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ON GENERALIZED STATISTICAL AND IDEAL CONVERGENCE OF METRIC-VALUED SEQUENCES

ПРО УЗАГАЛЬНЕНУ СТАТИСТИЧНУ ТА ІДЕАЛЬНУ ЗБІЖНІСТЬ МЕТРИЧНОЗНАЧНИХ ПОСЛІДОВНОСТЕЙ

We consider the notion of generalized density, namely, natural density of weight g recently introduced in [Balcerzak M., Das P., Filipczak M., Swaczyna J. Generalized kinds of density and the associated ideals // Acta Math. Hung. – 2015. – **147**, № 1. – P. 97 – 115] and primarily study some sufficient and almost converse necessary conditions for the generalized statistically convergent sequence under which the subsequence is also generalized statistically convergent. Some results are also obtained in more general form using the notion of ideals. The entire investigation is performed in the setting of general metric spaces extending the recent results of Kücükaslan M., Deger U., Dovgoshey O. On statistical convergence of metric valued sequences, see Ukr. Math. J. – 2014. – **66**, № 5. – P. 712 – 720.

Ми розглядаємо поняття узагальненої щільності, тобто натуральної щільності з вагою g, нещодавно введеної в статті [Balcerzak M., Das P., Filipczak M., Swaczyna J. Generalized kinds of density and the associated ideals // Acta Math. Hung. – 2015. – **147**, № 1. – P. 97 – 115], та переважно вивчаємо деякі достатні та майже протилежні необхідні умови для узагальненої статистично збіжної послідовності, за яких підпослідовність також є узагальненою та статистично збіжною. Деякі результати також отримано в більш загальному вигляді за допомогою поняття ідеалів. Наше дослідження виконано в постановці загальних метричних просторів і узагальнює нещодавні результати, отримані у статті Kücükaslan M., Deger U., Dovgoshey O. On statistical convergence of metric valued sequences (див. Укр. мат. журн. – 2014. – **66**, № 5. – С. 712 – 720).

1. Introduction. In recent years there have been rapid developments of the analytical studies in metric spaces which can be seen in [21, 27]. In [16] Dovgoshey and Martio introduced a new approach to the introduction of smooth structures for general metric spaces (one can see also [4, 5, 14, 15] where more references can be found). In the language of [16] this new approach is completely based on the convergence of metric-valued sequences but it is not *a priori* clear that the ordinary convergence is the best possible way to obtain smooth structures for arbitrary metric spaces.

From the beginnings of 1800's several methods have been introduced to make a divergent real or complex sequence convergent (for example Česaro, Nörlund, weighted mean, Abel etc.) but most of these convergence methods are dependent on the algebraic structures of the spaces of reals or complex numbers. It should be noted that in general metric spaces do not have algebraic structures. However if one considers the notion of statistical convergence introduced in [18, 29] and its extensions like statistical convergence of order α [3, 6] or more generally the notion of ideal convergence [22], it is clear that they can be readily extended to arbitrary metric spaces.

On the other direction the study of statistical convergence and its many extensions and in particular ideal convergence and its applications has been one of the most active areas of research in summability theory over the last 15 years.

Naturally it seems that the studies of these generalized methods of convergence may provide a natural foundation for the upbuilding of various tangent spaces to general metric spaces. The construction of tangent spaces in [4, 5, 14 – 16] is primarily based on the fundamental fact that for a convergent sequence (x_n) in a metric space, each of its subsequence $(x_{n(k)})$ is also convergent. However this is generally not true for the generalized methods of convergence mentioned above.

Very recently following the line of investigation of [25], in [23] conditions were studied for the density of a subsequence of a statistically convergent sequence under which the subsequence is also statistically convergent in metric space settings.

As a natural consequence, in this paper we continue the investigation proposed in [23] and investigate similar problems for metric-valued sequences by considering the notion of natural density of weight g which was very recently introduced in [1] as also for certain results we use the most general notion of ideals.

2. Basic facts and definitions. Let $\mathbb{N}$ denote the set of all positive integers. By $\mathrm{card}(A)$ we denote the cardinality of a set A. The natural density of a set $A \subset \mathbb{N}$ is defined as follows: The lower and the upper densities of A are given by the formulas

$$\underline{d}(A) = \liminf_{n\to\infty} \frac{\mathrm{card}(A \cap [1,n])}{n},$$

$$\overline{d}(A) = \limsup_{n\to\infty} \frac{\mathrm{card}(A \cap [1,n])}{n}.$$

If $\underline{d}(A) = \overline{d}(A)$, we say that the natural density of A exists and it is denoted by $d(A)$. The notion of statistical convergence was introduced by Fast [18] (see also [29]) using this notion of natural density.

Now recall that a family $\mathcal{I} \subset 2^Y$ of subsets of a nonempty set Y is said to be an ideal in Y if (i) $A,\ B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$, (ii) $A \in \mathcal{I},\ B \subset A$ implies $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$. Such ideals are also called free ideals. If $\mathcal{I}$ is a proper ideal in Y (i.e., $Y \notin \mathcal{I},\ \mathcal{I} \neq \{\varnothing\}$), then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y :$ there exists $A \in \mathcal{I}\colon M = Y \setminus A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$. Throughout the paper $\mathcal{I}$ will stand for a proper admissible ideal of $\mathbb{N}$. We denote the ideal of all finite subsets of $\mathbb{N}$ by $\mathcal{I}_{fin}$. For more example of different ideals see [22].

An admissible ideal $\mathcal{I}$ is said to satisfy the condition (AP) (or is called a P-ideal or sometimes AP-ideal) if for every countable family of mutually disjoint sets $(A_1, A_2, \ldots) \in \mathcal{I}$ there exists a countable family of sets $(B_1, B_2, \ldots)$ such that $A_j \triangle B_j$ is finite for each $j \in \mathbb{N}$ and $\bigcup_{k=1}^{\infty} B_k \in \mathcal{I}$. Several examples of P-ideals can be found in [1].

It is known that the density ideal

$$\mathcal{I}_d = \{A \subset \mathbb{N} \colon \overline{d}(A) = 0\}$$

is an $F_{\sigma\delta}$ P-ideal on $\mathbb{N}$. It is also an example of a so-called Erdős – Ulam ideal (for further information see [17]).

In [3] the authors proposed a modified version of density. Namely, for $0 < \alpha \leq 1$ and $A \subset \mathbb{N}$, they put

$$\overline{d_\alpha}(A) = \limsup_{n\to\infty} \frac{\mathrm{card}(A \cap [1,n])}{n^\alpha}$$

and $\underline{d_\alpha}(A)$ is defined analogously. It has been very recently observed in [1] that these density functions also generate P-ideals.

In this connection it can be mentioned that Kostyrko et al. [22] considered arbitrary ideals $\mathcal{I}$ on $\mathbb{N}$ and defined the notion of $\mathcal{I}$-convergence of sequences extending the idea of statistical convergence. Following the general line of [22], ideals were used to study sequences in topological spaces [9, 24], to study nets in topological and uniform spaces [11, 12]. More recent applications of ideals can be found in [8, 10, 13] where many more references can be found.

We now start our main discussions. In [1] the notion of natural density (as also natural density of order α) has further been extended as follows. Let $g\colon \mathbb{N} \to [0,\infty)$ be a function with $\lim_{n\to\infty} g(n) = \infty$. The upper density of weight g was defined in [1] by the formula

$$\overline{d}_g(A) = \limsup_{n\to\infty} \frac{\operatorname{card}(A \cap [1,n])}{g(n)}$$

for $A \subset \mathbb{N}$. Then the family

$$\mathcal{I}_g = \{A \subset \mathbb{N}\colon \overline{d}_g(A) = 0\}$$

forms an ideal. It has been observed in [1] that $\mathbb{N} \in \mathcal{I}_g$ iff $\dfrac{n}{g(n)} \to 0$. So we additionally assume that $n/g(n) \nrightarrow 0$ so that $\mathbb{N} \notin \mathcal{I}_g$ and in has been observed in [1] that $\mathcal{I}_g$ is a proper admissible P-ideal of $\mathbb{N}$. The collection of all such functions g satisfying the above mentioned properties will be denoted by G. As a natural consequence we can introduce the following definition.

Throughout (X,ρ) will stand for a metric space and $\widetilde{X}$ will denote the set of all sequences of points of X.

Definition 2.1. *A metric-valued sequence $\widetilde{x} = (x_n) \in \widetilde{X}$ is said to be d_g-statistically convergent to $a \in X$ if for any $\epsilon > 0$ we have $d_g(A(\varepsilon)) = 0$ where $A(\varepsilon) = \{n \in \mathbb{N}:\ \rho(x_n, a) \geq \varepsilon\}$.*

Below some more basic definitions are given which will be needed throughout the paper.

Definition 2.2. *A set $K \subset \mathbb{N}$ is called d_g-dense subset of $\mathbb{N}$ if $d_g(K^c) = 0$.*

Definition 2.3 (see [22]). *A metric-valued sequence $x = (x_n) \in \widetilde{X}$ is $\rho - \mathcal{I}$-convergent to $a \in X$ if for any $\varepsilon > 0$, $A(\varepsilon) = \{n \in N\colon \rho(x_n, a) \geq \varepsilon\} \in \mathcal{I}$.*

Definition 2.4. *A set $K \subset \mathbb{N}$ is called $\mathcal{I}$-dense subset of $\mathbb{N}$ if $K \in F(\mathcal{I})$.*

Definition 2.5. *If $\big(n(k)\big)$ is an infinite strictly increasing sequence of natural numbers and $x = (x_n) \in \widetilde{X}$, then we write $\widetilde{x}' = (x_{n(k)})$ and $K_{\tilde{x}'} = \{n(k)\colon k \in \mathbb{N}\}$. $\widetilde{x}'$ is called an $\mathcal{I}$-dense subsequence of $\tilde{x}$ if $K_{\tilde{x}}$ is an $\mathcal{I}$-dense subset of $\mathbb{N}$.*

Definition 2.6. *Two sequences $\tilde{x} = (x_n) \in \tilde{X}$ and $\tilde{y} = (y_n) \in \tilde{X}$ are $\mathcal{I}$-equivalent, $\tilde{x} \asymp \tilde{y}$ if there is an $\mathcal{I}$-dense set $M \subset N$ such that $x_n = y_n$ for every $n \in M$.*

The following definitions are special cases of the above two definitions.

Definition 2.7. *If $(n(k))$ is an infinite strictly increasing sequence of natural numbers and $x = (x_n) \in \widetilde{X}$, then we write $\widetilde{x}' = (x_{n(k)})$ and $K_{\tilde{x}'} = \{n(k)\colon k \in N\}$. $\widetilde{x}'$ is called d_g-dense subsequence of $\tilde{x}$ if $K_{\tilde{x}'}$ is d_g-dense in $\mathbb{N}$.*

Definition 2.8. *Two sequences $\tilde{x} = (x_n) \in \tilde{X}$ and $\tilde{y} = (y_n) \in \tilde{X}$ are d_g-statistically equivalent, $\tilde{x} \asymp \tilde{y}$ (d_g-statistically) if there is an d_g-dense set $M \subset N$ such that $x_n = y_n$ for every $n \in M$.*

3. Main results. The first result given below extends Theorem 2.1 [23] and shows that there is a one to one correspondence between metrizable topologies on X and the subsets of $\widetilde{X}$ consisting of all $\mathcal{I}$-convergent sequences for certain special types of ideals.

Theorem 3.1. *Let (X,ρ_1) and (X,ρ_2) be two metric spaces. Let $\mathcal{I}$ be a P-ideal which is not maximal. Then the following statements are equivalent*:

(i) *The set of all $\rho_1 - \mathcal{I}$-convergent sequences coincides with the set of all $\rho_2 - \mathcal{I}$-convergent sequences.*

(ii) *The set of all sequences convergent in* (X, ρ_1) *coincides with the set of all sequences convergent in* (X, ρ_2).

(iii) *The metrics* ρ_1 *and* ρ_2 *induce one and the same topology on* X.

Proof. (ii) $\Longleftrightarrow$ (iii). The result is well known.

(ii) $\Rightarrow$ (i). Let $\tilde{x} = (x_n)$ be $\rho_1 - \mathcal{I}$-convergent. Since $\mathcal{I}$ is a P-ideal so $\tilde{x}$ is $\rho_1 - \mathcal{I}^*$-convergent, i.e., there is a set $M \in F(\mathcal{I})$ such that $(\tilde{x})_M$ is ρ_1-convergent (see [22]). By (ii), $(\tilde{x})_M$ is ρ_2-convergent and so $\tilde{x}$ is $\rho_2 - \mathcal{I}^*$-convergent which consequently implies that $\tilde{x}$ is $\rho_2 - \mathcal{I}$-convergent (see [22]).

(i) $\Rightarrow$ (iii). Assume that (i) holds. But on the contrary assume that the topologies induced by the metrics ρ_1 and ρ_2 are distinct. Then there is a $x_0 \in X$ and $\varepsilon_0 > 0$ such that

$$\big\{x \in X : \rho_1(x, x_0) < \varepsilon_0\big\} \not\supset \big\{x \in X : \rho_2(x, x_0) < \delta\big\} \tag{3.1}$$

for all $\delta > 0$ or

$$\big\{x \in X : \rho_2(x, x_0) < \varepsilon_0\big\} \not\supset \big\{x \in X : \rho_1(x, x_0) < \delta\big\}$$

for all $\delta > 0$. Without any loss of generality assume that (3.1) holds. For each $n \in \mathbb{N}$ we can then choose $x_n \in X$ such that

$$\rho_2(x_n, x_0) < \frac{1}{n} \quad \text{and} \quad \rho_1(x_n, x_0) \geq \varepsilon_0 \tag{3.2}$$

for each $n \in \mathbb{N}$. Choose a set $K \subset \mathbb{N}$ such that $K \notin \mathcal{I}$ as well as $K^c \notin \mathcal{I}$ (since $\mathcal{I}$ is not maximal). Define a sequence $\tilde{y} = (y_n) \in \widetilde{X}$ by

$$y_n = \begin{cases} x_n & \text{if} \quad n \in K, \\ x_0 & \text{if} \quad n \notin K. \end{cases}$$

Clearly

$$\big\{n \in \mathbb{N} \colon \rho_1(y_n, x_0) \geq \varepsilon_0\big\} = K \notin \mathcal{I}. \tag{3.3}$$

Now observe that the sequence $\tilde{y} = (y_n)$ is convergent to x_0 in (X, ρ_2) and so is $\rho_2 - \mathcal{I}$-convergent. By (i), $\tilde{y} = (y_n)$ is then also $\rho_1 - \mathcal{I}$-convergent. Observe that $\tilde{y}$ must be $\rho_1 - \mathcal{I}$-convergent to x_0 for otherwise if $\tilde{y}$ is $\rho_1 - \mathcal{I}$-convergent to $y_0 \neq x_0$ then taking $0 < \varepsilon < \rho_1(x_0, y_0)$ we have

$$\big\{n \in \mathbb{N} \colon \rho_1(y_n, y_0) \geq \varepsilon\big\} \supset K^c.$$

Since $K^c \notin \mathcal{I}$ so $\big\{n \in \mathbb{N} \colon \rho_1(y_n, y_0) \geq \varepsilon\big\} \notin \mathcal{I}$ which is a contradiction to the fact that $\tilde{y} = (y_n)$ is $\rho_1 - \mathcal{I}$-convergent to y_0. But if $\tilde{y}$ is $\rho_1 - \mathcal{I}$-convergent to x_0 then we must have

$$\big\{n \in \mathbb{N} \colon \rho_1(y_n, x_0) \geq \varepsilon_0\big\} = K \in \mathcal{I}$$

which contradicts (3.3). This proves that (i) $\Rightarrow$ (iii) holds.

Lemma 3.1. *Let* (X, ρ) *be a metric space,* $x_0 \in X$ *and* $\tilde{x} = (x_n) \in \widetilde{X}$. *Then* $\tilde{x}$ *is* $\rho - \mathcal{I}$-*convergent to* x_0 *in* X *if and only if the sequence* $(\rho(x_n, x_0))$ *is* $\mathcal{I}$-*convergent to* 0 *in* $\mathbb{R}$.

The proof is straightforward and so is omitted.

Lemma 3.2. *Let* (X, ρ) *be a metric space,* $x_0 \in X$ *and let* $\tilde{x} = (x_n) \in \widetilde{X}$ *be* $\rho - \mathcal{I}$-*convergent to* x_0. *Then there is* $\tilde{y} = (y_n) \in \widetilde{X}$ *such that* $\tilde{y} \asymp \tilde{x}$ *and* $\tilde{y}$ *is convergent to* x_0 *in* (X, ρ) *provided* $\mathcal{I}$ *is a* P-*ideal.*

The result again follows from the fact that $\tilde{x} = (x_n)$ is $\rho - \mathcal{I}^*$-convergent to x_0 as $\mathcal{I}$ is a P-ideal [22] and consequently we get the required sequence $\tilde{y}$.

We now start our discussions on subsequences. The first natural question that arises is that if a sequence is d_g-statistically convergent which of its subsequences are also d_g-statistically convergent to the same limit. It is also natural to ask when the converse is also true. We prove the next two results in this direction.

Theorem 3.2. *Let (X, ρ) be a metric space, $\tilde{x} = (x_n) \in \widetilde{X}$ and $\widetilde{x}' = (x_{n(k)})$ be a subsequence of $\tilde{x}$ such that*

$$\liminf_{n\to\infty} \frac{g\big(|K_{\tilde{x}}(n)|\big)}{g(n)} > 0.$$

If $\tilde{x}$ is d_g-statistically convergent to $x_0 \in X$, then $\widetilde{x}'$ is also d_g-statistically convergent to x_0.

Proof. Assume that $\tilde{x}$ is d_g-statistically convergent to x_0. Now clearly

$$\big\{n(k)\colon n(k) \leq n,\ d(x_{n(k)}, x_0) \geq \varepsilon\big\} \subseteq \big\{m\colon m \leq n,\ d(x_m, x_0) \geq \varepsilon\big\}$$

for all $n \in \mathbb{N}$ where $\varepsilon > 0$ is given. Then we get

$$\frac{1}{g\left(|K_{\tilde{x}}(n)|\right)} \left|\left\{n(k)\colon n\left(k\right) \leq n,\ \rho\left(x_{n(k)}, x_0\right) \geq \varepsilon\right\}\right| \leq$$
$$\leq \frac{\left|\left\{m\colon m \leq n,\ \rho\left(x_m, x_0\right) \geq \varepsilon\right\}\right|}{g\left(|K_{\tilde{x}}\left(n\right)|\right)}. \tag{3.4}$$

In order to prove that $\widetilde{x}'$ is d_g-statistically convergent to x_0 we need to show that

$$\limsup_{n\to\infty} \frac{\left|\left\{n\left(k\right)\,:\, n\left(k\right) \leq n,\ \rho\left(x_{n(k)}, x_0\right) \geq \varepsilon\right\}\right|}{g\left(|K_{\tilde{x}}\left(n\right)|\right)} = 0.$$

Recall that for any two sequences (c_n) and (d_n) of nonnegative real numbers with $0 \neq \liminf_{n\to\infty} c_n < \infty$ we have (see [2])

$$\liminf_{n\to\infty} c_n \limsup_{n\to\infty} d_n \leq \limsup_{n\to\infty} c_n d_n. \tag{3.5}$$

In (3.5) let us take

$$c_n = \frac{g\left(|K_{\tilde{x}}\left(n\right)|\right)}{g(n)} \quad \text{and} \quad d_n = \frac{\left|\left\{m\colon m \leq n,\ \rho\left(x_m, x_0\right) \geq \varepsilon\right\}\right|}{g\left(|K_{\tilde{x}}\left(n\right)|\right)}$$

so that

$$c_n d_n = \frac{\left|\left\{m\colon m \leq n,\ \rho\left(x_m, x_0\right) \geq \varepsilon\right\}\right|}{g(n)}.$$

Therefore we get from (3.5)

$$\liminf_{n\to\infty} \frac{g\left(|K_{\tilde{x}}\left(n\right)|\right)}{g\left(n\right)} \limsup_{n\to\infty} \frac{\left|\left\{m\colon m \leq n,\ \rho\left(x_m, x_0\right) \geq \varepsilon\right\}\right|}{g\left(|K_{\tilde{x}}\left(n\right)|\right)} \leq$$
$$\leq \limsup_{n\to\infty} \frac{\left|\left\{m\colon m \leq n,\ \rho\left(x_m, x_0\right) \geq \varepsilon\right\}\right|}{g(n)}.$$

Since $\tilde{x}$ is d_g-statistically convergent to x_0 so the right-hand side of the above inequality is zero. Since by our assumption $\liminf_{n\to\infty} \dfrac{g\left(|K_{\tilde{x}}(n)|\right)}{g(n)} > 0$ so it follows that

$$\limsup_{n\to\infty} \frac{|\{m : m \leq n,\ \rho(x_m, x_0) \geq \varepsilon_0\}|}{g\left(|K_{\tilde{x}}(n)|\right)} = 0$$

and the result now follows from (3.4).

Theorem 3.3. *Let* (X, ρ) *be a metric space and* $\tilde{x} \in \widetilde{X}$. *Then the following statements are equivalent*:

(a) $\tilde{x}$ *is* d_g*-statistically convergent*;

(b) *every subsequence* $\widetilde{x}'$ *of* $\tilde{x}$ *with*

$$\liminf_{n\to\infty} \frac{g\left(|K_{\tilde{x}}(n)|\right)}{g(n)} > 0$$

is d_g*-statistically convergent*;

(c) *every* d_g*-statistically dense subsequence* $\widetilde{x}'$ *of* $\tilde{x}$ *is* d_g*-statistically convergent provided* $g \in G$ *is such that* $0 < \liminf_{n\to\infty} \dfrac{n}{g(n)} < \infty$.

Proof. From Theorem 3.2 it follows that (a) $\Rightarrow$ (b). Since evidently $\tilde{x}$ itself is a d_g-dense subsequence of itself so (c) $\Rightarrow$ (a).

(b) $\Rightarrow$ (c). Let $\widetilde{x}'$ be a d_g-statistically dense subsequence of $\tilde{x}$. This means that $K_{\widetilde{x}'}$ has the property that $d_g\left(K^c_{\widetilde{x}'}\right) = 0$, i.e.,

$$\limsup_{n\to\infty} \frac{g\left(\left|K^c_{\widetilde{x}'}(n)\right|\right)}{g(n)} = 0.$$

Since $|K_{\widetilde{x}'}(n)| + |K^c_{\widetilde{x}'}(n)| = n$, consequently we have

$$\frac{|K_{\widetilde{x}'}(n)|}{g(n)} + \frac{|K_{\widetilde{x}'}{}^c(n)|}{g(n)} = \frac{n}{g(n)} \qquad \forall\, n \in \mathbb{N}.$$

It now follows that

$$\liminf_{n\to\infty} \frac{|K_{\widetilde{x}'}(n)|}{g(n)} + \limsup_{n\to\infty} \frac{\left|K^c_{\widetilde{x}'}(n)\right|}{g(n)} \geq \liminf_{n\to\infty} \frac{n}{g(n)}$$

which implies that

$$\liminf_{n\to\infty} \frac{|K_{\widetilde{x}'}(n)|}{g(n)} \geq \liminf_{n\to\infty} \frac{n}{g(n)} > 0$$

is a finite positive number. Finally we obtain

$$\liminf_{n\to\infty} \frac{g(|K_{\widetilde{x}'}(n)|)}{g(n)} \geq \liminf_{n\to\infty} \frac{g(|K_{\widetilde{x}'}(n)|)}{|K_{\widetilde{x}'}(n)|} \liminf_{n\to\infty} \frac{|K_{\widetilde{x}'}(n)|}{g(n)} > 0$$

in view of the fact that $|K_{\widetilde{x}'}(n)| \to \infty$ as $n \to \infty$. By (b) it now readily follows that $\widetilde{x}'$ is d_g-statistically convergent. This completes the equivalence of the three statements.

The next three results are given in the most general version in terms of ideals. As a consequence, one should note that these results hold for natural density of weight g also.

Lemma 3.3. *Let (X,ρ) be a metric space with $|X|>2$. Let $\tilde{x}=(x_n)\in\widetilde{X}$ and $\widetilde{x}'=(x_{n(k)})$ be an infinite subsequence of $\tilde{x}$ such that $K_{\widetilde{x}'}\in\mathcal{I}$. Then there exists a sequence $\tilde{y}\in\widetilde{X}$ and a subsequence $\widetilde{y}'$ of $\tilde{y}$ such that $K_{\widetilde{x}'}=K_{\widetilde{y}'}$ where $\widetilde{y}'$ is not $\mathcal{I}$-convergent provided $\mathcal{I}$ is not a maximal ideal.*

Proof. Choose two distinct elements a and b from X. Choose a subset $M\subset\mathbb{N}$ such that $M\notin\mathcal{I}$ as well as $M\notin\mathcal{F}(\mathcal{I})$. Let us define a sequence $\tilde{y}=(y_n)\in\widetilde{X}$ by

$$y_n=\begin{cases}x_n & \text{if} \quad n\in\mathbb{N}\backslash K_{\widetilde{x}'},\\ a & \text{if} \quad n=n(k)\in K_{\widetilde{x}'}, \quad \text{where} \quad k\in M,\\ b & \text{if} \quad n=n(k)\in K_{\widetilde{x}'}, \quad \text{where} \quad k\notin M.\end{cases}$$

Since $K_{\tilde{x}'}\in\mathcal{I}$ so $\mathbb{N}\setminus K_{\widetilde{x}'}\in\mathcal{F}(\mathcal{I})$ which shows that $\tilde{x}\overset{\mathcal{I}}{\asymp}\tilde{y}$. Obviously taking $\widetilde{y}'=\big(y_{n(k)}\big)$ we see that $K_{\widetilde{x}'}=K_{\widetilde{y}'}$. Since for any $c\in X$, taking $0<\varepsilon<\max\big\{\rho(a,c),\ \rho(b,c)\big\}$ we observe that

$$\big\{k\colon\rho(y_{n(k)},c)\geq\varepsilon\big\}\supset M \quad \text{or} \quad M^c$$

and so cannot belong to $\mathcal{I}$. This shows that $\widetilde{y}'$ is not $\mathcal{I}$-convergent.

Lemma 3.4. *Let (X,ρ) be a metric space. Let $a\in X$, $\tilde{x}=(x_n)$ and $\tilde{y}=(y_n)$ belong to $\widetilde{X}$. If $\tilde{x}$ is $\mathcal{I}$-convergent to a and $\tilde{x}\overset{\mathcal{I}}{\asymp}\tilde{y}$, then $\tilde{y}$ is also $\mathcal{I}$-convergent to a.*

Proof. Since $\tilde{x}\overset{\mathcal{I}}{\asymp}\tilde{y}$ so there is $M\in\mathcal{F}(\mathcal{I})$ such that $x_n=y_n$ for all $n\in M$. Clearly for any $\varepsilon>0$,

$$\big\{n\colon\rho(y_n,a)\geq\varepsilon\big\}\subset M^c\cup\big\{n\colon\rho(x_n,a)\geq\varepsilon\big\}.$$

Since $\tilde{x}$ is $\mathcal{I}$-convergent to a so the set on the right-hand side belong to $\mathcal{I}$ which implies that $\big\{n\colon\rho(y_n,a)\geq\varepsilon\big\}\in\mathcal{I}$ and $\tilde{y}$ is also $\mathcal{I}$-convergent to a.

Theorem 3.4. *Let (X,ρ) be a metric space with $|X|>2$, $a\in X$ and $\mathcal{I}$ be not maximal. Let $\tilde{x}=(x_n)$ be $\mathcal{I}$-convergent to a. Then for every infinite subsequence $\widetilde{x}'$ of $\tilde{x}$ with $K_{\widetilde{x}'}\in\mathcal{I}$, there exist a sequence $\tilde{y}\in\widetilde{X}$ and a subsequence $\widetilde{y}'$ of $\tilde{y}$ such that*:

(i) *$\tilde{y}\overset{\mathcal{I}}{\asymp}\tilde{x}$ and $K_{\widetilde{x}'}=K_{\widetilde{y}'}$,*

(ii) *$\tilde{y}$ is $\mathcal{I}$-convergent to a,*

(iii) *$\widetilde{y}'$ is not $\mathcal{I}$-convergent.*

The result follows from Lemmas 3.3 and 3.4.

Lemma 3.5. *Let (X,ρ) be a metric space. Let $\tilde{x},\ \tilde{y}\in\widetilde{X}$ and $\tilde{x}\asymp\tilde{y}$ (d_g-statistically). If K is a subset of $\mathbb{N}$ such that*

$$0<\liminf_{n\to\infty}\frac{g\big(|K(n)|\big)}{g(n)}, \tag{3.6}$$

if $\widetilde{x}'=(x_{n(k)})$ and $\widetilde{y}'=(y_{n(k)})$ are subsequences of $\tilde{x}$ and $\tilde{y}$ respectively such that $K_{\widetilde{x}'}=K_{\widetilde{y}'}=K$, then the relation $\widetilde{y}'\asymp\widetilde{x}'$ (d_g-statistically) is true.

Proof. We have to show that

$$\limsup_{m\to\infty}\frac{\big|\big\{n\,(k)\in K\colon x_{n(k)}\neq y_{n(k)},n\,(k)\leq m\big\}\big|}{g\big(|K(m)|\big)}=0. \tag{3.7}$$

Observe that for any $m\in\mathbb{N}$ we obtain

$$\big\{n(k) \in K : x_{n(k)} \neq y_{n(k)}, n(k) \le m\big\} \subset \big\{n \in \mathbb{N} : x_n \neq y_n, n \le m\big\}.$$

Consequently we get

$$\limsup_{m\to\infty} \frac{\big|\big\{n(k) \in K : x_{n(k)} \neq y_{n(k)}, n(k) \le m\big\}\big|}{g(|K(m)|)} \le$$

$$\le \limsup_{m\to\infty} \frac{|\{n \in N : x_n \neq y_n, n \le m\}|}{g(|K(m)| \le)} \le$$

$$\le \limsup_{m\to\infty} \frac{g(m)}{g(|K(m)|)} \limsup_{m\to\infty} \frac{|\{n \in \mathbb{N} : x_n \neq y_n, n \le m\}|}{g(m)} =$$

$$= \limsup_{m\to\infty} \frac{|\{n \in N : x_n \neq y_n, n \le m\}|}{g(m)} \left(\liminf_{m\to\infty} \frac{g(|K(m)|)}{g(m)}\right)^{-1}. \tag{3.8}$$

From (3.6) it follows that

$$0 \le \left(\liminf_{m\to\infty} \frac{g(|K(m)|)}{g(m)}\right)^{-1} < \infty.$$

Also as $\tilde{x} \asymp \tilde{y}$ (d_g-statistically) we have

$$\limsup_{m\to\infty} \frac{|\{n \in \mathbb{N} : x_n \neq y_n, n \le m\}|}{g(m)} = 0.$$

Now (3.7) readily follows that (3.8) and this completes the proof.

Theorem 3.5. *Let (X, ρ) be a metric space, $a \in X$ and $\tilde{x} = (x_n)$ is d_g-statistically convergent to a. Suppose that $\widetilde{x}' = (x_{n(k)})$ is a subsequence of $\tilde{x}$ for which there are $\tilde{y} = (y_n) \in \widetilde{X}$ and $\widetilde{y}'$ such that*:

(i) *$\tilde{y} \asymp \tilde{x}$ (d_g-statistically) and $K_{\widetilde{x}'} = K_{\widetilde{y}'}$,*

(ii) *$\widetilde{y}'$ is not d_g-statistically convergent,*

then $\liminf_{n\to\infty} \dfrac{|K_{\widetilde{x}'}(n)|}{g(n)} = 0$, provided $g : \mathbb{N} \to [0, \infty)$ satisfies $0 < \liminf_{n\to\infty} \dfrac{n}{g(n)}$ and $\limsup_{n\to\infty} \dfrac{n}{g(n)} < \infty$.

Proof. On the contrary suppose that

$$\liminf_{n\to\infty} \frac{|K_{\widetilde{x}'}(n)|}{g(n)} > 0.$$

Then

$$\liminf_{n\to\infty} \frac{g(|K_{\widetilde{x}'}(n)|)}{g(n)} \ge$$

$$\ge \liminf_{n\to\infty} \frac{g\big(|K_{\widetilde{x}'}(n)|\big)}{|K_{\widetilde{x}'}(n)|} \liminf_{n\to\infty} \frac{|K_{\widetilde{x}'}(n)|}{g(n)} > 0.$$

Let $\tilde{y} \in \widetilde{X}$ and $\widetilde{y}'$ be a subsequence of $\tilde{y}$ such that (i) and (ii) hold. Then we have $K_{\widetilde{x}'} = K_{\widetilde{y}'}$ and $\tilde{x} \asymp \tilde{y}$ (d_g-statistically). Then from Lemma 3.5 it follows that $\widetilde{x}' \asymp \widetilde{y}'$ (d_g-statistically). Applying Theorem 3.2 we observe that $\widetilde{x}'$ is d_g-statistically convergent to a. Since $\widetilde{x}' \asymp \widetilde{y}'$ (d_g-statistically) so by Lemma 3.4 $\widetilde{y}'$ is also d_g-statistically convergent to a which contradicts (ii).

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