \quad z \in D^{\pm}.$$

Similarly, we get the estimations for (16) and (17), i.e.,

$$\left| \Psi^+(z) - \sum_{k=0}^{n} c_k \Phi_k(z) \right| \le K_4 \frac{\ln n}{n^{r+\mu}}, \quad z \in \overline{D^+},$$

$$\left| \Psi^-(z) - \sum_{k=1}^{n} \frac{d_k}{\Phi^k(z)} \right| \le K_5 \frac{\ln n}{n^{r+\mu}}, \quad z \in \overline{D^-},$$

whenever $g(t) \in W^r H^\mu$. To estimate the modulus $\big|\Psi^{\pm}(z) - \Psi_n^{\pm}(z)\big|$, we first estimate the difference

$$\sum_{k=0}^{n} c_k \Phi_k(z) - \sum_{k=0}^{n} c_k^* \Phi_k(z).$$

By [25, p. 155], we have

$$\left|\sum_{k=0}^{n} c_k \Phi_k(z) - \sum_{k=0}^{n} c_k^* \Phi_k(z)\right| \leq$$

$$\leq \frac{1}{2\pi} \int\limits_L \left| g(\tau) \frac{X_n^+(\tau) - X^+(\tau)}{X^+(\tau) X_n^+(\tau)} \sum_{k=0}^{n} \frac{\Phi_k(z)}{\Phi^{k+1}(\tau)} \Phi'(\tau) \right| |d\tau| \leq$$

$$\leq K_6 \frac{\ln n}{n^{r+\mu}} \int\limits_L \left| \sum_{k=0}^{n} \frac{\Phi_k(z)}{\Phi^{k+1}(\tau)} \right| \left|\Phi'(\tau)\right| |d\tau| \leq K_7 \frac{\ln^2 n}{n^{r+\mu}}, \quad z \in L.$$

From the above, we obtain

$$\left| \sum_{k=1}^{n} \frac{d_k}{\Phi^k(z)} - \sum_{k=1}^{n} \frac{d_k^*}{\Phi^k(z)} \right| \leq K_8 \frac{\ln^2 n}{n^{r+\mu}}, \quad z \in L.$$

Hence, we get

$$\left|\Psi^+(z) - \Psi_n^+(z)\right| = \left| \sum_{k=0}^{\infty} c_k \Phi_k(z) - \sum_{k=0}^{n} c_k^* \Phi_k(z) \right| \leq$$

$$\leq \left| \sum_{k=0}^{\infty} c_k \Phi_k(z) - \sum_{k=0}^{n} c_k \Phi_k(z) \right| + \left| \sum_{k=0}^{n} c_k \Phi_k(z) - \sum_{k=0}^{n} c_k^* \Phi_k(z) \right| \leq$$

$$\leq K_9 \frac{\ln^2 n}{n^{r+\mu}}, \quad z \in \overline{D^+},$$

and

$$\left|\Psi^-(z) - \Psi_n^-(z)\right| = \left| \sum_{k=1}^{\infty} \frac{d_k}{\Phi^k(z)} - \sum_{k=1}^{n} \frac{d_k^*}{\Phi^k(z)} \right| \leq K_{10} \frac{\ln^2 n}{n^{r+\mu}}, \quad z \in \overline{D^-}.$$

Now we can estimate the modulus $\left|F^{\pm}(z) - F_n^{\pm}(z)\right|$. We obtain

$$\left|F^{\pm}(z) - F_n^{\pm}(z)\right| =$$

$$= \left|X^{\pm}(z)\left(\Psi^{\pm}(z) + P_{\varkappa-1}(z)\right) - X_n^{\pm}(z)\left(\Psi_n^{\pm}(z) + P_{\varkappa-1}(z)\right)\right| =$$

$$= \left| \left(X^{\pm}(z)\Psi^{\pm}(z) - X_n^{\pm}(z)\Psi^{\pm}(z)\right) + \left(X_n^{\pm}(z)\Psi^{\pm}(z) - X_n^{\pm}(z)\Psi_n^{\pm}(z)\right) + \right.$$

$$\left. + \left(X^{\pm}(z) - X_n^{\pm}(z)\right) P_{\varkappa-1}(z) \right| \leq K_{11} \frac{\ln^2 n}{n^{r+\mu}}, \quad z \in \overline{D^{\pm}}. \tag{18}$$

***Remark* 1.** If $\varkappa < 0$ then the estimation (18) holds as well.

We have thereby proved the following theorem.

Theorem 1. *Let the functions* $G^*(t) = \ln(t^{-\varkappa} G(t))$ *and* $g(t)$ *appearing in* (1) *belong to the class* $W^r H^\mu$, $r \geq 0$, $0 < \mu < 1$, *and let* $F^{\pm}(z)$, $F_n^{\pm}(z)$ *denote the exact and approximate solutions* (2) *and* (12), *respectively. Then the estimation* (18) *holds.*

4. Numerical experiment. Let L be an ellipse with foci ± 1 and semiaxes $a = \dfrac{5}{4}$ and $b = \dfrac{3}{4}$. We find the exact and approximate solutions of the following Riemann problem:

$$F^+(t) = \frac{t}{t^2-1}F^-(t) + \frac{t^3-t^2+1}{t(t-1)}, \quad t \in L.$$

Here $G(t) = \dfrac{t}{t^2-1}$, $g(t) = \dfrac{t^3-t^2+1}{t(t-1)}$. Moreover, the ellipse L can be expressed by the equation $t = \dfrac{1}{2}\left(2e^{i\theta} + \dfrac{1}{2}e^{-i\theta}\right)$, $0 \le \theta \le 2\pi$.

Functions $\Phi(z)$ and $\varphi(w)$ have the forms

$$w = \Phi(z) = \frac{1}{2}\left(z + \sqrt{z^2-1}\right),$$

$$z = \varphi(w) = \frac{1}{2}\left(2w + \frac{1}{2w}\right).$$

Furthermore, in this case Faber polynomials have the form

$$\Phi_n(z) = \frac{2}{R^n}T_n(z), \quad n = 1, 2, \ldots,$$

where $T_n(x)$, $-1 \le x \le 1$ are the Chebyshev polynomials of the first kind.

It can be easily obtained that the index $\varkappa = -1$ and the conditions of solvability (6) are satisfied. Applying the formulae (3)–(5) we have

$$\Gamma^+(z) = 0, \quad z \in D^+, \qquad \Gamma^-(z) = -\ln\frac{z^2}{z^2-1}, \quad z \in D^-,$$

$$X^+(z) = 1, \quad z \in D^+, \qquad X^-(z) = z - \frac{1}{z}, \quad z \in D^-,$$

$$\Psi^+(z) = z, \quad z \in D^+, \qquad \Psi^-(z) = -\frac{1}{z(z-1)}, \quad z \in D^-.$$

Finally from (2), with $P_{\varkappa-1}(z) \equiv 0$, we obtain

$$F^+(z) = z, \quad z \in D^+, \qquad F^-(z) = -\frac{z+1}{z^2}, \quad z \in D^-.$$

To find the approximate solution, we determine the functions $\Gamma_n^+(z)$, $\Gamma_n^-(z)$ and $\Psi_n^+(z)$, $\Psi_n^-(z)$ as follows:

$$\Gamma_n^+(z) = 0, \quad z \in D^+, \qquad \Gamma_n^-(z) = \sum_{k=1}^{\mathrm{Ent}\,\frac{n+2}{4}} \frac{-4}{(2k-1)(z+\sqrt{z^2-1})^{4k-2}}, \quad z \in D^-,$$

$$\Psi^+(z) = z, \quad z \in D^+, \qquad \Psi^-(z) = \sum_{k=1}^{n} \frac{i^{k-1}(1-(-1)^k) - 2k}{(z+\sqrt{z^2-1})^k}, \quad z \in D^-.$$

The approximate solution $F_n^{\pm}(z)$ can be obtained from (15).

The exact and approximate values of the function $F^-(z)$ for chosen points in D^- with $n = 20$, are presented in Table 1.

Table 1

$t = 2.00$	$F^-(t) = -0.7500000000$ $F_n^-(t) = -0.7499999999$
$t = 1.62 + 0.71i$	$F^-(t) = -0.7378050367 + 0.4615476936i$ $F_n^-(t) = -0.7378050374 + 0.4615476937i$
$t = 0.618 + 1.141i$	$F^-(t) = -0.0424777057 + 1.1747032784i$ $F_n^-(t) = -0.0424777052 + 1.1747033045i$
$t = -0.618 + 1.141i$	$F^-(t) = 0.6913297244 + 0.1803529499i$ $F_n^-(t) = 0.6913297119 + 0.1803529346i$
$t = -1.62 + 0.71i$	$F^-(t) = 0.3008804422 - 0.0087581894i$ $F_n^-(t) = 0.3008804418 - 0.0087581893i$
$t = -2.00$	$F^-(t) = 0.2500000000$ $F_n^-(t) = 0.2500000000$
$t = -1.62 - 0.71i$	$F^-(t) = 0.3008804422 + 0.0087581894i$ $F_n^-(t) = 0.3008804418 + 0.0087581893i$
$t = -0.618 - 1.141i$	$F^-(t) = 0.6913297244 - 0.1803529499i$ $F_n^-(t) = 0.6913297119 - 0.1803529346i$
$t = 0.618 - 1.141i$	$F^-(t) = -0.0424777057 - 1.1747032784i$ $F_n^-(t) = -0.0424777052 - 1.1747033045i$
$t = 1.62 - 0.71i$	$F^-(t) = -0.7378050367 - 0.4615476936i$ $F_n^-(t) = -0.7378050374 - 0.4615476937i$

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