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L_p-DUAL MIXED AFFINE SURFACE AREAS*

L_p-ДУАЛЬНІ МІШАНІ АФІННІ ПОВЕРХНЕВІ ПЛОЩІ

Lutwak proposed the notion of L_p-affine surface area according to the L_p-mixed volume. Recently, Wang and He introduced the concept of L_p-dual affine surface area combined with the L_p-dual mixed volume. In the article, we give the concept of L_p-dual mixed affine surface areas associated with the L_p-dual mixed quermassintegrals. Further, some inequalities for the L_p-dual mixed affine surface areas are obtained.

Лутвок запропонував поняття L_p-афінної поверхневої площі, що відповідає поняттю L_p-мішаного об'єму. Нещодавно Ванг і Хе ввели поняття L_p-дуальної афінної поверхневої площі, пов'язаної з L_p-дуальним мішаним об'ємом. В роботі запропоновано поняття L_p-дуальної мішаної афінної поверхневої площі, що відповідає L_p-дуальним мішаним квермасінтегралам. Крім того, наведено деякі нерівності для L_p-дуальних мішаних афінних поверхневих площ.

1. Introduction and main results. Let $\mathcal{K}^n$ denote the set of convex bodies (compact, convex subsets with nonempty interiors) in Euclidean space $\mathbb{R}^n$. For the set of convex bodies containing the origin in their interiors, the set of centroid of convex bodies is the origin and the set of origin-symmetric convex bodies in $\mathbb{R}^n$, we write $\mathcal{K}^n_o$, $\mathcal{K}^n_c$ and $\mathcal{K}^n_{os}$, respectively. Let $\mathcal{S}^n_o$ denotes the set of star bodies (about the origin) in $\mathbb{R}^n$. Let S^{n-1} denotes the unit sphere in $\mathbb{R}^n$, denote by $V(K)$ the n-dimensional volume of body K, for the standard unit ball B in $\mathbb{R}^n$, denote $\omega_n = V(B)$.

The studies of the classical affine surface area went back to Blaschke [1]. The notion of classical affine surface area was extended to convex bodies by Leichtweiß [5]. For $K \in \mathcal{K}^n$, the affine surface area, $\Omega(K)$, of K is defined by

$$n^{-1/n}\Omega(K)^{\frac{n+1}{n}} = \inf\{nV_1(K, Q^*)V(Q)^{1/n} : Q \in S^n_o\}. \tag{1.1}$$

Here Q^* denotes the polar of body Q. Subsequently, Lutwak [10] introduced mixed affine surface areas. On the researches of classical affine surface areas, also see [6].

The L_p-affine surface areas were introduced by Lutwak [13]: for $K \in \mathcal{K}^n_o$, $p \geq 1$, the L_p-affine surface area, $\Omega_p(K)$, of K is defined by

$$n^{-p/n}\Omega_p(K)^{\frac{n+p}{n}} = \inf\{nV_p(K, Q^*)V(Q)^{p/n} : Q \in S^n_o\}.$$

Here $V_p(M, N)$ denotes the L_p-mixed volume of $M, N \in \mathcal{K}^n_o$ (see [12, 13]). Obviously, if $p = 1$, $\Omega_p(K)$ is just the affine surface area $\Omega(K)$ of K.

In addition, Lutwak [13] also gave the notion of L_p-mixed affine surface areas. Moreover, Wang and Leng in [16] defined L_p-mixed affine surface area, $\Omega_{p,i}(K)$, of K (for $i = 0$, $\Omega_{p,i}(K)$ is just the L_p-affine surface area $\Omega_p(K)$) and extended some Lutwak's results. Regarding the studies of L_p-affine surface areas, besides see [13, 16], also see [17 – 21]. Recently, Ludwig [7, 8] extended L_p-affine surface areas to L_ϕ-affine surface areas.

Because the definition of L_p-affine surface area base on the L_p-mixed volume. In 2008, Wang and He [14] showed the notion of L_p-dual affine surface area associated with the L_p-dual mixed

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volume. For $K \in \mathcal{S}_o^n$, and $1 \leq p < n$, the L_p-dual affine surface area, $\widetilde{\Omega}_{-p}(K)$, of K is defined by

$$n^{p/n}\widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} = \inf\{n\widetilde{V}_{-p}(K,Q^*)V(Q)^{-p/n} : Q \in \mathcal{K}_c^n\}. \tag{1.2}$$

Here $\widetilde{V}_{-p}(M,N)$ denotes the L_p-dual mixed volume of $M,N \in \mathcal{S}_o^n$ [13].

Associated with the L_p-dual affine surface areas, Wang and He [14] proved the following dual forms of Lutwak's results:

Theorem 1.A. *If $K \in \mathcal{S}_o^n$, $n > p \geq 1$, then*

$$\widetilde{\Omega}_{-p}(K)^{n-p} \geq n^{n-p}\omega_n^{-2p}V(K)^{n+p}$$

with equality if and only if K is an ellipsoid.

Theorem 1.B. *If $K \in \mathcal{K}_c^n$, $n > p \geq 1$, then*

$$\widetilde{\Omega}_{-p}(K)\widetilde{\Omega}_{-p}(K^*) \leq n^2\omega_n^2$$

with equality if and only if K is an ellipsoid.

Theorem 1.C. *If $K \in \mathcal{S}_o^n$, $1 \leq p \leq q \leq n$, then*

$$\left(\frac{\Omega_{-p}(K)^{n-p}}{n^{n-p}V(K)^{n+p}}\right)^{1/p} \leq \left(\frac{\Omega_{-q}(K)^{n-q}}{n^{n-q}V(K)^{n+q}}\right)^{1/q}.$$

Here

$$\left(\frac{\Omega_{-p}(K)^{n-p}}{n^{n-p}V(K)^{n+p}}\right)^{1/p} \tag{1.3}$$

be called the L_p-dual affine area ratio of $K \in \mathcal{S}_o^n$ (see [14]*).*

Recall that Wang and Leng in [15] extended the notion of L_p-dual mixed volume and gave the definition of L_p-dual mixed quermassintegrals. The main aim of this article is to define the L_p-dual mixed affine surface area by the L_p-dual mixed quermassintegrals. Further, we extend Wang and He's results.

Now we give the concept of L_p-dual mixed affine surface areas as follows: for $K \in \mathcal{S}_o^n$, $p \geq 1$, real $i \neq n$, the L_p-dual mixed affine surface area, $\widetilde{\Omega}_{-p,i}(K)$, of K is defined by

$$n^{\frac{p}{n-i}}\widetilde{\Omega}_{-p,i}(K)^{\frac{n-p-i}{n-i}} = \inf\left\{n\widetilde{W}_{-p,i}(K,Q^*)\widetilde{W}_i(Q)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n\right\}. \tag{1.4}$$

Here $\widetilde{W}_{-p,i}(M,N)$ denote the L_p-dual mixed quermassintegrals of $M,N \in \mathcal{S}_o^n$.

According to definitions (1.2), (1.4) and equality (2.11), we easily know that for $K \in \mathcal{S}_o^n$,

$$\widetilde{\Omega}_{-p,0}(K) = \widetilde{\Omega}_{-p}(K). \tag{1.5}$$

Associated with the L_p-dual mixed affine surface areas, we give the general forms of Theorems 1.A, 1.B and 1.C. Our main results can be stated as follows, respectively.

Theorem 1.1. *If $K \in \mathcal{S}_o^n$, $p \geq 1$ and $0 \leq i < n$, then*

$$\widetilde{\Omega}_{-p,i}(K)^{n-p-i} \geq n^{n-p-i}\omega_n^{-2p}\widetilde{W}_i(K)^{n+p-i} \tag{1.6}$$

with equality for $i = 0$ if and only if K is an ellipsoid, for $0 < i < n$ if and only if K is a ball.

Theorem 1.2. *If $K \in \mathcal{K}_c^n$, $p \geq 1$, and $0 \leq i < n-p$, then*

$$\widetilde{\Omega}_{-p,i}(K)\widetilde{\Omega}_{-p,i}(K^*) \leq n^2\omega_n^2 \tag{1.7}$$

with equality for $i=0$ if and only if K is an ellipsoid, for $0<i<n$ if and only if K is a ball.

Theorem 1.3. *If $K \in \mathcal{S}_o^n$, $1 \leq p \leq q$, i is a real and $0 \leq i < n$, then*

$$\left(\frac{\widetilde{\Omega}_{-p,i}(K)^{n-p-i}}{n^{n-p-i}\widetilde{W}_i(K)^{n+p-i}}\right)^{1/p} \leq \left(\frac{\widetilde{\Omega}_{-q,i}(K)^{n-q-i}}{n^{n-q-i}\widetilde{W}_i(K)^{n+q-i}}\right)^{1/q}. \tag{1.8}$$

Similar to the definitions (1.1) and (1.3),

$$\left(\frac{\widetilde{\Omega}_{-p,i}(K)^{n-p-i}}{n^{n-p-i}\widetilde{W}_i(K)^{n+p-i}}\right)^{1/p}$$

may be called the L_p-dual mixed affine area ratio of $K \in \mathcal{S}_o^n$.

Finally, we give the following Brunn – Minkowski-type inequality for the L_p-dual mixed affine surface areas.

Theorem 1.4. *If $K, L \in S_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), real i satisfies $n+p < i <$ $< n+2p$, then*

$$\widetilde{\Omega}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L)^{-\frac{p(n-p-i)}{(n-i)(n+p-i)}} \geq \lambda\widetilde{\Omega}_{-p,i}(K)^{-\frac{p(n-p-i)}{(n-i)(n+p-i)}} + \mu\widetilde{\Omega}_{-p,i}(L)^{-\frac{p(n-p-i)}{(n-i)(n+p-i)}} \tag{1.9}$$

with equality if and only if K and L are dilates. Here $\lambda \cdot K +_{-p} \mu \cdot L$ denotes the L_p-harmonic radial combination of K and L.

2. Preliminaries. ***2.1. Radial function and polar of convex bodies.*** If K is a compact star-shaped (about the origin) in R^n, then its radial function, $\rho_K = \rho(K, \cdot)\colon R^n \setminus \{0\} \longrightarrow [0, \infty)$, is defined by (see [2, 20])

$$\rho(K,u) = \max\{\lambda \geq 0\colon \lambda \cdot u \in K\}, \quad u \in S^{n-1}.$$

If ρ_K is continuous and positive, then K will be called a star body. Two star bodies K, L are said to be dilates (of one another) if $\rho_K(u)/\rho_L(u)$ is independent of $u \in S^{n-1}$.

If $K \in \mathcal{K}_o^n$, the polar body, K^*, of K is defined by (see [2, 20])

$$K^* = \{x \in R^n\colon x \cdot y \leq 1, y \in K\}.$$

Obviously, for $K \in \mathcal{K}_o^n$,

$$(K^*)^* = K. \tag{2.1}$$

2.2. L_p-dual mixed quermassintegrals. For $K, L \in S_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-harmonic radial combination, $\lambda \cdot K +_{-p} \mu \cdot L \in S_o^n$, of K and L is defined by (see [13])

$$\rho(\lambda \cdot K +_{-p} \mu \cdot L, \cdot)^{-p} = \lambda\rho(K,\cdot)^{-p} + \mu\rho(L,\cdot)^{-p}. \tag{2.2}$$

For $K \in \mathcal{S}_o^n$ and any real i, the dual quermassintegrals, $\widetilde{W}_i(K)$, of K are defined by (see [9])

$$\widetilde{W}_i(K) = \frac{1}{n}\int\limits_{S^{n-1}} \rho(K,u)^{n-i} dS(u). \tag{2.3}$$

Obviously,

$$\widetilde{W}_0(K) = \frac{1}{n} \int\limits_{S^{n-1}} \rho(K,u)^n dS(u) = V(K). \tag{2.4}$$

For $K \in \mathcal{K}_o^n$ and its polar body, Ghandehari (see [3]) established an upper bound of the dual quermassintegrals product as follows:

Theorem 2.A. *If $K \in \mathcal{K}_c^n$, i is any real and $0 \le i < n$, then*

$$\widetilde{W}_i(K)\widetilde{W}_i(K^*) \le \omega_n^2 \tag{2.5}$$

with equality for $i = 0$ if and only if K is an ellipsoid centered at the origin, for $0 < i < n$ if and only if K is a ball centered at the origin.

Note that the case $i = 0$ of (2.5) is just the well-known Blaschke – Santaló inequality (see [4]).

Associated with the L_p-harmonic radial combination of star bodies, Wang and Leng (see [15]) introduced the notion of L_p-dual mixed quermassintegrals as follows: for $K, L \in S_o^n$, $p \ge 1$, $\varepsilon > 0$, real $i \ne n$, the L_p-dual mixed quermassintegrals, $\widetilde{W}_{-p,i}(K,L)$, of the K and L be defined by

$$\frac{n-i}{-p}\widetilde{W}_{-p,i}(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{\widetilde{W}_i(K \widetilde{+}_{-p} \varepsilon \cdot L) - \widetilde{W}_i(K)}{\varepsilon}. \tag{2.6}$$

The definition above and Hospital's role give the following integral representation of the L_p-dual mixed quermassintegrals (see [15]):

$$\widetilde{W}_{-p,i}(K,L) = \frac{1}{n} \int\limits_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_L^{-p}(u) dS(u), \tag{2.7}$$

where the integration with respect to spherical Lebesgue measure S on S^{n-1}. From the formula (2.7) and definition (2.3), we get

$$\widetilde{W}_{-p,i}(K,K) = \widetilde{W}_i(K). \tag{2.8}$$

Theorem 2.B. *Let $K, L \in \mathcal{S}_o^n$, $p \ge 1$, and real $i \ne n$, then for $i < n$ or $n < i < n+p$*

$$\widetilde{W}_{-p,i}(K,L) \ge \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}} \widetilde{W}_i(L)^{-\frac{p}{n-i}}, \tag{2.9}$$

for $i > n+p$, inequality (2.9) *is reverse. Equality holds in every inequality if and only if K and L are dilates. For $i = n+p$,* (2.9) *is identic.*

Recall that Lutwak in [13] gave the concept of L_p-dual mixed volume: for $K, L \in S_o^n$, $p \ge 1$, the L_p-dual mixed volume, $\widetilde{V}_{-p}(K,L)$, of the K and L is defined by

$$\frac{n}{-p}\widetilde{V}_{-p}(K,L) = \lim_{\varepsilon \longrightarrow 0^+} \frac{V(K \widetilde{+}_{-p} \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.10}$$

From (2.10), (2.6) and (2.4), we see that

$$\widetilde{W}_{-p,0}(K,L) = \widetilde{V}_{-p}(K,L). \tag{2.11}$$

3. L_p-dual mixed affine surface areas. In this section, we will complete the proofs of theorems.

***Proof of Theorem* 1.1.** For $i = 0$, Theorem 1.1 is just Theorem 1.A.

For $0 < i < n$, from (2.9) and (2.5), we have

$$\widetilde{W}_{-p,i}(K, Q^*)\widetilde{W}_i(Q)^{-\frac{p}{n-i}} \geq \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}} \left[\widetilde{W}_i(Q^*)\widetilde{W}_i(Q)\right]^{-\frac{p}{n-i}} \geq$$
$$\geq \omega_n^{-\frac{2p}{n-i}} \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}}.$$

Hence, using definition (1.4), we know

$$\widetilde{\Omega}_{-p,i}(K)^{\frac{n-p-i}{n-i}} \geq n^{\frac{n-p-i}{n-i}} \omega_n^{-\frac{2p}{n-i}} \widetilde{W}_i(K)^{\frac{n+p-i}{n-i}},$$

this yields inequality (1.6).

According to the equality condition of (2.5), we see that equality holds in (1.6) if and only if K is a ball when $0 < i < n$.

Theorem 1.1 is proved.

***Proof of Theorem* 1.2.** For the case $i = 0$, the proof of Theorem 1.2 see Theorem 1.B.

For $0 < i < n - p$, from definition (1.4), it follows that for $Q \in \mathcal{K}_c^n$,

$$\widetilde{\Omega}_{-p,i}(K)^{\frac{n-p-i}{n-i}} \leq n^{\frac{n-p-i}{n-i}} \widetilde{W}_{-p,i}(K, Q^*)\widetilde{W}_i(Q)^{-\frac{p}{n-i}}.$$

Since $K \in \mathcal{K}_c^n$, taking K^* for Q and using (2.1), we can get

$$\widetilde{\Omega}_{-p,i}(K)^{n-p-i} \leq n^{n-p-i}\widetilde{W}_{-p,i}(K, K)^{n-i}\widetilde{W}_i(K^*)^{-p}.$$

Thus by (2.8),

$$\widetilde{\Omega}_{-p,i}(K)^{n-p-i} \leq n^{n-p-i}\widetilde{W}_i(K)^{n-i}\widetilde{W}_i(K^*)^{-p}. \tag{3.1}$$

Similarly,

$$\widetilde{\Omega}_{-p,i}(K^*)^{n-p-i} \leq n^{n-p-i}\widetilde{W}_i(K^*)^{n-i}\widetilde{W}_i(K)^{-p}. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\left[\widetilde{\Omega}_{-p,i}(K)\widetilde{\Omega}_{-p,i}(K^*)\right]^{n-p-i} \leq n^{2(n-p-i)} \left[\widetilde{W}_i(K)\widetilde{W}_i(K^*)\right]^{n-p-i}.$$

Hence, using (2.5), we have

$$\left[\widetilde{\Omega}_{-p,i}(K)\widetilde{\Omega}_{-p,i}(K^*)\right]^{n-p-i} \leq$$
$$\leq n^{2(n-p-i)} \left[\omega_n^{\frac{2i}{n}} (V(K)V(K^*))^{\frac{n-i}{n}}\right]^{n-p-i} \leq (n\omega_n)^{2(n-p-i)}, \quad 0 < i < n - p.$$

Because of $0 < i < n - p$, so inequality (1.7) is given.

According to the equality condition of (2.5), we see that equality holds in (1.7) if and only if K is a ball.

Theorem 1.2 is proved.

***Proof of Theorem* 1.3.** For $K, L \in \mathcal{K}_o^n$, since $1 \le p \le q$, i is a real and $0 \le i < n$, and

$$\rho_K^{n+p-i}(u)\rho_{L^*}^{-p}(u) = \left[\rho_K^{n+q-i}(u)\rho_{L^*}^{-q}(u)\right]^{p/q}\left[\rho_K^{n-i}(u)\right]^{\frac{q-p}{q}},$$

then using the Hölder inequality, (2.3) and (2.7) we obtain

$$\widetilde{W}_{-p,i}(K, L^*) = \frac{1}{n}\int\limits_{S^{n-1}} \rho_K^{n+p-i}(u)\rho_{L^*}^{-p}(u)dS(u) =$$

$$= \frac{1}{n}\int\limits_{S^{n-1}} \left[\rho_K^{n+q-i}(u)\rho_{L^*}^{-q}(u)\right]^{p/q}\left[\rho_K^{n-i}(u)\right]^{\frac{q-p}{q}} dS(u) \le$$

$$\le \widetilde{W}_{-q,i}(K, L^*)^{p/q}\widetilde{W}_i(K)^{\frac{q-p}{q}},$$

that is

$$\left(\frac{\widetilde{W}_{-p,i}(K, L^*)}{\widetilde{W}_i(K)}\right)^{1/p} \le \left(\frac{\widetilde{W}_{-q,i}(K, L^*)}{\widetilde{W}_i(K)}\right)^{1/q}. \tag{3.3}$$

The definition of $\widetilde{\Omega}_{-p,i}(K)$ can be rewritten as

$$\frac{1}{\widetilde{W}_i(K)}\left(\frac{\widetilde{\Omega}_{-p,i}(K)}{n\widetilde{W}_i(K)}\right)^{\frac{n-p-i}{p}} = \inf\left\{\left(\frac{\widetilde{W}_{-p,i}(K, Q^*)}{\widetilde{W}_i(K)}\right)^{\frac{n-i}{p}}\widetilde{W}_i(Q)^{-1} : Q \in \mathcal{K}_c^n\right\}.$$

Associated with (3.3) and notice $n - i > 0$, we can get (1.8).

Theorem 1.3 is proved.

4. Brunn – Minkowski-type inequality. In this section, we give Brunn – Minkowski-type inequality for the L_p-dual mixed affine surface areas. First, we prove Theorem 1.4. Next, associated with the L_p-radial combination of star bodies, we get another Brunn – Minkowski-type inequality. Here the proof of Theorem 1.4 require a lemma as follows:

Lemma 4.1. *If $K, L \in S_o^n$, $p \ge 1$ and $\lambda, \mu \ge 0$ (not both zero), real $i < n+2p$ and $i \ne n, n+p$, then for any $Q \in S_o^n$,*

$$\widetilde{W}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L, Q)^{-\frac{p}{n+p-i}} \ge \lambda\widetilde{W}_{-p,i}(K, Q)^{-\frac{p}{n+p-i}} + \mu\widetilde{W}_{-p,i}(L, Q)^{-\frac{p}{n+p-i}} \tag{4.1}$$

with equality if and only if K and L are dilates.

Proof. Since $i < n + 2p$ and $i \ne n, n + p$, thus $-(n + p - i)/p < 0$ when $i < n + p$ and $i \ne n$, or $0 < -(n+p-i)/p < 1$ when $n+p < i < n+2p$. Hence by (2.2), (2.7) and Minkowski's integral inequality (see [2]), we have

$$\widetilde{W}_{-p,i}(\lambda \cdot K +_{-p} \mu \cdot L, Q)^{-\frac{p}{n+p-i}} =$$

$$= \left[\frac{1}{n}\int\limits_{S^{n-1}} \rho(\lambda \cdot K +_{-p} \mu \cdot L, u)^{n+p-i}\rho(Q, u)^{-p}du\right]^{-\frac{p}{n+p-i}} =$$

$$= \left[\frac{1}{n}\int\limits_{S^{n-1}} [\rho(\lambda \cdot K +_{-p} \mu \cdot L, u)^{-p}\rho(Q, u)^{\frac{p^2}{n+p-i}}]^{-\frac{n+p-i}{p}}du\right]^{-\frac{p}{n+p-i}} =$$

$$= \left[\frac{1}{n}\int\limits_{S^{n-1}} [(\lambda\rho(K,u)^{-p} + \mu\rho(L,u)^{-p})\rho(Q,u)^{\frac{p^2}{n+p-i}}]^{-\frac{n+p-i}{p}} du\right]^{-\frac{p}{n+p-i}} \geq$$

$$\geq \lambda\left[\frac{1}{n}\int\limits_{S^{n-1}} \rho(K,u)^{n+p-i}\rho(Q,u)^{-p}du\right]^{-\frac{p}{n+p-i}} +$$

$$+\mu\left[\frac{1}{n}\int\limits_{S^{n-1}} \rho(L,u)^{n+p-i}\rho(Q,u)^{-p}du\right]^{-\frac{p}{n+p-i}} =$$

$$= \lambda\widetilde{W}_{-p,i}(K,Q)^{-\frac{p}{n+p-i}} + \mu\widetilde{W}_{-p,i}(L,Q)^{-\frac{p}{n+p-i}} \qquad \text{for any} \quad Q \in S_o^n.$$

According to the equality condition of Minkowski's integral inequality, we see that equality holds in (4.1) if and only if K and L are dilates.

Lemma 4.1 is proved.

***Proof of Theorem* 1.4.** Since $n+p < i < n+2p$, thus $-\dfrac{p}{n+p-i} > 1$. Then by definition (1.4) and inequality (4.1), we obtain

$$\left[n^{\frac{p}{n-i}}\widetilde{\Omega}_{-p,i}(\lambda\cdot K +_{-p} \mu\cdot L)^{\frac{n-p-i}{n-i}}\right]^{-\frac{p}{n+p-i}} =$$

$$= \inf\left\{\left[n\widetilde{W}_{-p,i}(\lambda\cdot K +_{-p} \mu\cdot L, Q^*)\widetilde{W}_i(Q)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} =$$

$$= \inf\left\{\left[n\widetilde{W}_{-p,i}(\lambda\cdot K +_{-p} \mu\cdot L, Q^*)\right]^{-\frac{p}{n+p-i}} \widetilde{W}_i(Q)^{\frac{p^2}{(n-i)(n+p-i)}} : Q \in \mathcal{K}_c^n\right\} \geq$$

$$\geq \inf\left\{\left[\lambda(n\widetilde{W}_{-p,i}(K,Q^*))^{-\frac{p}{n+p-i}} + \mu(n\widetilde{W}_{-p,i}(L,Q^*))^{-\frac{p}{n+p-i}}\right] \widetilde{W}_i(Q)^{\frac{p^2}{(n-i)(n+p-i)}} : Q \in \mathcal{K}_c^n\right\} \geq$$

$$\geq \inf\left\{\lambda\left[n\widetilde{W}_{-p,i}(K,Q^*)\widetilde{W}_i(Q)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} +$$

$$+ \inf\left\{\mu\left[n\widetilde{W}_{-p,i}(L,Q^*)\widetilde{W}_i(Q)^{-\frac{p}{n-i}}\right]^{-\frac{p}{n+p-i}} : Q \in \mathcal{K}_c^n\right\} =$$

$$= \lambda\left[n^{-\frac{p}{n-i}}\widetilde{\Omega}_{-p,i}(K)^{\frac{n-p-i}{n-i}}\right]^{-\frac{p}{n+p-i}} + \mu\left[n^{-\frac{p}{n-i}}\widetilde{\Omega}_{-p,i}(L)^{\frac{n-p-i}{n-i}}\right]^{-\frac{p}{n+p-i}}.$$

This yields inequality (1.9).

By the equality condition of (4.1) we know that equality holds in (1.9) if and only if K and L are dilates.

Theorem 1.4 is proved.

For $K, L \in \mathcal{S}_o^n$, $p \geq 1$ and $\lambda, \mu \geq 0$ (not both zero), the L_p-radial combination, $\lambda\circ K\tilde{+}_p\mu\circ L \in \mathcal{S}_o^n$, of K and L is defined by (see [20])

$$\rho(\lambda\circ K\tilde{+}_p\mu\circ L, \cdot)^p = \lambda\rho(K,\cdot)^p + \mu\rho(L,\cdot)^p. \tag{4.2}$$

According to definition (4.2) of the L_p-radial combination, Wang and He in [14] showed the Brunn–Minkowski-type inequality for the L_p-dual affine surface area as follows:

Theorem 4.A. *If* $K, L \in K_c^n$, $n > p \geq 1$, *then*

$$\widetilde{\Omega}_{-p}(K \tilde{+}_{n+p} L)^{\frac{n-p}{n}} \geq \widetilde{\Omega}_{-p}(K)^{\frac{n-p}{n}} + \widetilde{\Omega}_{-p}(L)^{\frac{n-p}{n}} \tag{4.3}$$

with equality if and only if K *and* L *are dilates.*

Associated with the L_p-radial combination of star bodies, we establish a Brunn–Minkowski inequality for the L_p-dual mixed affine surface areas. Under the definition (4.2) of L_p-radial combination, we have the following theorem.

Theorem 4.1. *If* $K, L \in K_c^n$, $p \geq 1$, *real* $i < n - p$, *then*

$$\widetilde{\Omega}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L)^{\frac{n-p-i}{n-i}} \geq \lambda \widetilde{\Omega}_{-p,i}(K)^{\frac{n-p-i}{n-i}} + \mu \widetilde{\Omega}_{-p,i}(L)^{\frac{n-p-i}{n-i}} \tag{4.4}$$

with equality if and only if K *and* L *are dilates.*

Proof. Since $i < n - p$, thus $\dfrac{n-p-i}{n-i} > 0$ and $n + p - i > 2p > 1$. Therefore, from definition (1.4) and formula (2.7), we have

$$n^{\frac{p}{n-i}} \widetilde{\Omega}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L)^{\frac{n-p-i}{n-i}} =$$

$$= \inf\left\{ n \widetilde{W}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L, Q^*) \widetilde{W}_i(Q)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} =$$

$$= \inf\left\{ n \left[\lambda \widetilde{W}_{-p,i}(K, Q^*) + \mu \widetilde{W}_{-p,i}(L, Q^*) \right] \widetilde{W}_i(Q)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} =$$

$$= \inf\left\{ n\lambda \widetilde{W}_{-p,i}(K, Q^*) \widetilde{W}_i(Q)^{-\frac{p}{n-i}} + n\mu \widetilde{W}_{-p,i}(L, Q^*) \widetilde{W}_i(Q)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} \geq$$

$$\geq \inf\left\{ n\lambda \widetilde{W}_{-p,i}(K, Q^*) \widetilde{W}_i(Q)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} +$$

$$+ \inf\left\{ n\mu \widetilde{W}_{-p,i}(L, Q^*) \widetilde{W}_i(Q)^{-\frac{p}{n-i}} : Q \in \mathcal{K}_c^n \right\} =$$

$$= n^{\frac{p}{n-i}} \lambda \widetilde{\Omega}_{-p,i}(K)^{\frac{n-p-i}{n-i}} + n^{\frac{p}{n-i}} \mu \widetilde{\Omega}_{-p,i}(L)^{\frac{n-p-i}{n-i}}.$$

Thus

$$\widetilde{\Omega}_{-p,i}(\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L)^{\frac{n-p-i}{n-i}} \geq \lambda \widetilde{\Omega}_{-p,i}(K)^{\frac{n-p-i}{n-i}} + \mu \widetilde{\Omega}_{-p,i}(L)^{\frac{n-p-i}{n-i}}.$$

The equality holds if and only if $\lambda \circ K \tilde{+}_{n+p-i} \mu \circ L$ are dilates with K and L, respectively. This mean that equality holds in (4.4) if and only if K and L are dilates.

Theorem 4.1 is proved.

Obviously, by (1.3) we know that if $i = 0$ and $\lambda = \mu = 1$ in Theorem 4.1, then inequality (4.4) is just inequality (4.3).

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