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LEHMER SEQUENCES IN FINITE GROUPS*

ПОСЛІДОВНОСТІ ЛЕМЕРА У СКІНЧЕННИХ ГРУПАХ

We study the Lehmer sequences modulo m. Moreover, we define the Lehmer orbit and the basic Lehmer orbit of a 2-generator group G for a generating pair $(x, y) \in G$ and examine the lengths of the periods of these orbits. Furthermore, we obtain the Lehmer lengths and the basic Lehmer lengths of the Fox groups $G_{1,t}$ for $t \geq 3$.

Вивчаються послідовності Лемера за модулем m. Крім того, визначено поняття орбіти Лемера та базової орбіти Лемера двогенераторної групи G для породжуючої пари $(x, y) \in G$ та досліджено довжини періодів для цих орбіт. Також встановлено довжини Лемера та базові довжини Лемера для груп Фокса $G_{1,t}$ при $t \geq 3$.

1. Introduction and preliminaries. The Lehmer sequence $U = U(L, M) = \{U_n\}_0^\infty$ is the sequence of integers which is defined by integer constants $L,\ M,\ U_0 = 0,\ U_1 = 1$ and the recurrence

$$U_n = \begin{cases} LU_{n-1} - MU_{n-2} & \text{for} \quad n \quad \text{odd}, \\ U_{n-1} - MU_{n-2} & \text{for} \quad n \quad \text{even}, \end{cases} \tag{1}$$

where $LM \neq 0$ and $K = L - 4M \neq 0$. The sequence U is called a Lehmer sequence and U_n is a Lehmer number. For more information on this sequence, see [6]. The Lehmer numbers and their properties have been studied by some authors (see, for example, [5, 7, 8]).

It is well-known that a sequence is periodic if, after a certain point, it consists only of repetitions of a fixed subsequence. The number of elements in the repeating subsequence is the period of the sequence. A sequence is simply periodic with period *k* if the first *k* elements in the sequence form a repeating subsequence.

The study of Fibonacci sequences in groups began with the earlier work of Wall [9]. In the mid eighties, Wilcox extended the problem to Abelian groups [10]. Campbell, Doostie and Robertson [1] expanded the theory to some simple groups. There they defined the Fibonacci length of the Fibonacci orbit and the basic Fibonacci length of the basic Fibonacci orbit in a 2-generator group. Deveci and Karaduman [4] defined the generalized order-*k* Pell sequences in finite groups and obtained the periods of the generalized order-*k* Pell sequences in dihedral groups D_n. Deveci [3] expanded the concept to the Pell – Padovan sequence and the Jacobsthal – Padovan sequence. Now we extend the concept to the Lehmer sequences.

In this paper, the usual notation p is used for a prime number and the notation $\left\{U^{M,L}\right\}$ is used for the Lehmer sequence U.

2. The Lehmer sequences modulo α. Reducing the Lehmer sequence by a modulus α, we can get a repeating sequence, denoted by

$$\left\{U^{M,L}(\alpha)\right\} = \left\{U_0^{M,L}(\alpha),\ U_1^{M,L}(\alpha),\ U_2^{M,L}(\alpha), \ldots, U_i^{M,L}(\alpha), \ldots\right\},$$

where $U_i^{M,L}(\alpha) = U_i^{M,L} \pmod{\alpha}$. It has the same recurrence relation as in (1).

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Theorem 2.1. *The sequence* $\{U^{M,L}(\alpha)\}$ *is simply periodic if* $M = \pm 1$, *and is periodic otherwise.*

Proof. The sequence repeats since there are only a finite number α^2 of pairs of terms possible, and the recurrence of a pair results in recurrence of all following terms, which impliest that the sequence $\{U^{M,L}(\alpha)\}$ is periodic. From definition of the Lehmer sequence we have

$$MU_{n-2} = \begin{cases} LU_{n-1} - U_n & \text{for } n \text{ odd,} \\ U_{n-1} - U_n & \text{for } n \text{ even,} \end{cases}$$

so if $U_{i+1}^{M,L}(\alpha) \equiv U_{j+1}^{M,L}(\alpha)$, $U_i^{M,L}(\alpha) \equiv U_j^{M,L}(\alpha)$ and $M = \pm 1$, then $U_{i-j+1}^{M,L}(\alpha) \equiv U_1^{M,L}(\alpha)$ and $U_{i-j}^{M,L}(\alpha) \equiv U_0^{M,L}(\alpha)$, which implies that the sequence $\{U^{M,L}(\alpha)\}$ is simply periodic.

Let $k^{M,L}(\alpha)$ denote the smallest period of the sequence $\{U^{M,L}(\alpha)\}$, called the period of the Lehmer sequences modulo α.

Example. We have $\{U^{1,5}(7)\} = \{0, 1, 1, 4, 3, 4,\ 1, 1, 0, 6, 6, 3, 4, 3, 6, 6, 0, 1, 1, 4, \ldots\}$. So, we get $k^{1,5}(7) = 16$.

Theorem 2.2. *If* $m = \prod_{i=1}^{t} p_i^{e_i}$, $t \geq 1$, *where* p_i *are distinct primes, then* $k^{M,L}(m) = lcm[k^{M,L}(p_i^{e_i})]$ $\Big($*where the least common multiple of* $k^{M,L}(p_1^{e_1})$, $k^{M,L}(p_2^{e_2}), \ldots, k^{M,L}(p_t^{e_t})$ *is denoted by* $lcm\,[k^{M,L}(p_i^{e_i})]\Big)$.

Proof. The statement, "$k^{M,L}(p_i^{e_i})$ is the length of the period of $\{U^{M,L}(p_i^{e_i})\}$", implies that the sequence $\{U^{M,L}(p_i^{e_i})\}$ repeats only after blocks of length $u \cdot k^{M,L}(p_i^{e_i})$, $u \in N$, and the statement, "$k^{M,L}(m)$ is the length of the period $\{U^{M,L}(m)\}$", implies that $\{U^{M,L}(p_i^{e_i})\}$ repeats after $k^{M,L}(m)$ terms for all values i. Thus, $k^{M,L}(m)$ is of the form $u \cdot k^{M,L}(p_i^{e_i})$ for all values of i, and since any such number gives a period of $\{U^{M,L}(m)\}$. Then we get that $k^{M,L}(m) = lcm[k^{M,L}(p_i^{e_i})]$.

Theorem 2.3. *If* $k^{M,L}(p^2) \neq k^{M,L}(p)$ *and* $M = \pm 1$, *then* $k^{M,L}(p^2) = p.k^{M,L}(p)$.

Proof. Let $k^{M,L}(p^2) \neq k^{M,L}(p)$ and $M = \pm 1$, then the sequence $\{U^{M,L}\}$ is

$$U_0^{M,L} = 0, \qquad U_1^{M,L} = 1, \ldots,$$

$$U_{k^{M,L}(p)}^{M,L} = \lambda_1.p, \qquad U_{k^{M,L}(p)+1}^{M,L} = \lambda_2.p + 1, \ldots,$$

$$U_{2.k^{M,L}(p)}^{M,L} = \lambda_1.2p, \qquad U_{2.k^{M,L}(p)+1}^{M,L} = \lambda_2.2p + 1, \ldots,$$

$$U_{p.k^{M,L}(p)}^{M,L} = \lambda_1.p^2, \qquad U_{p.k^{M,L}(p)+1}^{M,L} = \lambda_2.p^2 + 1, \ldots,$$

where $\lambda_1, \lambda_2 \in N$ such that $p \nmid gcd(\lambda_1, \lambda_2)$ $\big($where by $p \nmid gcd(\lambda_1, \lambda_2)$ we mean that p not divides greatest common divisor λ_1 and $\lambda_2\big)$. Since the elements succeeding $U_{p.k^{M,L}(p)}^{M,L} \equiv 0$ and $U_{p.k^{M,L}(p)+1}^{M,L} \equiv 1$, the cycles begins again with the $p^{2\text{nd}}$ element, i.e., $U_{p.k^{M,L}(p)}^{M,L} \equiv U_0^{M,L}$ and $U_{p.k^{M,L}(p)+1}^{M,L} \equiv U_1^{M,L}$. Then we get that $k^{M,L}(p^2) = p.k^{M,L}(p)$.

Conjecture 2.1. (i) *If* $p \neq 2$, $k^{M,L}(p^{t+1}) \neq k^{M,L}(p^t)$, $t \geq 1$, *and* $M = \pm 1$, *then* $k^{M,L}(p^{t+1}) = p.k^{M,L}(p^t)$.

(ii) *If* $k^{M,L}(2^{t+1}) \neq k^{M,L}(2^t)$, $t \geq 2$, *and* $M = \pm 1$, *then* $k^{M,L}(2^{t+1}) = 2.k^{M,L}(2^t)$.

3. The Lehmer length and the basic Lehmer length of generating pairs in groups. Let G be a group and let $x, y \in G$. If every element of G can be written as a word

$$x^{u_1}\, y^{u_2}\, x^{u_3}\, y^{u_4} \ldots x^{u_{m-1}}\, y^{u_m}, \tag{2}$$

where $u_i \in Z$, $1 \leq i \leq m$, then we say that x and y generate G and that G is a 2-generator group. Let G be a finite 2-generator group and X be the subset of $G \times G$ such that $(x, y) \in X$ if, and only if, G is generated by x and y. We call (x, y) a generating pair for G.

Definition 3.1. *For a generating pair $(x, y) \in G$, we define the Lehmer orbit $U_{x,y}^{M,L}(G) = \{x_i\}$ as follows*:

$$x_0 = x, \qquad x_1 = y, \qquad x_{i+1} = \begin{cases} (x_{i-1})^{-M}(x_i)^L & for\ i\ \ even, \\ (x_{i-1})^{-M}(x_i) & for\ i\ \ odd, \end{cases} \qquad i \geq 1.$$

Theorem 3.1. *A Lehmer orbit $U_{x,y}^{M,L}(G)$ of a finite group is simply periodic if $M = \pm 1$, and is periodic otherwise.*

Proof. Let n be the order of G. Since there are n^2 distinct 2-tuples of elements of G, at least one of the 2-tuples appears twice in a Lehmer orbit of G. Thus, the subsequence following this 2-tuples. Because of the repeating, the Lehmer orbit is periodic.

Since the Lehmer orbit is periodic, there exist natural numbers u and v, with $u > v$, such that

$$x_{u+1} = x_{v+1}, \qquad x_{u+2} = x_{v+2}.$$

By the defining relation of the Lehmer orbit, we know that

$$(x_u)^{-M} = \begin{cases} (x_{u+2})(x_{u+1})^{-L} & \text{for } v \quad \text{odd}, \\ (x_{u+2})(x_{u+1})^{-1} & \text{for } v \quad \text{even}, \end{cases} \quad \text{and} \quad (x_v)^{-M} = \begin{cases} (x_{v+2})(x_{v+1})^{-L} & \text{for } v \quad \text{odd}, \\ (x_{v+2})(x_{v+1})^{-1} & \text{for } v \quad \text{even}. \end{cases}$$

Hence, $x_u = x_v$ for $M = \pm 1$, and it then follows that

$$x_{u-v} = x_{v-v} = x_0, \qquad x_{u-v+1} = x_{v-v+1} = x_1.$$

Thus, the Lehmer orbit $U_{x,y}^{M,L}(G)$ is simply periodic for $M = \pm 1$.

In this paper, we denote the length of the period of the Lehmer orbit $U_{x,y}^{M,L}(G)$ by $\operatorname{Len} U_{x,y}^{M,L}(G)$ and we call the Lehmer length of G with respect to generating pair (x, y) and integer constants L, M.

Lemma 3.1. *If $M = \pm 1$ and the Lehmer orbit $U_{x,y}^{M,L}(G)$ of $(x, y) \in X$ has length n_1, then for any i, $0 \leq i \leq n_1 - 1$, we have $(x_i, x_{i+1}) \in X$. Also we have $U_{x,y}^{M,L}(G) = U_{x_i,y_i}^{M,L}(G)$.*

Proof. We will use the induction method on i to show $(x_i, x_{i+1}) \in X$. The case $i = 0$ is trivially true. Suppose by way of inductive hypothesis that $(x_k, x_{k+1}) \in X$ and consider (x_{k+1}, x_{k+2}). Now

$$(x_k)^{-M} = \begin{cases} (x_{k+2})(x_{k+1})^{-L} & \text{for } k \quad \text{odd}, \\ (x_{k+2})(x_{k+1})^{-1} & \text{for } k \quad \text{even}, \end{cases}$$

so, since every element of G has an expression of the form (2) with $x_k = x$, $x_{k+1} = y$, we see that, on replacing $(x_k)^{-M}$ by

$$\begin{cases} (x_{k+2})(x_{k+1})^{-L} & \text{for } k \ \text{odd}, \\ (x_{k+2})(x_{k+1})^{-1} & \text{for } k \ \text{even}, \end{cases}$$

every element of G is generated by x_{k+1} and x_{k+2}.

Finally suppose $U_{x,y}^{M,L}(G) = \{x_i\}$ and $U_{r,s}^{M,L}(G) = \{b_i\}$. Then again an inductive argument proves that if $x_0 = b_j$, $x_1 = b_{j+1}$, then $U_{x,y}^{M,L}(G) = U_{r,s}^{M,L}(G)$.

For suppose $x_i = b_{i+j}$, $i < t$. Then $x_t = (x_{t-2})^{-1}(x_{t-1})^2 = (b_{t-2+j})^{-1}(b_{t-1+j})^2 = b_{j+t}$ and the result is proved.

Lemma 3.1 gives immediately the following theorem.

Theorem 3.2. *If $M = \pm 1$ and G is a finite group, then X partitioned by and the Lehmer orbits $U_{x,y}^{M,L}(G)$ for $(x, y) \in X$.*

To examine the concept more fully we study the action of the automorphism group $\operatorname{Aut} G$ of G on X and on the Lehmer orbits $U_{x,y}^{M,L}(G)$, $(x, y) \in X$. Now $\operatorname{Aut} G$ consist of all isomorphisms $\theta\colon G \to G$ and if $\theta \in \operatorname{Aut} G$ and $(x, y) \in X$, then $(x\theta, y\theta) \in X$.

For a subset $A \subseteq G$ and $\theta \in \operatorname{Aut} G$ the image of A under θ is $A\theta = \{a\theta \colon a \in A\}$.

Lemma 3.2. *Let $(x, y) \in X$ and $\theta \in \operatorname{Aut} G$. If $M = \pm 1$, then $U_{x,y}^{M,L}(G)\theta = U_{x\theta,y\theta}^{M,L}(G)$.*

Proof. Let $U_{x,y}^{M,L}(G) = \{x_i\}$. Now $\{x_i\}\theta = \{x_i\theta\}$ and since

$$\left((x_{i-1})^{-M}(x_i)^L\right)\theta = (x_{i-1})^{-M}\theta(x_i)^L\theta \quad \text{and} \quad \left((x_{i-1})^{-M}(x_i)\right)\theta = (x_{i-1})^{-M}\theta(x_i)\theta$$

the result follows.

If $M = \pm 1$ and n of the elements of $\operatorname{Aut} G$ map $U_{x,y}^{M,L}(G)$ into itself. Then there are $|\operatorname{Aut} G|/n$ distinct Lehmer orbits $U_{x\theta,y\theta}^{M,L}(G)$ for $\theta \in \operatorname{Aut} G$.

Definition 3.2. *For a generating pair $(x, y) \in X$ and $M = \pm 1$, we define the basic Lehmer orbits $\overline{U_{x,y}^{M,L}}(G)$ of basic length m to be the sequence $\{x_i\}$ of elements of G such that*

$$x_0 = x, \qquad x_1 = y, \qquad x_{i+1} = \begin{cases} (x_{i-1})^{-M}(x_i)^L & \text{for } i \text{ even}, \\ (x_{i-1})^{-M}(x_i) & \text{for } i \text{ odd}, \end{cases} \qquad i \geq 1,$$

where $m \geq 1$ is least integer with

$$x_0 = x_m\theta, \qquad x_1 = x_{m+1}\theta,$$

for some $\theta \in \operatorname{Aut} G$.

Since x_m, x_{m+1} generate G, it follows that θ is uniquely determined.

In this paper, we denote the length of the period of the the basic Lehmer orbit $\overline{U_{x,y}^{M,L}}(G)$ by $\overline{\operatorname{Len} U_{x,y}^{M,L}}(G)$ and we call the basic Lehmer length of G with respect to generating pair (x, y) and integer constants L, M.

From the definitions it is clear that the Lehmer lengths and the basic Lehmer lengths of a group depend on the chosen generating set and the order in which the assignments of x_0, x_1 are made.

Theorem 3.3. *Let G be a finite group and $(x, y) \in X$. If $M = \pm 1$, the orbit $U_{x,y}^{M,L}(G)$ has length n_1 and the basic orbit $\overline{U_{x,y}^{M,L}}(G)$ has length m_1, then m_1 divides n_1 and there n_1/m_1 elements of $\operatorname{Aut} G$ which map $U_{x,y}^{M,L}(G)$ into itself.*

Proof. Since $U_{x,y}^{M,L}(G) = \overline{U_{x,y}^{M,L}}(G) \cup \overline{U_{x\theta,y\theta}^{M,L}}(G) \cup \overline{U_{x\theta^2,y\theta^2}^{M,L}}(G) \cup \ldots$ and $\overline{\operatorname{Len} U_{x,y}^{M,L}}(G) = \overline{\operatorname{Len} U_{x\theta,y\theta}^{M,L}}(G)$ we have $n_1 = m_1 \cdot \lambda$, where λ is order of automorphism $\theta \in \operatorname{Aut} G$. Clearly $1, \theta, \theta^2, \ldots, \theta^{\lambda-1}$ map $U_{x,y}^{M,L}(G)$ into itself.

4. The Lehmer lengths and the basic Lehmer lengths of the Fox groups. The Fox groups $G_{1,t}$, are finite metacyclic groups of order $|t-1|^3$, having generators of order $(t-1)^2$ (see [2]). They are presented by

$$\langle x,\, y : xy = y^t x,\ yx = x^t y\rangle.$$

The relations of $G_{1,t}$ imply the relation $x^{t-1} = y^{1-t}$.

In this section, we obtain the Lehmer lengths and the basic Lehmer lengths of $G_{1,t}$ for $M = \pm 1$ and $t \geq 3$.

Theorem 4.1. (i) *Let $t = 3$, then tree cases occur*:

(1) *If $M = 1$ and L is an integer such that $L \neq 0$, then*

$$\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = \begin{cases} 4, & L \equiv 0 \pmod{4}, \\ 3, & L \equiv 1 \pmod{4}, \\ 8, & L \equiv 2 \pmod{4}, \\ 6, & L \equiv 3 \pmod{4} \end{cases}$$

and

$$\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = \begin{cases} 2, & L \equiv 0 \pmod{4}, \\ 1, & L \equiv 1 \pmod{4}, \\ 2, & L \equiv 2 \pmod{4}, \\ 2, & L \equiv 3 \pmod{4}. \end{cases}$$

(2) *If $M = -1$ and L is an integer such that $L > 0$, then*

$$\operatorname{Len} U_{x,y}^{-1,L}(G_{1,3}) = \begin{cases} 8, & L \equiv 0 \pmod{4}, \\ 3, & L \equiv 1 \pmod{4}, \\ 4, & L \equiv 2 \pmod{4}, \\ 3, & L \equiv 3 \pmod{4} \end{cases}$$

and

$$\overline{\operatorname{Len} U_{x,y}^{-1,L}}(G_{1,3}) = \begin{cases} 2, & L \equiv 0 \pmod{4}, \\ 1, & L \equiv 1 \pmod{4}, \\ 2, & L \equiv 2 \pmod{4}, \\ 1, & L \equiv 3 \pmod{4}. \end{cases}$$

(3) *If $M = -1$ and L is an integer such that $L < 0$, then*

$$\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = \begin{cases} 8, & L \equiv 0 \pmod{4}, \\ 3, & L \equiv 1 \pmod{4}, \\ 4, & L \equiv 2 \pmod{4}, \\ 6, & L \equiv 3 \pmod{4} \end{cases}$$

and

$$\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = \begin{cases} 2, & L \equiv 0 \pmod 4, \\ 1, & L \equiv 1 \pmod 4, \\ 2, & L \equiv 2 \pmod 4, \\ 2, & L \equiv 3 \pmod 4. \end{cases}$$

(ii) *Let* $t \geq 4$, *then two cases occur*:

($1'$) *If* $k^{M,L}\big((t-1)^2\big) = k^{M,L}(t-1)$, *then* $\operatorname{Len} U_{x,y}^{M,L}(G_{1,t}) = \overline{\operatorname{Len} U_{x,y}^{M,L}}(G_{1,t}) = k^{M,L}(t-1)$.

($2'$) *If* $k^{M,L}\big((t-1)^2\big) \neq k^{M,L}(t-1)$, *then* $\operatorname{Len} U_{x,y}^{M,L}(G_{1,t}) = k^{M,L}\big((t-1)^2\big)$ *and* $\overline{\operatorname{Len} U_{x,y}^{M,L}}(G_{1,t}) = k^{M,L}(t-1)$.

Proof. i (1) Let $M = 1$ and L is an integer such that $L \neq 0$.

If $L \equiv 0 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ yx,\ y^{-1},\ x,\ y,\ \ldots .$$

So we get $\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = 4$ and $\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = 2$ since $x\theta = yx$ and $y\theta = y^{-1}$, where θ is a outher automorphism of order 2.

If $L \equiv 1 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ yx,\ x,\ y, \ldots .$$

So we get $\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = 3$ and $\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = 1$ since $x\theta = yx$ and $y\theta = x$, where θ is a outher automorphism of order 3.

If $L \equiv 2 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ yx,\ y,\ x^{-1},\ y,\ xy,\ y,\ x,\ y, \ldots .$$

So we get $\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = 8$ and $\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = 2$ since $x\theta = xy$ and $y\theta = y$, where θ is a outher automorphism of order 4.

If $L \equiv 3 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ yx,\ x^{-1},\ y^{-1},\ xy,\ x,\ y,\ \ldots .$$

So we get $\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = 6$ and $\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = 2$ since $x\theta = y^{-1}$ and $y\theta = xy$, where θ is a outher automorphism of order 3.

(2) Let $M = -1$ and L is an integer such that $L > 0$.

If $L \equiv 0 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ xy,\ y,\ x^{-1},\ y,\ yx,\ y,\ x,\ y,\ \ldots .$$

So we get $\operatorname{Len} U_{x,y}^{-1,L}(G_{1,3}) = 8$ and $\overline{\operatorname{Len} U_{x,y}^{-1,L}}(G_{1,3}) = 2$ since $x\theta = yx$ and $y\theta = y$, where θ is a outher automorphism of order 4.

If $L \equiv 2 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ xy,\ y^{-1},\ x,\ y,\ \ldots.$$

So we get $\operatorname{Len} U_{x,y}^{-1,L}(G_{1,3}) = 4$ and $\overline{\operatorname{Len} U_{x,y}^{-1,L}}(G_{1,3}) = 2$ since $x\theta = xy$ and $y\theta = y^{-1}$, where θ is a outher automorphism of order 2.

If $L \equiv 1 \pmod 4$ or $L \equiv 3 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ xy,\ x,\ y, \ldots.$$

So we get $\operatorname{Len} U_{x,y}^{-1,L}(G_{1,3}) = 3$ and $\overline{\operatorname{Len} U_{x,y}^{-1,L}}(G_{1,3}) = 1$ since $x\theta = xy$ and $y\theta = x$, where θ is a outher automorphism of order 3.

(3) Let $M = -1$ and L is an integer such that $L < 0$.

If $L \equiv 0 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ xy,\ y,\ x^{-1},\ y,\ yx,\ y,\ x,\ y, \ldots.$$

So we get $\operatorname{Len} U_{x,y}^{-1,L}(G_{1,3}) = 8$ and $\overline{\operatorname{Len} U_{x,y}^{-1,L}}(G_{1,3}) = 2$ since $x\theta = yx$ and $y\theta = y$, where θ is a outher automorphism of order 4.

If $L \equiv 1 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ xy,\ x,\ y, \ldots.$$

So we get $\operatorname{Len} U_{x,y}^{-1,L}(G_{1,3}) = 3$ and $\overline{\operatorname{Len} U_{x,y}^{-1,L}}(G_{1,3}) = 1$ since $x\theta = xy$ and $y\theta = x$, where θ is a outher automorphism of order 3.

If $L \equiv 2 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ xy,\ y^{-1},\ x,\ y, \ldots.$$

So we get $\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = 4$ and $\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = 2$ since $x\theta = xy$ and $y\theta = y^{-1}$, where θ is a outher automorphism of order 2.

If $L \equiv 3 \pmod 4$, then the Lehmer orbit is

$$x,\ y,\ xy,\ x^{-1},\ y^{-1},\ yx,\ x,\ y, \ldots.$$

So we get $\operatorname{Len} U_{x,y}^{1,L}(G_{1,3}) = 6$ and $\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,3}) = 2$ since $x\theta = y^{-1}$ and $y\theta = yx$, where θ is a outher automorphism of order 3.

ii (1′) The proof is similar to the proof of the Theorem 3.1 in [2] and is omitted.

(2′) If $k^{M,L}\big((t-1)^2\big) \neq k^{M,L}(t-1)$, then there are 3 subcases:

Case 1. If $M = 1$ and L is an integer such that $L > 0$ and $M = -1$ and L is an integer such that $L < 0$, then the Lehmer orbit $U_{x,y}^{M,L}(G_{1,t})$ is

$$x_0 = x, \qquad x_1 = y, \ldots,$$

$$x_{k^{M,L}(t-1)} = x^{-t^2+3t-1}, \qquad x_{k^{M,L}(t-1)+1} = y, \ldots,$$

$$x_{(t-a)k^{M,L}(t-1)} = x^{(-a+1)t+a}, \qquad x_{(t-a)k^{M,L}(t-1)+1} = y, \ldots,$$

$$x_{(t-1)k^{M,L}(t-1)} = x_{k^{M,L}((t-1)^2)} = x, \qquad x_{(t-1)k^{M,L}(t-1)+1} = x_{k^{M,L}\left((t-1)^2\right)+1} = y, \ldots,$$

where $2 \leq a \leq t-2$.

So we get $\operatorname{Len} U_{x,y}^{M,L}(G_{1,t}) = k^{M,L}\left((t-1)^2\right)$ and $\overline{\operatorname{Len} U_{x,y}^{M,L}}(G_{1,t}) = k^{M,L}(t-1)$ since $x\theta = x^{-t+2}$ and $y\theta = y$, where θ is the inner automorphism induced by conjugation by y^{t-2}.

Case 2. If $M = 1$ and L is an integer such that $L < 0$, then the Lehmer orbit $U_{x,y}^{1,L}(G_{1,t})$ is

$$x_0 = x, \qquad x_1 = y, \ldots,$$

$$x_{k^{1,L}(t-1)} = x^{(t)^{t-2}}, \qquad x_{k^{1,L}(t-1)+1} = y, \ldots,$$

$$x_{(t-a)k^{1,L}(t-1)} = x^{(t)^{a-1}}, \qquad x_{(t-a)k^{1,L}(t-1)+1} = y, \ldots,$$

$$x_{(t-1)k^{1,L}(t-1)} = x_{k^{1,L}\left((t-1)^2\right)} = x, \qquad x_{(t-1)k^{1,L}(t-1)+1} = x_{k^{1,L}\left((t-1)^2\right)+1} = y, \ldots,$$

where $2 \le a \le t-2$.

So we get $\operatorname{Len} U_{x,y}^{1,L}(G_{1,t}) = k^{1,L}\left((t-1)^2\right)$ and $\overline{\operatorname{Len} U_{x,y}^{1,L}}(G_{1,t}) = k^{1,L}(t-1)$ since $x\theta = x^t$ and $y\theta = y$, where θ is the inner automorphism induced by conjugation by y.

Case 3. If $M = -1$ and L is an integer such that $L > 0$, then the Lehmer orbit $U_{x,y}^{1,L}(G_{1,t})$ is

$$x_0 = x, \qquad x_1 = y, \ldots,$$

$$x_{k^{1,L}(t-1)} = x^{(t)^{t-2}}, \qquad x_{k^{1,L}(t-1)+1} = y^{-t^2+3t-1}, \ldots,$$

$$x_{(t-a)k^{1,L}(t-1)} = x^{(t)^{a-1}}, \qquad x_{(t-a)k^{1,L}(t-1)+1} = y^{(-a+1)t+a}, \ldots,$$

$$x_{(t-1)k^{1,L}(t-1)} = x_{k^{1,L}((t-1)^2)} = x, \qquad x_{(t-1)k^{1,L}(t-1)+1} = x_{k^{1,L}((t-1)^2)+1} = y, \ldots,$$

where $2 \le a \le t-2$.

So we get $\operatorname{Len} U_{x,y}^{-1,L}(G_{1,t}) = k^{-1,L}\left((t-1)^2\right)$ and $\overline{\operatorname{Len} U_{x,y}^{-1,L}}(G_{1,t}) = k^{-1,L}(t-1)$ since $x\theta = x^t$ and $y\theta = y^{-t+2}$, where θ is a outher automorphism of order $t-1$.

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