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ON THE UNIQUENESS OF REPRESENTATION BY LINEAR SUPERPOSITIONS*

ПРО ЄДИНІСТЬ ЗОБРАЖЕННЯ ЧЕРЕЗ ЛІНІЙНІ СУПЕРПОЗИЦІЇ

Let Q be a set such that every function on Q can be represented by linear superpositions. This representation is, in general, not unique. However, for some sets, it may be unique provided that the initial values of the representing functions are prescribed at some point of Q. We study the properties of these sets.

Нехай Q — така множина, що кожну функцію на Q можна зобразити в термінах лінійних суперпозицій. У загальному випадку таке зображення не є єдиним. Проте для деяких множин воно може бути єдиним, якщо початкові значення функцій з цього зображення задано в деякій точці Q. Вивчаються деякі властивості таких множин.

1. Introduction. Let $X, X_1, \ldots, X_r$ be sets and $h_i \colon X \to X_i,\ i = 1, \ldots, r$, be arbitrarily fixed mappings. Consider the set

$$\mathcal{L} = \mathcal{L}(h_1, \ldots, h_r) = \left\{ \sum_{i=1}^{r} g_i(h_i(x)) \colon x \in X, g_i \colon X_i \to \mathbb{R}, i = 1, \ldots, r \right\}.$$

Members of this set will be called linear superpositions (see [12]). Linear superpositions were begun to be systematically studied after the famous result of A. N. Kolmogorov [6] on Hilbert's 13th problem. The result states that for the unit cube $\mathbb{I}^d$, $\mathbb{I} = [0, 1]$, $d \geq 2$, there exists $2d + 1$ functions $\{s_q\}_{q=1}^{2d+1} \subset C(\mathbb{I}^d)$ of the form

$$s_q(x_1, \ldots, x_d) = \sum_{p=1}^{d} \varphi_{pq}(x_p), \quad \varphi_{pq} \in C(\mathbb{I}), \quad p = 1, \ldots, d, \quad q = 1, \ldots, 2d + 1, \tag{1.1}$$

such that each function $f \in C(\mathbb{I}^d)$ admits the representation

$$f(x) = \sum_{q=1}^{2d+1} g_q(s_q(x)), \quad x = (x_1, \ldots, x_d) \in \mathbb{I}^d, \quad g_q \in C(\mathbb{R}). \tag{1.2}$$

This surprising and deep result was improved and generalized in several directions. It was first observed by G. G. Lorentz [7] that the functions g_q can be replaced with a single continuous function g. D. A. Sprecher [9] showed that the theorem can be proven with constant multiples of a single function φ and translations. Specifically, φ_{pq} in (1.1) can be chosen as $\lambda^p \varphi(x_p + \varepsilon q)$, where ε and λ are some positive constants. B. L. Fridman [1] succeeded in showing that the functions φ_{pq} can be constructed to belong to the class Lip(1). A. G. Vitushkin and G. M. Henkin [12] showed that φ_{pq} cannot be taken to be continuously differentiable. Y. Sternfeld [11] showed that the number $2d + 1$ in (1.2) cannot be reduced.

Kolmogorov's result shows that continuous functions admit representation by linear superpositions of form (1.2). Y. Sternfeld [10] proved that bounded functions also admit such representation

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with the natural proviso that the functions g_q are bounded. In [2], we start to study properties of linear superpositions on topology-free spaces and showed that every multivariate function f can be represented in form (1.2), where g_q are univariate functions depending on f. In the current paper, we continue our research on the representation capabilities of linear superpositions.

Let T be the set of all real functions on X. Note that the above set $\mathcal{L}$ is a linear subspace of T. For a set $Q \subset X$, let $T(Q)$ and $\mathcal{L}(Q)$ denote the restrictions of T and $\mathcal{L}$ to Q respectively. We are interested in sets Q with the property that $\mathcal{L}(Q) = T(Q)$. Such sets will be called representation sets. For a representation set Q, we will also use the notation $Q \in RS$. Here, RS stands for the set of all representation sets in X.

Let $Q \in RS$. Clearly for a function f defined on Q the representation

$$f(x) = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in Q, \tag{1.3}$$

is not unique. We are interested in the uniqueness of such representation under some reasonable restrictions on the functions $g_i \circ h_i$. These restrictions may be various, but in the current paper, we require that the values of the representing functions in (1.3) are prescribed at some point $x_0 \in Q$. That is, we require that

$$g_i(h_i(x_0)) = a_i, \quad i = 1, \ldots, r-1, \tag{1.4}$$

where a_i are arbitrarily fixed real numbers. Is representation (1.3) subject to initial conditions (1.4) always unique? Obviously, not. We are going to identify those representation sets Q for which representation (1.3) subject to conditions (1.4) is unique for all functions $f\colon Q \to \mathbb{R}$. In the sequel, such sets Q will be called unicity sets.

2. Main results. In our earlier paper [2], we characterized representation sets in terms of rather practical objects called closed paths. A closed path (with respect to the functions $h_1, \ldots, h_r$) is a set of points $\{x_1, \ldots, x_n\}$ in X such that there exists a vector $\lambda = (\lambda_1, \ldots \lambda_n)$ with $\lambda_i \in \mathbb{R}\backslash\{0\}$, $i = 1, \ldots, n$, satisfying the equations

$$\sum_{j=1}^{n} \lambda_j \delta_{h_i(x_j)}(t) = 0 \quad \text{for all} \quad t \in X_i, \quad i = 1, \ldots, r.$$

Here δ_a is the characteristic function of a single point set $\{a\}$.

For example, the set $l = \{(0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (1,1,1)\}$ is a closed path in $\mathbb{R}^3$ with respect to the functions $h_i(z_1, z_2, z_3) = z_i$, $i = 1, 2, 3$. The vector λ above can be taken as $(-2, 1, 1, 1, -1)$.

In the case $r = 2$, the picture of closed path becomes more clear. Let, for example, h_1 and h_2 be the coordinate functions on $\mathbb{R}^2$. In this case, a closed path is the union of some sets A_k with the property: each A_k consists of vertices of a closed broken line with the sides parallel to the coordinate axis. These objects (sets A_k) have been exploited in practically all works devoted to the approximation of bivariate functions by univariate functions, although under the different names (see, for example, [3], Chapter 2). If X and the functions h_1 and h_2 are arbitrary, the sets A_k can be described as a trace of some point traveling alternatively in the level sets of h_1 and h_2, and then returning to its primary position.

A result of [2] states that $Q \in RS$ if and only if there is no closed path in Q. From this result it is easy to obtain the following set-theoretic properties of representation sets:

(1) $Q \in RS \iff A \in RS$ for every finite set $A \subset Q$.

(2) The union of any linearly ordered (under inclusion) system of representation sets is also a representation set.

(3) For any representation set Q there is a maximal representation set, that is, a set $M \in RS$ such that $Q \subset M$ and for any $P \supset M$, $P \in RS$ we have $P = M$.

(4) If $M \subset X$ is a maximal representation set, then $h_i(M) = h_i(X)$, $i = 1, \ldots, r$.

Properties (1) and (2) are obvious, since any closed path is a finite set. The property (3) follows from (2) and Zorn's lemma. To prove (4) note that if $x_0 \in X$ and $h_i(x_0) \notin h_i(M)$ for some i, one can construct the representation set $M \cup \{x_0\}$, which is bigger than M. But this is impossible, since M is maximal.

Definition 2.1. *A set $Q \subset X$ is called a complete representation set if Q itself is a representation set and there is no other representation set P such that $Q \subset P$ and $h_i(P) = h_i(Q)$, $i = 1, \ldots, r$.*

The set of all complete representation sets of X will be denoted by CRS. Obviously, every representation set is contained in a complete representation set. That is, if $A \in RS$, then there exists $B \in CRS$ such that $h_i(B) = h_i(A)$, $i = 1, \ldots, r$. It turns out that for the functions $h_1, \ldots, h_r$, complete representation sets entirely characterize unicity sets. To prove this fact we need some auxiliary lemmas.

Lemma 2.1. *Let $Q \subset X$ be a representation set and for some point $x_0 \in Q$ the zero function representation*

$$0 = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in Q,$$

is unique, provided that $g_i(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$. That is, all the functions $g_i \equiv 0$ on the sets $h_i(Q)$, $i = 1, \ldots, r$. Then $Q \in CRS$.

Proof. Assume that $Q \notin CRS$. Then there exists a point $p \in X$ such that $p \notin Q$, $h_i(p) \in h_i(Q)$ for all $i = 1, \ldots, r$ and $Q' = Q \cup \{p\}$ is also a representation set. Consider a function $f_0\colon Q' \to \mathbb{R}$ such that $f_0(q) = 0$ for any $q \in Q$ and $f_0(p) = 1$. Since $Q' \in RS$,

$$f_0(x) = \sum_{i=1}^{r} s_i(h_i(x)), \quad x \in Q'.$$

Then

$$f_0(x) = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in Q', \tag{2.1}$$

where

$$g_i(h_i(x)) = s_i(h_i(x)) - s_i(h_i(x_0)), \quad i = 1, \ldots, r-1,$$

and

$$g_r(h_r(x)) = s_r(h_r(x)) + \sum_{i=1}^{r-1} s_i(h_i(x_0)).$$

A restriction of representation (2.1) to the set Q gives the equality

$$\sum_{i=1}^{r} g_i(h_i(x)) = 0 \quad \text{for all} \quad x \in Q. \tag{2.2}$$

Note that $g_i(h_i(x_0)) = 0,\ i = 1, \ldots, r-1$. It follows from the hypothesis of the lemma that representation (2.2) is unique. Hence, $g_i(h_i(x)) = 0$ for all $x \in Q$ and $i = 1, \ldots, r$. But from (2.1) it follows that

$$\sum_{i=1}^{r} g_i(h_i(p)) = f_0(p) = 1.$$

Since $h_i(p) \in h_i(Q)$ for all $i = 1, \ldots, r$, the above relation contradicts that the functions g_i are identically zero on the sets $h_i(Q),\ i = 1, \ldots, r$. This means that our assumption is not true and $Q \in CRS$.

The following lemma is a strengthened general version of Lemma 2.1.

Lemma 2.1B. *Let $Q \in RS$ and for some point $x_0 \in Q$, numbers $c_1, c_2, \ldots, c_{r-1} \in \mathbb{R}$ and a function $v \in T(Q)$ the representation*

$$v(x) = \sum_{i=1}^{r} v_i(h_i(x))$$

is unique under the initial conditions $v_i(h_i(x_0)) = c_i,\ i = 1, \ldots, r-1$. Then for any numbers $b_1, b_2, \ldots, b_{r-1} \in \mathbb{R}$ and an arbitrary function $f \in T(Q)$ the representation

$$f(x) = \sum_{i=1}^{r} f_i(h_i(x))$$

is also unique, provided that $f_i(h_i(x_0)) = b_i,\ i = 1, \ldots, r-1$. Besides, $Q \in CRS$.

Proof. Assume the contrary. Assume that there exists a function $f \in T(Q)$ having two different representations subject to the same initial conditions. That is,

$$f(x) = \sum_{i=1}^{r} f_i(h_i(x)) = \sum_{i=1}^{r} f_i'(h_i(x))$$

with $f_i(h_i(x_0)) = f_i'(h_i(x_0)) = b_i,\ i = 1, \ldots, r-1$, and $f_i \neq f_i'$ for some indice $i \in \{1, \ldots, r\}$. In this case, the function $v(x)$ will possess the following two different representations:

$$v(x) = \sum_{i=1}^{r} v_i(h_i(x)) = \sum_{i=1}^{r} \left[v_i(h_i(x)) + f_i(h_i(x)) - f_i'(h_i(x)) \right]$$

both satisfying the initial conditions. The obtained contradiction and above Lemma 2.1 complete the proof.

In the sequel, we will assume that for any points $t_i \in h_i(X),\ i = 1, \ldots, r$, the system of equations $h_i(x) = t_i,\ i = 1, \ldots, r$, has at least one solution.

Lemma 2.2. *Let $Q \in CRS$. Then for any point $x_0 \in Q$ the representation*

$$0 = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in Q, \tag{2.3}$$

subject to the conditions

$$g_i(h_i(x_0)) = 0, \quad i = 1, \ldots, r-1, \tag{2.4}$$

is unique. That is, $g_i \equiv 0$ on the sets $h_i(Q),\ i = 1, \ldots, r$.

Proof. Assume the contrary. Assume that representation (2.3) subject to (2.4) is not unique, or in other words, not all of g_i are identically zero. Without loss of generality, we may suppose that $g_r(h_r(y)) \neq 0$ for some $y \in Q$. Let $\xi \in X$ be a solution of the system of equations $h_i(x) = h_i(x_0)$, $i = 1, \ldots, r-1$, and $h_r(x) = h_r(y)$. Therefore, $g_i(h_i(\xi)) = 0$, $i = 1, \ldots, r-1$, and $g_r(h_r(\xi)) \neq 0$. Obviously, $\xi \notin Q$. Otherwise, we may have $g_r(h_r(\xi)) = 0$.

We are going to prove that $Q' = Q \cup \{\xi\}$ is a representation set. For this purpose, consider an arbitrary function $f\colon Q' \to \mathbb{R}$. The restriction of f to the set Q admits a decomposition

$$f(x) = \sum_{i=1}^{r} t_i(h_i(x)), \quad x \in Q.$$

One is allowed to fix the values $t_i(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$. Note that then $t_i(h_i(\xi)) = 0$, $i = 1, \ldots, r-1$. Consider now the functions

$$v_i(h_i(x)) = t_i(h_i(x)) + \frac{f(\xi) - t_r(h_r(\xi))}{g_r(h_r(\xi))} g_i(h_i(x)), \quad x \in Q', \quad i = 1, \ldots, r.$$

It can be easily verified that

$$f(x) = \sum_{i=1}^{r} v_i(h_i(x)), \quad x \in Q'.$$

Since f is arbitrary, we obtain that $Q' \in RS$, where $Q' \supset Q$ and $h_i(Q') = h_i(Q)$, $i = 1, \ldots, r$. But this contradicts the hypothesis of the lemma that $Q \in CRS$.

Theorem 2.1. *$Q \in CRS$ if and only if for any $x_0 \in Q$, any $f \in T(Q)$ and any $a_1, \ldots, a_{r-1} \in \mathbb{R}$ the representation*

$$f(x) = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in Q,$$

subject to the conditions $g_i(h_i(x_0)) = a_i$, $i = 1, \ldots, r-1$, is unique. Equivalently, a set $Q \in CRS$ if and only if it is a unicity set.

Theorem 2.1 is an obvious consequence of Lemmas 2.1B and 2.2.

***Remark* 2.1.** In Theorem 2.1, all the words "any" can be replaced with the word "some".

***Remark* 2.2.** For the case $X = X_1 \times \ldots \times X_n$, the possibility and uniqueness of the representation by sums $\sum_{i=1}^{n} u_i(x_i)$, $u_i\colon X_i \to \mathbb{R}$, $i = 1, \ldots, n$, were investigated in [4] and [5].

Example. Let $r = 2$, $X = \mathbb{R}^2$, $h_1(x_1, x_2) = x_1 + x_2$, $h_2(x_1, x_2) = x_1 - x_2$, Q be the graph of the function $x_2 = \arcsin(\sin x_1)$. The set Q has no closed paths with respect to the functions h_1 and h_2. Therefore, $Q \in RS$. By adding a point $p \notin Q$, we obtain the set $Q \cup \{p\}$, which contain a closed path and hence is not a representation set. Thus, the set $Q \in CRS$ and representation on Q is unique.

Let now $r = 2$, $X = \mathbb{R}^2$, $h_1(x_1, x_2) = x_1$, $h_2(x_1, x_2) = x_2$, and Q be the graph of the function $x_2 = x_1$. Clearly, $Q \in RS$ and $Q \notin CRS$. By the definition of complete representation sets, there is a set $P \supset Q$ such that $P \in RS$ and any set $T \supset P$ is not a representation set. There are many sets P with this property. One of them can be obtained by adding to Q any straight line l parallel to one of the coordinate axes. Indeed, if $y \notin Q \cup l$, then the set $Q_1 = Q \cup l \cup \{y\}$ contains a four-point closed path (with one vertex y, two vertices lying on l and one vertex lying on Q). This means that $Q_1 \notin RS$ and hence $Q \cup l \in CRS$.

The following corollary can be easily obtained from Theorem 2.1 and Lemma 2.1B.

***Corollary* 2.1.** *$Q \in CRS$ if and only if $Q \in RS$ and in the representation*

$$0 = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in Q,$$

all the functions g_i, $i = 1, \ldots, r$, are constants.

We have seen that complete representation sets enjoy the unicity property. Let us study some other properties of the following sets:

(a) If $Q_1, Q_2 \in CRS$, $Q_1 \cap Q_2 \neq \varnothing$ and $Q_1 \cup Q_2 \in RS$, then $Q_1 \cup Q_2 \in CRS$.

(b) Let $\{Q_\alpha\}$, $\alpha \in \Phi$, be a family of complete representation sets such that $\cap_{\alpha\in\Phi} Q_\alpha \neq \varnothing$ and $\cup_{\alpha\in\Phi} Q_\alpha \in RS$. Then $\cup_{\alpha\in\Phi} Q_\alpha \in CRS$.

The above two properties follow from Corollary 2.1. Note that (b) is a generalization of (a). The following property is a consequence of (b) and property (2) of representation sets.

(c) Let $\{Q_\alpha\}$, $\alpha \in \Phi$, be a totally ordered (under inclusion) family of complete representation sets. Then $\cup_{\alpha\in\Phi} Q_\alpha \in CRS$.

We know that every representation set A is contained in a complete representation set Q such that $h_i(A) = h_i(Q)$, $i = 1, \ldots, r$. What can we say about the set $Q\backslash A$? Clearly, $Q\backslash A \in RS$. But can we chose Q so that $Q\backslash A \in CRS$? The following theorem answers this question.

Theorem 2.2. *Let $A \in RS$ and $A \notin CRS$. Then there exists a set $B \in CRS$ such that $A \subset B$, $h_i(A) = h_i(B)$, $i = 1, \ldots, r$, and $B\backslash A \in CRS$.*

Proof. Since the representation set A is not complete, there exists a point $p \notin A$ such that $h_i(p) \in h_i(A)$, $i = 1, \ldots, r$, and $A' = A \cup \{p\} \in RS$. By $\mathcal{M}$ denote the collection of sets M such that

(1) $A \subset M$ and $M \in RS$;

(2) $h_i(M) = h_i(A)$ for all $i = 1, \ldots, r$;

(3) $M\backslash A \in CRS$.

Obviously, $\mathcal{M}$ is not empty. It contains the above set A'. Consider the partial order on $\mathcal{M}$ defined by inclusion. Let $\{M_\beta\}$, $\beta \in \Gamma$, be any chain in $\mathcal{M}$. The set $\cup_{\beta\in\Gamma} M_\beta$ is an upper bound for this chain. To see this, let us check that $\cup_{\beta\in\Gamma} M_\beta$ belongs to $\mathcal{M}$. That is, all the above conditions (1)–(3) are satisfied. Indeed,

(1) $A \subset \cup_{\beta\in\Gamma} M_\beta$ and $\cup_{\beta\in\Gamma} M_\beta \in RS$. This follows from property (2) of representation sets;

(2) $h_i(\cup_{\beta\in\Gamma} M_\beta) = \cup_{\beta\in\Gamma} h_i(M_\beta) = \cup_{\beta\in\Gamma} h_i(A) = h_i(A)$, $i = 1, \ldots, r$;

(3) $\cup_{\beta\in\Gamma} M_\beta\backslash A \in CRS$. This follows from property (c) of complete representation sets and the facts that $M_\beta\backslash A \in CRS$ for any $\beta \in \Gamma$ and the system $\{M_\beta\backslash A\}$, $\beta \in \Gamma$, is totally ordered under inclusion.

Thus we see that any chain in $\mathcal{M}$ has an upper bound. By Zorn's lemma, there are maximal sets in $\mathcal{M}$. Let B be one of such sets. Let us now prove that $B \in CRS$.

Assume on the contrary that $B \notin CRS$. Then by Lemma 2.1B, for any point $x_0 \in B$ the representation

$$0 = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in B, \tag{2.5}$$

subject to the conditions $g_i(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$, is not unique. That is, there is a point $y \in B$ such that for some index i, $g_i(h_i(y)) \neq 0$. Without loss of generality we may assume that $g_r(h_r(y)) \neq 0$. Clearly, y cannot belong to $B\backslash A$, since $B\backslash A \in CRS$ and over complete representation sets, the zero function has a trivial representation provided that conditions (2.4) hold. Thus, $y \in A$. Let $\xi \in X$ be a point such that $h_i(\xi) = h_i(x_0)$, $i = 1, \ldots, r-1$, and $h_r(\xi) = h_r(y)$. The point $\xi \notin B$, otherwise from (2.5) we would obtain that $g_r(h_r(y)) = g_r(h_r(\xi)) = 0$. Following the techniques in the proof of Lemma 2.2, it can be shown that $B_1 = B \cup \{\xi\} \in RS$. Now prove that $B_1 \backslash A \in CRS$. For this purpose, consider the representation

$$0 = \sum_{i=1}^{r} g_i'(h_i(x)), \quad x \in B_1 \backslash A, \tag{2.6}$$

subject to the conditions $g_i'(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$, where x_0 is some point of $B \backslash A$. Such representation holds uniquely on $B \backslash A$, since $B \backslash A \in CRS$. That is, all the functions g_i' are identically zero on $h_i(B \backslash A)$, $i = 1, \ldots, r$. On the other hand, since $g_i'(h_i(\xi)) = g_i'(h_i(x_0)) = 0$ for all $i = 1, \ldots, r-1$ we obtain that $g_r'(h_r(\xi)) = 0$. This means that representation (2.6) subject to the conditions $g_i'(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$, is unique on $B_1 \backslash A$. That is, all the functions g_i' in (2.6) are zero functions on $h_i(B_1 \backslash A)$, $i = 1, \ldots, r$. Hence by Lemma 2.1, $B_1 \backslash A \in CRS$. Thus, $B_1 \in \mathcal{M}$. But the set B was chosen as a maximal set in $\mathcal{M}$. We see that the above assumption $B \notin CRS$ leads us to the contradiction that there is a set $B_1 \in \mathcal{M}$ bigger than the maximal set B. Thus, in fact, $B \in CRS$.

Let A be a representation set. The relation on A defined by setting $x \sim y$ if there is a finite complete representation subset of A containing both x and y, is an equivalence relation. Indeed, it is reflexive and symmetric. It is transitive on the basis of property (a) of complete representation sets. The equivalence classes we call C-orbits. In the case $r = 2$, C-orbits turn into classical orbits considered by D. E. Marshall and A. G. O'Farrell [8], which have a very nice geometric interpretation in terms of bolts (for this terminology see [3, 8]). A classical orbit consists of all possible traces of an arbitrary point in it traveling alternatively in the level sets of h_1 and h_2. In the general setting, one partial case of C-orbits are introduced by A. Klopotowski, M. G. Nadkarni, K. P. S. Rao [5] under the name of related components. The case considered in [5] requires that $A \subset X = X_1 \times \ldots \times X_n$ and h_i be the canonical projections of X onto X_i, $i = 1, \ldots, r$, respectively. Finite complete representation sets containing x and y will be called C-trips connecting x and y. A C-trip of the smallest cardinality connecting x and y will be called a minimal C-trip.

Theorem 2.3. *Let A be a representation set and x and y be any two points of some C-orbit in A. Then there is only one minimal C-trip connecting them.*

Proof. Assume that L_1 and L_2 are two minimal C-trips connecting x and y. By definition, L_1 and L_2 are complete representation sets. Note that $L_1 \cup L_2$ is also complete. Let us prove that the set $L_1 \cap L_2$ is complete. Clearly, $L_1 \cap L_2 \in RS$. Let $x_0 \in L_1 \cap L_2$. In particular, x_0 can be one of the points x and y. Consider the representation

$$0 = \sum_{i=1}^{r} g_i(h_i(x)), \quad x \in L_1 \cap L_2, \tag{2.7}$$

subject to $g_i(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$. On the strength of Lemma 2.1, it is enough to prove that this representation is unique. For $i = 1, \ldots, r$, let g_i' be any extension of g_i from the set $h_i(L_1 \cap L_2)$ to the set $h_i(L_1)$. Construct the function

$$f'(x) = \sum_{i=1}^{r} g_i'(h_i(x)), \quad x \in L_1. \tag{2.8}$$

Since $f'(x) = 0$ on $L_1 \cap L_2$, the following function is well defined:

$$f(x) = \begin{cases} f'(x), & x \in L_1, \\ 0, & x \in L_2. \end{cases}$$

Since $L_1 \cup L_2 \in CRS$, the representation

$$f(x) = \sum_{i=1}^{r} w_i(h_i(x)), \quad x \in L_1 \cup L_2, \tag{2.9}$$

subject to

$$w_i(h_i(x_0)) = 0, \quad i = 1, \ldots, r-1, \tag{2.10}$$

is unique. Besides, since $L_1 \in CRS$ and $g_i'(h_i(x_0)) = g_i(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$, representation (2.8) is unique. This means that for each function g_i, there is only one extension g'. Note that

$$f(x) = f'(x) = \sum_{i=1}^{r} w_i(h_i(x)), \quad x \in L_1.$$

Now from uniqueness of representation (2.8) we obtain

$$w_i(h_i(x)) = g_i'(h_i(x)), \quad i = 1, \ldots, r, \quad x \in L_1. \tag{2.11}$$

A restriction of formula (2.9) to the set L_2 gives

$$0 = \sum_{i=1}^{r} w_i(h_i(x)), \quad x \in L_2. \tag{2.12}$$

Since $L_2 \in CRS$, representation (2.12) subject to conditions (2.10) is unique, whence

$$w_i(h_i(x)) = 0, \quad i = 1, \ldots, r, \quad x \in L_2. \tag{2.13}$$

From (2.11) and (2.13) it follows that

$$g_i(h_i(x)) = g_i'(h_i(x)) = 0, \quad i = 1, \ldots, r, \quad x \in L_1 \cap L_2.$$

Thus, we see that representation (2.7) subject to the conditions $g_i(h_i(x_0)) = 0$, $i = 1, \ldots, r-1$, is unique on the intersection $L_1 \cap L_2$. Therefore by Lemma 2.1, $L_1 \cap L_2 \in CRS$.

Let the cardinalities of L_1 and L_2 be equal to n. Since $x, y \in L_1 \cap L_2$ and $L_1 \cap L_2 \in CRS$, we obtain from the definition of minimal C-trips that the cardinality of $L_1 \cap L_2$ is also n. Hence, $L_1 \cap L_2 = L_1 = L_2$.

Let Q be a representation set. That is, each function $f : Q \to \mathbb{R}$ enjoys representation (1.3). Can we find g_i, $i = 1, \ldots, r$, for a given f? There is a procedure for finding one certain collection of g_i, provided that Q consists of a single C-orbit. That is, any two points of Q can be connected by a C-trip. To show this procedure, take some point $x_0 \in Q$ and fix it. We are going to find g_i from (1.3) and conditions (1.4). Let y be any point Q. To find the values of g_i at the points $h_i(y)$, $i = 1, \ldots, r$, connect x_0 and y by a minimal C-trip $S = \{x_1, \ldots, x_n\}$, where $x_0 = x_1$ and $x_n = y$.

Since S is a complete representation set, equation (1.3) subject to (1.4) has a unique solution on S. That is, we can find $g_i(h_i(y)),\ i = 1, \ldots, r,$ by solving the system of linear equations

$$\sum_{i=1}^{r} g_i(h_i(x_j)) = f(x_j), \quad j = 1, \ldots, n.$$

We see that each minimal C-trip containing x_0 generates a system of linear equations, which is uniquely solvable. Since any point of Q can be connected with x_0 by such a trip, we can find $g_i(t)$ at each point $t \in h_i(Q),\ i = 1, \ldots, r.$

The above procedure can still be effective for some particular representation sets Q consisting of many C-orbits. Let $\{C_\alpha\},\ \alpha \in \Lambda,$ denote the set of all C-orbits of Q. Fix some points $x_\alpha \in C_\alpha,$ $\alpha \in \Lambda,$ one in each orbit. Let y_α be any points of $C_\alpha,\ \alpha \in \Lambda,$ respectively. We can apply the above procedure of finding the values of g_i at each y_α if $h_i(y_\alpha) \neq h_i(y_\beta)$ for all i and $\alpha \neq \beta$. For $h_i(y_\alpha) = h_i(y_\beta)$, one cannot guarantee that after solving the corresponding systems of linear equations (associated with y_α and y_β), the solutions $g_i(h_i(y_\alpha)$ and $g_i(h_i(y_\beta))$ will be equal. That is, for the case $h_i(y_\alpha) = h_i(y_\beta)$, the constructed functions g_i may not be well defined.

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