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Cui Chen (Tianjin Univ. Finance and Economics, China),
Ze-Hua Zhou (Tianjin Univ., China)

DIFFERENCES OF THE WEIGHTED DIFFERENTIATION COMPOSITION OPERATORS FROM MIXED-NORM SPACES TO WEIGHTED-TYPE SPACES*

РІЗНИЦІ ЗВАЖЕНИХ ДИФЕРЕНЦІАЛЬНИХ ОПЕРАТОРІВ КОМПОЗИЦІЇ З ПРОСТОРІВ ІЗ МІШАНОЮ НОРМОЮ У ПРОСТОРАХ ЗВАЖЕНОГО ТИПУ

We characterize the boundedness and compactness of the differences of weighted differentiation composition operators $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2}$, where $n \in \mathbb{N}_0$, $u_1, u_2 \in H(\mathbb{D})$, and $\varphi_1, \varphi_2 \in S(\mathbb{D})$, from mixed-norm spaces $H(p,q,\phi)$, where $0 < p, q < \infty$ and ϕ is normal, to weighted-type spaces H^∞_v.

Проаналізовано обмеженість і компактність різниць зважених диференціальних операторів композиції $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2}$, де $n \in \mathbb{N}_0$, $u_1, u_2 \in H(\mathbb{D})$ та $\varphi_1, \varphi_2 \in S(\mathbb{D})$, із просторів із мішаною нормою $H(p,q,\phi)$, де $0 < p, q < \infty$, а ϕ є нормальним, у просторах зваженого типу H^∞_v.

1. Introduction. Let $\mathbb{N}_0$ denote the set of all nonnegative integers, $H(\mathbb{D})$ and $S(\mathbb{D})$ represent the class of analytic functions and analytic self-maps on the unit disk $\mathbb{D}$ of the complex plane of $\mathbb{C}$, respectively.

A positive continuous function ϕ is called normal [13] if there exist $\delta \in [0,1)$ and s, t $(0 < s < t)$ such that

$$\frac{\phi(r)}{(1-r)^s} \quad \text{is decreasing on} \quad [\delta,1) \quad \text{and} \quad \lim_{r\to 1}\frac{\phi(r)}{(1-r)^s} = 0,$$

$$\frac{\phi(r)}{(1-r)^t} \quad \text{is increasing on} \quad [\delta,1) \quad \text{and} \quad \lim_{r\to 1}\frac{\phi(r)}{(1-r)^t} = \infty.$$

For $0 < p, q < \infty$ and a normal weight ϕ, the mixed-norm space denoted by $H(p,q,\phi)$ is the space of all functions $f \in H(\mathbb{D})$ satisfying

$$\|f\|^p_{H(p,q,\phi)} := \int\limits_0^1 M^p_q(f,r)\frac{\phi^p(r)}{1-r}dr < \infty,$$

where

$$M_q(f,r) = \left(\int\limits_0^{2\pi} |f(re^{i\theta})|^q d\theta\right)^{1/q}, \quad 0 \le r < 1.$$

For $1 \le p < \infty$, $H(p,q,\phi)$ is a Banach space equipped with the norm $\|\cdot\|_{H(p,q,\phi)}$. But when $0 < p < 1$, $\|\cdot\|_{H(p,q,\phi)}$ is just a quasinorm on $H(p,q,\phi)$, and then $H(p,q,\phi)$ is a Fréchet space but not a Banach space. If $0 < p = q < \infty$, $H(p,q,\phi)$ becomes a Bergman-type space, and moreover if $\phi(r) = (1-r)^{\frac{\alpha+1}{p}}$ for $\alpha > -1$, $H(p,q,\phi)$ is equivalent to the classical weighted Bergman space A^p_α defined by

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$$A^p_\alpha = \left\{ f \in H(\mathbb{D})\colon \|f\|^p_{A^p_\alpha} = (\alpha+1)\int\limits_{\mathbb{D}} |f(z)|^p(1-|z|^2)^\alpha dm(z) < \infty \right\},$$

and the norms $\|f\|_{A^p_\alpha}$ and $\|f\|_{H(p,q,\phi)}$ are equivalent in this case. Recently there has been a great interest in studying mixed norm spaces and operators on them on various domains in the complex plane or in the n-dimensional complex vector space $\mathbb{C}^n$ (see, for example, [4, 7, 9, 14 – 16, 18, 19, 21, 23, 24] and the related references therein).

Let v be a strictly positive continuous and bounded function (weight) on $\mathbb{D}$. The weighted-type space H^∞_v is defined to be the collection of all functions $f \in H(\mathbb{D})$ that satisfy

$$\|f\|_v := \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty.$$

With this norm the weighted-type space becomes a Banach space.

Let φ be a holomorphic self-map of $\mathbb{D}$, the composition operator C_φ induced by φ is defined by

$$(C_\varphi f)(z) = f(\varphi(z)), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

Let $D = D^1$ be the differentiation operator, i.e., $Df = f'$. If $n \in \mathbb{N}_0$ then the operator D^n is defined by $D^0 f = f$, $D^n f = f^{(n)}$, $f \in H(\mathbb{D})$. Some of the first product-type operators studied in the literature were products of the composition and differentiation operators (see, e.g., [3, 5 – 7, 17, 20, 25] and the related references therein).

The weighted differentiation composition operator, denoted by $D^n_{\varphi,u}$, is defined by $(D^n_{\varphi,u} f)(z) = u(z)f^{(n)}(\varphi(z))$, which was studied in some recent papers such as [8, 10, 22 – 24].

Recently, there have been an increasing interest in studying the compact difference of operators acting on different spaces of holomorphic functions. Motivated by some recent papers such as [1, 7, 21, 23, 25, 26], here we characterize the boundedness and compactness of the operators $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2}\colon H(p,q,\phi) \to H^\infty_v$.

Our results involve the pseudohyperbolic metric. For $a \in \mathbb{D}$, let φ_a be the automorphism of $\mathbb{D}$ exchanging 0 and a, that is, $\varphi_a(z) = \dfrac{a-z}{1-\overline{a}z}$. For $z, w \in \mathbb{D}$, the pseudohyperbolic distance between z and w is given by $\rho(z,w) = |\varphi_z(w)|$.

Throughout this paper, we will use the symbol C to denote a finite positive number, and it may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2. Background and some lemmas. Now let us state a couple of lemmas, which are used in the proofs of the main results in the next sections. The first lemma is taken from [15] and [23].

Lemma 2.1. *Assume that $0 < p, q < \infty$, ϕ is normal and $f \in H(p,q,\phi)$. Then for every $n \in \mathbb{N}_0$, there is a constant C independent of f such that*

$$|f^{(n)}(z)| \le C\frac{\|f\|_{H(p,q,\phi)}}{\phi(|z|)(1-|z|^2)^{\frac{1}{q}+n}}, \quad z \in \mathbb{D}. \tag{2.1}$$

The next lemma can be found in [13].

Lemma 2.2. *For $\beta > -1$ and $m > 1 + \beta$, one has*

$$\int_0^1 \frac{(1-r)^\beta}{(1-\rho r)^m} dr \leq C(1-\rho)^{1+\beta-m}, \quad 0 < \rho < 1. \tag{2.2}$$

Lemma 2.3. *Assume that $0 < p, q < \infty$, ϕ is normal and $n \in \mathbb{N}_0$. Then for each $f \in H(p,q,\phi)$, there is a constant C independent of f such that*

$$\left|\phi(|z|)(1-|z|^2)^{\frac{1}{q}+n} f^{(n)}(z) - \phi(|\omega|)(1-|\omega|^2)^{\frac{1}{q}+n} f^{(n)}(\omega)\right| \leq$$
$$\leq C\|f\|_{H(p,q,\phi)}\rho(z,\omega). \tag{2.3}$$

Proof. For $f \in H(p,q,\phi)$, let $u(z) = \phi(|z|)(1-|z|^2)^{\frac{1}{q}+n}$, by Lemma 2.1, we obtain $f^{(n)} \in \in H_u^\infty$, so from Lemma 3.2 in [2] and Lemma 2.1, there is a constant $C > 0$ such that

$$|u(z)f^{(n)}(z) - u(\omega)f^{(n)}(\omega)| \leq C\|f^{(n)}\|_u \rho(z,\omega) \leq C\|f\|_{H(p,q,\phi)}\rho(z,\omega)$$

for all $z, \omega \in \mathbb{D}$.

Remark. From the proof of Lemma 2.3, it is not difficult to see that for any $z, \omega \in r\mathbb{D} = \{z \in \mathbb{D} : |z| < r < 1\}$, then

$$\left|\phi(|z|)(1-|z|^2)^{\frac{1}{q}+n} f^{(n)}(z) - \phi(|\omega|)(1-|\omega|^2)^{\frac{1}{q}+n} f^{(n)}(\omega)\right| \leq$$
$$\leq C\rho(z,\omega) \sup_{\zeta \in r\mathbb{D}} \phi(|\zeta|)(1-|\zeta|^2)^{\frac{1}{q}+n}|f^{(n)}(\zeta)| \tag{2.4}$$

for any $f \in H(p,q,\phi)$.

The next Schwartz-type lemma can be proved in a standard way [12].

Lemma 2.4. *Suppose $n \in \mathbb{N}_0$, $0 < p, q < \infty$, $u_1, u_2 \in H(\mathbb{D})$, $\varphi_1, \varphi_2 \in S(\mathbb{D})$ and ϕ is normal. Then the operator $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$ is compact if and only if $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $H(p,q,\phi)$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|(D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2})f_k\|_v \to 0$, as $k \to \infty$.*

The following result is well-known. It can be proved by a slight modification of the proof of Theorem 2 in [4].

Lemma 2.5. *Assume that $0 < p, q < \infty$, ϕ is normal and $n \in \mathbb{N}_0$. Then for each $f \in H(p,q,\phi)$,*

$$\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}dr \asymp \sum_{j=0}^{n-1}|f^{(j)}(0)| + \int_0^1 M_q^p(f^{(n)},r)\frac{\phi^p(r)}{1-r}(1-r)^{np}dr. \tag{2.5}$$

3. Boundedness of $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$. In this section we will characterize the boundedness of the operator $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$. For the purpose, we list the following three conditions which we will use below:

$$M_1 = \sup_{z\in\mathbb{D}} \frac{v(z)|u_1(z)|\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} < \infty, \tag{3.1}$$

$$M_2 = \sup_{z\in\mathbb{D}} \frac{v(z)|u_2(z)|\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} < \infty, \tag{3.2}$$

$$M_3 = \sup_{z\in\mathbb{D}} \left| \frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} - \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \right| < \infty. \tag{3.3}$$

Theorem 3.1. *Suppose* $n \in \mathbb{N}_0$, $0 < p,q < \infty$, $u_1, u_2 \in H(\mathbb{D})$, $\varphi_1, \varphi_2 \in S(\mathbb{D})$ *and* ϕ *is normal. Then the following statements are equivalent*:

(i) $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} \colon H(p,q,\phi) \to H^\infty_v$ *is bounded.*

(ii) *The conditions* (3.1) *and* (3.3) *hold.*

(iii) *The conditions* (3.2) *and* (3.3) *hold.*

Proof. First, we prove the implication (i) $\Rightarrow$ (ii). Assume that $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} \colon H(p,q,\phi) \to$ $\to H^\infty_v$ is bounded. Fix $w \in \mathbb{D}$, we consider the function f_w defined by

$$f_w(z) = \int_0^z \int_0^{t_n} \cdots \int_0^{t_2} \frac{(1-|\varphi_1(w)|^2)^{t+1}}{\phi(|\varphi_1(w)|)(1-\overline{\varphi_1(w)}t_1)^{\frac{1}{q}+t+1+n}} \varphi_{\varphi_2(w)}(t_1)\,dt_1 dt_2 \cdots dt_n. \tag{3.4}$$

Next we show that $f_w \in H(p,q,\phi)$. Notice that

$$f_w^{(n)}(z) = \frac{(1-|\varphi_1(w)|^2)^{t+1}}{\phi(|\varphi_1(w)|)(1-\overline{\varphi_1(w)}z)^{\frac{1}{q}+t+1+n}} \varphi_{\varphi_2(w)}(z),$$

according to Lemma 1.4.10 in [11],

$$M_q^p(f_w^{(n)}, r) = \left(\int_0^{2\pi} \frac{(1-|\varphi_1(w)|^2)^{q(t+1)}}{\phi^q(|\varphi_1(w)|)|1-\overline{\varphi_1(w)}re^{i\theta}|^{1+q(t+1+n)}} |\varphi_{\varphi_2(w)}(re^{i\theta})|^q d\theta \right)^{p/q} \le$$

$$\le \frac{(1-|\varphi_1(w)|^2)^{p(t+1)}}{\phi^p(|\varphi_1(w)|)} \left(\int_0^{2\pi} \frac{d\theta}{|1-\overline{\varphi_1(w)}re^{i\theta}|^{1+q(t+1+n)}} \right)^{p/q} \asymp$$

$$\asymp \frac{(1-|\varphi_1(w)|^2)^{p(t+1)}}{\phi^p(|\varphi_1(w)|)(1-r|\varphi_1(w)|)^{p(t+1+n)}}. \tag{3.5}$$

By using Lemma 2.5, (3.5), the fact that $f_w^{(j)}(0) = 0$, $j = 1, \ldots, n-1$, the normality of ϕ, Lemma 1.4.10 in [11] and Lemma 2.2, we have

$$\|f_w\|^p_{H(p,q,\phi)} \le C \int_0^1 \frac{(1-|\varphi_1(w)|^2)^{p(t+1)}}{\phi^p(|\varphi_1(w)|)(1-r|\varphi_1(w)|)^{p(t+1+n)}} \frac{\phi^p(r)}{1-r}(1-r)^{np} dr \le$$

$$\le C \int_0^1 \frac{(1-|\varphi_1(w)|^2)^{p(t+1)}}{\phi^p(|\varphi_1(w)|)(1-r|\varphi_1(w)|)^{p(t+1)}} \frac{\phi^p(r)}{1-r} dr \le$$

$$\leq C\left(\int\limits_0^{|\varphi_1(w)|}\frac{(1-|\varphi_1(w)|^2)^{p(t+1)}}{\phi^p(|\varphi_1(w)|)(1-r|\varphi_1(w)|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr+\right.$$

$$\left.+\int\limits_{|\varphi_1(w)|}^{1}\frac{(1-|\varphi_1(w)|^2)^{p(t+1)}}{\phi^p(|\varphi_1(w)|)(1-r|\varphi_1(w)|)^{p(t+1)}}\frac{\phi^p(r)}{1-r}dr\right)\leq$$

$$\leq C(1-|\varphi_1(w)|^2)^p\int\limits_0^{|\varphi_1(w)|}\frac{(1-r)^{pt-1}}{(1-r|\varphi_1(w)|)^{p(t+1)}}dr+$$

$$+C(1-|\varphi_1(w)|^2)^p\int\limits_{|\varphi_1(w)|}^{1}\frac{(1-r)^{ps-1}}{(1-r|\varphi_1(w)|)^{p(s+1)}}dr\leq C.$$

Therefore $f_w\in H(p,q,\phi)$, and moreover $\sup_{w\in\mathbb{D}}\|f_w\|_{H(p,q,\phi)}\leq C$. Note that

$$f_w^{(n)}(\varphi_1(w))=\frac{\rho(\varphi_1(w),\varphi_2(w))}{\phi(|\varphi_1(w)|)(1-|\varphi_1(w)|^2)^{\frac{1}{q}+n}}\quad\text{and}\quad f_w^{(n)}(\varphi_2(w))=0.$$

So by the boundedness of $D^n_{\varphi_1,u_1}-D^n_{\varphi_2,u_2}\colon H(p,q,\phi)\to H_v^\infty$, we obtain

$$\infty>\|(D^n_{\varphi_1,u_1}-D^n_{\varphi_2,u_2})f_w\|_v=$$
$$=\sup_{z\in\mathbb{D}}v(z)|u_1(z)f_w^{(n)}(\varphi_1(z))-u_2(z)f_w^{(n)}(\varphi_2(z))|\geq$$
$$\geq v(w)|u_1(w)f_w^{(n)}(\varphi_1(w))-u_2(w)f_w^{(n)}(\varphi_2(w))|=$$
$$=\frac{v(w)|u_1(w)|\rho(\varphi_1(w),\varphi_2(w))}{\phi(|\varphi_1(w)|)(1-|\varphi_1(w)|^2)^{\frac{1}{q}+n}}.\tag{3.6}$$

Since $w\in\mathbb{D}$ is an arbitrary element, (3.1) comes from (3.6).

Next we prove (3.3). Fix $w\in\mathbb{D}$, let

$$g_w(z)=\int\limits_0^z\int\limits_0^{t_n}\ldots\int\limits_0^{t_2}\frac{(1-|\varphi_2(w)|^2)^{t+1}}{\phi(|\varphi_2(w)|)(1-\overline{\varphi_2(w)}t_1)^{\frac{1}{q}+t+1+n}}dt_1dt_2\ldots dt_n.$$

Similarly as for the test functions in (3.4), we obtained that $g_w\in H(p,q,\phi)$ with $g_w^{(n)}(\varphi_2(w))=$ $=\dfrac{1}{\phi(|\varphi_2(w)|)(1-|\varphi_2(w)|^2)^{\frac{1}{q}+n}}$. Then

$$\infty>\|(D^n_{\varphi_1,u_1}-D^n_{\varphi_2,u_2})g_w\|_v\geq$$
$$\geq v(w)|u_1(w)g_w^{(n)}(\varphi_1(w))-u_2(w)g_w^{(n)}(\varphi_2(w))|=|I(w)+J(w)|,\tag{3.7}$$

where

$$I(w) = \phi(|\varphi_2(w)|)(1-|\varphi_2(w)|^2)^{\frac{1}{q}+n} g_w^{(n)}(\varphi_2(w)) \left[\frac{v(w)u_1(w)}{\phi(|\varphi_1(w)|)(1-|\varphi_1(w)|^2)^{\frac{1}{q}+n}} - \right.$$

$$\left. - \frac{v(w)u_2(w)}{\phi(|\varphi_2(w)|)(1-|\varphi_2(w)|^2)^{\frac{1}{q}+n}} \right]$$

and

$$J(w) = \frac{v(w)u_1(w)}{\phi(|\varphi_1(w)|)(1-|\varphi_1(w)|^2)^{\frac{1}{q}+n}} \Big[\phi(|\varphi_1(w)|)(1-|\varphi_1(w)|^2)^{\frac{1}{q}+n} g_w^{(n)}(\varphi_1(w)) -$$

$$- \phi(|\varphi_2(w)|)(1-|\varphi_2(w)|^2)^{\frac{1}{q}+n} g_w^{(n)}(\varphi_2(w)) \Big].$$

By Lemma 2.3 and (3.1), we conclude that $|J(w)| < \infty$. From this along with (3.7) we get

$$|I(w)| = \left| \frac{v(w)u_1(w)}{\phi(|\varphi_1(w)|)(1-|\varphi_1(w)|^2)^{\frac{1}{q}+n}} - \frac{v(w)u_2(w)}{\phi(|\varphi_2(w)|)(1-|\varphi_2(w)|^2)^{\frac{1}{q}+n}} \right| < \infty$$

for all $w \in \mathbb{D}$, thus (3.3) holds.

(ii) $\Rightarrow$(iii). Assume that (3.1) and (3.3) hold, we only need to show that (3.2) holds. In fact,

$$\frac{v(z)|u_2(z)|\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \le \frac{v(z)|u_1(z)|\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} +$$

$$+ \left| \frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} - \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \right| \rho(\varphi_1(z),\varphi_2(z)).$$

From which, using (3.1) and (3.3), the desired condition (3.2) holds.

(iii) $\Rightarrow$(i). Assume that (3.2) and (3.3) hold. By Lemma 2.1 and Lemma 2.3, for any $f \in$ $\in H(p,q,\phi)$, we have

$$v(z)|u_1(z)f^{(n)}(\varphi_1(z)) - u_2(z)f^{(n)}(\varphi_2(z))| =$$

$$= \left| \phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n} f^{(n)}(\varphi_1(z)) \left[\frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} - \right. \right.$$

$$\left. - \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \right] + \Big[\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n} f^{(n)}(\varphi_1(z)) -$$

$$\left. - \phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n} f^{(n)}(\varphi_2(z)) \Big] \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \right| \le$$

$$\le C\|f\|_{H(p,q,\phi)} + C \frac{v(z)|u_2(z)|\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \|f\|_{H(p,q,\phi)} \le$$

$$\leq C\|f\|_{H(p,q,\phi)}$$

for each $z \in \mathbb{D}$. From which it follows that $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} \colon H(p,q,\phi) \to H^\infty_v$ is bounded.

Theorem 3.1 is proved.

4. Compactness of $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} \colon H(p,q,\phi) \to H^\infty_v$. In this section, we turn our attention to the problem of the compactness of the operator. Here we consider the following conditions:

$$M_4 = \frac{v(z)u_1(z)\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} \to 0 \quad \text{as} \quad |\varphi_1(z)| \to 1, \tag{4.1}$$

$$M_5 = \frac{v(z)u_2(z)\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \to 0 \quad \text{as} \quad |\varphi_2(z)| \to 1, \tag{4.2}$$

$$M_6 = \frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} - \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \to 0$$
$$\text{as} \quad |\varphi_1(z)| \to 1 \quad \text{and} \quad |\varphi_2(z)| \to 1. \tag{4.3}$$

Theorem 4.1. *Suppose $n \in \mathbb{N}_0$, $0 < p,q < \infty$, $u_1,u_2 \in H(\mathbb{D})$, $\varphi_1,\varphi_2 \in S(\mathbb{D})$ and ϕ is normal. Then $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} \colon H(p,q,\phi) \to H^\infty_v$ is compact if and only if $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} \colon H(p,q,\phi) \to H^\infty_v$ is bounded and the conditions* (4.1)–(4.3) *hold.*

Proof. First we suppose that $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} \colon H(p,q,\phi) \to H^\infty_v$ is bounded and the conditions (4.1)–(4.3) hold. It is clear that the conditions (3.1)–(3.3) hold by Theorem 3.1. From (4.1)–(4.3), it follows that for any $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$\frac{v(z)|u_1(z)|\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} \leq \varepsilon \quad \text{for} \quad |\varphi_1(z)| > r, \tag{4.4}$$

$$\frac{v(z)|u_2(z)|\rho(\varphi_1(z),\varphi_2(z))}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \leq \varepsilon \quad \text{for} \quad |\varphi_2(z)| > r, \tag{4.5}$$

$$\left|\frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} - \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}}\right| \leq \varepsilon$$
$$\text{for} \quad |\varphi_1(z)| > r, \quad |\varphi_2(z)| > r. \tag{4.6}$$

Now, let $(f_k)_{k\in\mathbb{N}}$ be a bounded sequence in $H(p,q,\phi)$ with $\|f_k\|_{H(p,q,\phi)} \leq 1$ and $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$. By Lemma 2.4 we need only to show that $\|(D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2})f_k\|_v \to 0$ as $k \to \infty$. A direct calculation shows that

$$v(z)|u_1(z)f_k^{(n)}(\varphi_1(z)) - u_2(z)f_k^{(n)}(\varphi_2(z))| = |I_k(z) + J_k(z)|, \tag{4.7}$$

where

$$I_k(z) = \phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n} f_k^{(n)}(\varphi_2(z)) \left[\frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} - \right.$$

$$\left. - \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}}\right]$$

and

$$J_k(z) = \frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} \Big[\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n} f_k^{(n)}(\varphi_1(z)) -$$

$$- \phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n} f_k^{(n)}(\varphi_2(z))\Big].$$

We divide the argument into four cases:

Case 1: $|\varphi_1(z)| \le r$ and $|\varphi_2(z)| \le r$.

By the assumption, note that f_k converges to zero uniformly on $E = \{w\colon |w| \le r\}$ as $k \to \infty$, and using (3.3), it is easy to check that $I_k(z) \to 0,\ k \to \infty$ uniformly for all z with $|\varphi_2(z)| \le r$.

On the other hand, from (2.4), (3.1) and since f_k converges to zero uniformly on E, we have that

$$|J_k(z)| \le C \frac{v(z)|u_1(z)|\rho(\varphi_1(z), \varphi_2(z))}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} \sup_{|\zeta| \le r} \phi(|\zeta|)(1-|\zeta|^2)^{\frac{1}{q}+n} |f^{(n)}(\zeta)| \le C\varepsilon.$$

Case 2: $|\varphi_1(z)| > r$ and $|\varphi_2(z)| \le r$.

As in the proof of Case 1, $I_k(z) \to 0$ uniformly as $k \to \infty$. On the other hand, using Lemma 2.3 and (4.4) we obtain $|J_k(z)| \le C\varepsilon$.

Case 3: $|\varphi_1(z)| > r$ and $|\varphi_2(z)| > r$.

For k sufficiently large, by Lemma 2.1 and (4.6) we obtain that $|I_k(z)| \le C\varepsilon$. Meanwhile, $|J_k(z)| \le C\varepsilon$ by Lemma 2.3 and (4.4).

Case 4: $|\varphi_1(z)| \le r$ and $|\varphi_2(z)| > r$. We rewrite

$$v(z)|u_1(z) f_k^{(n)}(\varphi_1(z)) - u_2(z) f_k^{(n)}(\varphi_2(z))| = |P_k(z) + Q_k(z)|,$$

where

$$P_k(z) = \phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n} f_k^{(n)}(\varphi_1(z)) \left[\frac{v(z)u_1(z)}{\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n}} - \right.$$

$$\left. - \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}}\right]$$

and

$$Q_k(z) = \frac{v(z)u_2(z)}{\phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n}} \Big[\phi(|\varphi_1(z)|)(1-|\varphi_1(z)|^2)^{\frac{1}{q}+n} f_k^{(n)}(\varphi_1(z)) -$$

$$- \phi(|\varphi_2(z)|)(1-|\varphi_2(z)|^2)^{\frac{1}{q}+n} f_k^{(n)}(\varphi_2(z))\Big].$$

The desired result follows by an argument analogous to that in the proof of Case 2. Thus, together with the above cases, we conclude that

$$\|(D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2})f_k\|_v =$$
$$= \sup_{z\in\mathbb{D}} v(z)|u_1(z)f_k^{(n)}(\varphi_1(z)) - u_2(z)f_k^{(n)}(\varphi_2(z))| \le C\varepsilon \tag{4.8}$$

for sufficiently large k. Employing Lemma 2.4 combining with the arbitrariness of ε, we obtain the compactness of $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$.

For the converse direction, we suppose that $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$ is compact. From which we can easily obtain the boundedness of $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$. Next we only need to show that (4.1) – (4.3) hold.

Let $(z_k)_{k\in\mathbb{N}}$ be a sequence of points in $\mathbb{D}$ such that $|\varphi_1(z_k)| \to 1$ as $k\to\infty$. Define the functions

$$f_k(z) = \int_0^z\int_0^{t_n}\ldots\int_0^{t_2} \frac{(1-|\varphi_1(z_k)|^2)^{t+1}}{\phi(|\varphi_1(z_k)|)(1-\overline{\varphi_1(z_k)}t_1)^{\frac{1}{q}+t+1+n}}\varphi_{\varphi_2(z_k)}(t_1)dt_1dt_2\ldots dt_n. \tag{4.9}$$

Clearly, $f_k \in H(p,q,\phi)$ with $\sup_{k\in\mathbb{N}}\|f_k\|_{H(p,q,\phi)} \le C$, and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$. Moreover,

$$f_k^{(n)}(\varphi_1(z_k)) = \frac{\rho(\varphi_1(z_k),\varphi_2(z_k))}{\phi(|\varphi_1(z_k)|)(1-|\varphi_1(z_k)|^2)^{\frac{1}{q}+n}} \quad \text{and} \quad f_k^{(n)}(\varphi_2(z_k)) = 0. \tag{4.10}$$

Then

$$\|(D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2})f_k\|_v = \sup_{z\in\mathbb{D}} v(z)|u_1(z)f_k^{(n)}(\varphi_1(z)) - u_2(z)f_k^{(n)}(\varphi_2(z))| \ge$$
$$\ge v(z_k)|u_1(z_k)f_k^{(n)}(\varphi_1(z_k)) - u_2(z_k)f_k^{(n)}(\varphi_2(z_k))| =$$
$$= \frac{v(z_k)|u_1(z_k)|\rho(\varphi_1(z_k),\varphi_2(z_k))}{\phi(|\varphi_1(z_k)|)(1-|\varphi_1(z_k)|^2)^{\frac{1}{q}+n}}. \tag{4.11}$$

On the other hand, since $D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2} : H(p,q,\phi) \to H_v^\infty$ is compact, by Lemma 2.4, it follows that $\|(D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2})f_k\|_v \to 0,\ k\to\infty$. Letting $k\to\infty$ in (4.11), it follows that (4.1) holds. The condition (4.2) holds for the similar arguments.

Now it remains to show that condition (4.3) holds. Assume that $(z_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{D}$ such that $|\varphi_1(z_k)| \to 1$ and $|\varphi_2(z_k)| \to 1$ as $k\to\infty$. Define the function

$$g_k(z) = \int_0^z\int_0^{t_n}\ldots\int_0^{t_2} \frac{(1-|\varphi_2(z_k)|^2)^{t+1}}{\phi(|\varphi_2(z_k)|)(1-\overline{\varphi_2(z_k)}t_1)^{\frac{1}{q}+t+1+n}}dt_1dt_2\ldots dt_n.$$

It is easy to check that g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k\to\infty$ and $g_k \in H(p,q,\phi)$ with $\|g_k\|_{H(p,q,\phi)} \le C$ for all $k\in\mathbb{N}$. It follows from Lemma 2.4 that $\|(D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2})g_k\|_v \to 0,\ k\to\infty$. On the other hand, we have

$$\|(D^n_{\varphi_1,u_1} - D^n_{\varphi_2,u_2})g_k\|_v \ge v(z_k)|u_1(z_k)g_k^{(n)}(\varphi_1(z_k)) - u_2(z_k)g_k^{(n)}(\varphi_2(z_k))| =$$
$$= |I(z_k) + J(z_k)|, \tag{4.12}$$

where

$$I(z_k) = \phi(|\varphi_2(z_k)|)(1-|\varphi_2(z_k)|^2)^{\frac{1}{q}+n} g_k^{(n)}(\varphi_2(z_k)) \left[\frac{v(z_k)u_1(z_k)}{\phi(|\varphi_1(z_k)|)(1-|\varphi_1(z_k)|^2)^{\frac{1}{q}+n}} - \right.$$

$$\left. - \frac{v(z_k)u_2(z_k)}{\phi(|\varphi_2(z_k)|)(1-|\varphi_2(z_k)|^2)^{\frac{1}{q}+n}}\right],$$

$$J(z_k) = \frac{v(z_k)u_1(z_k)}{\phi(|\varphi_1(z_k)|)(1-|\varphi_1(z_k)|^2)^{\frac{1}{q}+n}} \left[\phi(|\varphi_1(z_k)|)(1-|\varphi_1(z_k)|^2)^{\frac{1}{q}+n} g_k^{(n)}(\varphi_1(z_k)) - \right.$$

$$\left. - \phi(|\varphi_2(z_k)|)(1-|\varphi_2(z_k)|^2)^{\frac{1}{q}+n} g_k^{(n)}(\varphi_2(z_k))\right].$$

By Lemma 2.3 and the condition (4.1) that has been proved, we get $J(z_k) \to 0,\ k \to \infty$. This along with (4.12) shows that $I(z_k) \to 0, k \to \infty$. Hence (4.3) is true since $g_k^{(n)}(\varphi_2(z_k)) = = \dfrac{1}{\phi(|\varphi_2(z_k)|)(1-|\varphi_2(z_k)|^2)^{\frac{1}{q}+n}}$.

Theorem 4.1 is proved.

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