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SPECTRAL PROBLEM FOR STURM – LIOUVILLE OPERATOR WITH RETARDED ARGUMENT WHICH CONTAINS A SPECTRAL PARAMETER IN BOUNDARY CONDITION

СПЕКТРАЛЬНА ЗАДАЧА ДЛЯ ОПЕРАТОРА ШТУРМА – ЛІУВІЛЛЯ З АРГУМЕНТОМ, ЩО ЗАПІЗНЮЄТЬСЯ, ТА СПЕКТРАЛЬНИМ ПАРАМЕТРОМ У ГРАНИЧНІЙ УМОВІ

We consider a discontinuous Sturm – Liouville problem with retarded argument that contains a spectral parameter in the boundary condition. First, we investigate the simplicity of eigenvalues and then prove the existence theorem. As a result, we obtain the asymptotic formulas for eigenvalues and eigenfunctions.

Розглянуто розривну задачу Штурма – Ліувілля з аргументом, що запізнюється, та спектральним параметром у граничній умові. Спочатку ми вивчаємо простоту власних значень, а потім доводимо теорему про існування. Як результат, отримано асимптотичні формули для власних значень і власних функцій.

1. Preliminaries. Boundary-value problems for differential equations of the second order with retarded argument were studied in [1 – 9], and various physical applications of such problems can be found in [2]. The asymptotic formulas for the eigenvalues and eigenfunctions of boundary problem of Sturm – Liouville type for second order differential equation with retarded argument were obtained in [1, 2, 5 – 9]. The asymptotic formulas for the eigenvalues and eigenfunctions of classical Sturm – Liouville problem with the spectral parameter in the boundary condition were obtained in [10 – 13].

In this paper we study the eigenvalues and eigenfunctions of discontinuous boundary-value problem with retarded argument and a spectral parameter in the boundary condition. That is, we consider the boundary-value problem for the differential equation

$$p(x)y''(x) + q(x)y(x - \Delta(x)) + \lambda y(x) = 0 \tag{1.1}$$

on $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$, with boundary conditions

$$y'(0) = 0, \tag{1.2}$$

$$y'(\pi) + \lambda y(\pi) = 0, \tag{1.3}$$

and jump conditions

$$\gamma_1 y(r_1 - 0) = \delta_1 y(r_1 + 0), \tag{1.4}$$

$$\gamma_2 y'(r_1 - 0) = \delta_2 y'(r_1 + 0), \tag{1.5}$$

$$\theta_1 y(r_2 - 0) = \eta_1 y(r_2 + 0), \tag{1.6}$$

$$\theta_2 y'(r_2 - 0) = \eta_2 y'(r_2 + 0), \tag{1.7}$$

where $p(x) = p_1^2$, if $x \in [0, r_1)$, $p(x) = p_2^2$, if $x \in (r_1, r_2)$, and $p(x) = p_3^2$, if $x \in (r_2, \pi]$, the real-valued function $q(x)$ is continuous in $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$; and has finite limits $q(r_1 \pm \pm 0) = \lim_{x \to r_1 \pm 0} q(x)$, $q(r_2 \pm 0) = \lim_{x \to r_2 \pm 0} q(x)$; the real valued function $\Delta(x) \geq 0$ continuous in $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$ and has finite limits $\Delta(r_1 \pm 0) = \lim_{x \to r_1 \pm 0} \Delta(x)$, $\Delta(r_2 \pm 0) = = \lim_{x \to r_2 \pm 0} \Delta(x)$, $x - \Delta(x) \geq 0$, if $x \in \left[0, \dfrac{\pi}{2}\right)$; $x - \Delta(x) \geq \dfrac{\pi}{2}$, if $x \in \left(\dfrac{\pi}{2}, \pi\right]$; λ is a real spectral parameter; p_1, p_2, p_3, γ_1, γ_2, δ_1, δ_2, θ_1, θ_2, η_1, η_2 are arbitrary real numbers; $|\gamma_i| + |\delta_i| \neq 0$ and $|\theta_i| + |\eta_i| \neq 0$ for $i = 1, 2$. Also $\gamma_1 \delta_2 p_1 = \gamma_2 \delta_1 p_2$ and $\theta_1 \eta_2 p_2 = \theta_2 \eta_1 p_3$ hold.

It must be noted that some problems with jump conditions which arise in mechanics (thermal condition problem for a thin laminated plate) were studied in [14].

Let $w_1(x, \lambda)$ be a solution of Eq. (1.1) on $[0, r_1]$, satisfying the initial conditions

$$w_1(0, \lambda) = 1, \qquad w_1'(0, \lambda) = 0. \tag{1.8}$$

Conditions (1.8) define a unique solution of Eq. (1.1) on $[0, r_1]$ [2, p. 12].

After defining above solution we shall define the solution $w_2(x, \lambda)$ of Eq. (1.1) on $[r_1, r_2]$ by means of the solution $w_1(x, \lambda)$ by the initial conditions

$$w_2(r_1, \lambda) = \gamma_1 \delta_1^{-1} w_1(r_1, \lambda), \qquad w_2'(r_1, \lambda) = \gamma_2 \delta_2^{-1} \omega_1'(r_1, \lambda). \tag{1.9}$$

Conditions (1.9) are defined as a unique solution of Eq. (1.1) on $[r_1, r_2]$.

After defining above solution we shall define the solution $w_3(x, \lambda)$ of Eq. (1.1) on $[r_2, \pi]$ by means of the solution $w_2(x, \lambda)$ by the initial conditions

$$w_3(r_2, \lambda) = \theta_1 \eta_1^{-1} w_2(r_2, \lambda), \qquad w_3'(r_2, \lambda) = \theta_2 \eta_2^{-1} \omega_2'(r_2, \lambda). \tag{1.10}$$

Conditions (1.10) are defined as a unique solution of Eq. (1.1) on $[r_2, \pi]$.

Consequently, the function $w(x, \lambda)$ is defined on $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$ by the equality

$$w(x, \lambda) = \begin{cases} w_1(x, \lambda), & x \in [0, r_1), \\ w_2(x, \lambda), & x \in (r_1, r_2), \\ w_3(x, \lambda), & x \in (r_2, \pi], \end{cases}$$

is a such solution of Eq. (1.1) on $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$; which satisfies one of the boundary conditions and both transmission conditions.

Lemma 1.1. *Let $w(x, \lambda)$ be a solution of Eq.* (1.1) *and $\lambda > 0$. Then the following integral equations hold*:

$$w_1(x, \lambda) = \cos \frac{s}{p_1} x - \frac{1}{s} \int_0^x \frac{q(\tau)}{p_1} \sin \frac{s}{p_1} (x - \tau) w_1(\tau - \Delta(\tau), \lambda) \, d\tau, \qquad s = \sqrt{\lambda}, \quad \lambda > 0, \tag{1.11}$$

$$w_2(x, \lambda) = \frac{\gamma_1}{\delta_1} w_1(r_1, \lambda) \cos \frac{s}{p_2} (x - r_1) + \frac{\gamma_2 p_2 w_1'(r_1, \lambda)}{s \delta_2} \sin \frac{s}{p_2} (x - r_1) -$$

$$- \frac{1}{s} \int_{r_1}^x \frac{q(\tau)}{p_2} \sin \frac{s}{p_2} (x - \tau) w_2(\tau - \Delta(\tau), \lambda) \, d\tau, \qquad s = \sqrt{\lambda}, \quad \lambda > 0, \tag{1.12}$$

$$w_3(x,\lambda)=\frac{\theta_1}{\eta_1}w_2\left(r_2,\lambda\right)\cos\frac{s}{p_3}\left(x-r_2\right)+\frac{\theta_2 p_3 w_2'\left(r_2,\lambda\right)}{s\eta_2}\sin\frac{s}{p_3}\left(x-r_2\right)-$$

$$-\frac{1}{s}\int\limits_{r_2}^{x}\frac{q(\tau)}{p_3}\sin\frac{s}{p_3}(x-\tau)w_3\left(\tau-\Delta\left(\tau\right),\lambda\right)d\tau,\qquad s=\sqrt{\lambda},\quad \lambda>0. \tag{1.13}$$

Proof. To prove this, it is enough to substitute

$$-\frac{s^2}{p_1^2}w_1(\tau,\lambda)-w_1''(\tau,\lambda),\qquad -\frac{s^2}{p_2^2}w_2(\tau,\lambda)-w_2''(\tau,\lambda)$$

and

$$-\frac{s^2}{p_3^2}w_3(\tau,\lambda)-w_3''(\tau,\lambda)$$

instead of

$$-\frac{q(\tau)}{p_1^2}w_1(\tau-\Delta(\tau),\lambda),\qquad -\frac{q(\tau)}{p_2^2}w_2(\tau-\Delta(\tau),\lambda)\qquad\text{and}\qquad -\frac{q(\tau)}{p_3^2}w_3(\tau-\Delta(\tau),\lambda)$$

in the integrals in (1.11), (1.12) and (1.13) respectively and integrate by parts twice.

Theorem 1.1. *Problem* (1.1) – (1.7) *can have only simple eigenvalues.*

Proof. Let $\widetilde{\lambda}$ be an eigenvalue of problem (1.1) – (1.7) and

$$\widetilde{u}(x,\widetilde{\lambda})=\begin{cases}\widetilde{u_1}(x,\widetilde{\lambda}), & x\in[0,r_1),\\ \widetilde{u_2}(x,\widetilde{\lambda}), & x\in(r_1,r_2),\\ \widetilde{u_3}(x,\widetilde{\lambda}), & x\in(r_2,\pi],\end{cases}$$

be a corresponding eigenfunction. Then from (1.2) and (1.8) the determinant

$$W\left[\widetilde{u}_1(0,\widetilde{\lambda}),w_1(0,\widetilde{\lambda})\right]=\begin{vmatrix}\widetilde{u}_1(0,\widetilde{\lambda}) & 1\\ \widetilde{u}_1'(0,\widetilde{\lambda}) & 0\end{vmatrix}=0,$$

and by Theorem 2.2.2 in [2] the functions $\widetilde{u}_1(x,\widetilde{\lambda})$ and $w_1(x,\widetilde{\lambda})$ are linearly dependent on $[0,r_1]$. We can also prove that the functions $\widetilde{u}_2(x,\widetilde{\lambda})$ and $w_2(x,\widetilde{\lambda})$ are linearly dependent on $[r_1,r_2]$ and the functions $\widetilde{u}_3(x,\widetilde{\lambda})$ and $w_3(x,\widetilde{\lambda})$ are linearly dependent on $[r_2,\pi]$. Hence

$$\widetilde{u}_i(x,\widetilde{\lambda})=K_i w_i(x,\widetilde{\lambda}),\quad i=1,2,3, \tag{1.14}$$

for some $K_1\neq 0$, $K_2\neq 0$ and $K_3\neq 0$. We first show that $K_2=K_3$. Suppose that $K_2\neq K_3$. From equalities (1.6) and (1.14), we have

$$\theta_1\widetilde{u}(r_2-0,\widetilde{\lambda})-\eta_1\widetilde{u}(r_2+0,\widetilde{\lambda})=\theta_1\widetilde{u_2}(r_2,\widetilde{\lambda})-\eta_1\widetilde{u_3}(r_2,\widetilde{\lambda})=$$

$$=\theta_1K_2w_2(r_2,\widetilde{\lambda})-\eta_1K_3w_3(r_2,\widetilde{\lambda})=$$

$$=\theta_1K_2\eta_1\theta_1^{-1}w_3(r_2,\widetilde{\lambda})-\eta_1K_3w_3(r_2,\widetilde{\lambda})=$$

$$=\eta_1\left(K_2-K_3\right)w_3(r_2,\widetilde{\lambda})=0.$$

Since $\eta_1 (K_2 - K_3) \neq 0$, we obtain

$$w_3\left(r_2, \widetilde{\lambda}\right) = 0. \tag{1.15}$$

By the same procedure arising from (1.7), we see that

$$w_3'\left(r_2, \widetilde{\lambda}\right) = 0. \tag{1.16}$$

From the fact that $w_3(x, \widetilde{\lambda})$ is a solution of the differential Eq. (1.1) on $[r_2, \pi]$ and satisfies the initial conditions (1.15) and (1.16), $w_3(x, \widetilde{\lambda}) = 0$ identically on $[r_2, \pi]$ (cf. [2, p. 12], Theorem 1.2.1).

By using this procedure, we may also find

$$w_1\left(r_1, \widetilde{\lambda}\right) = w_1'\left(r_1, \widetilde{\lambda}\right) = w_2\left(r_2, \widetilde{\lambda}\right) = w_2'\left(r_2, \widetilde{\lambda}\right) = 0.$$

Thus, we have $w_2(x, \widetilde{\lambda}) = 0$ and $w_1(x, \widetilde{\lambda}) = 0$ identically on $[0, r_1) \cup (r_1, r_2) \cup (r_2, \pi]$. But this contradicts (1.8), thus completing the proof.

2. An existence theorem. The function $w(x, \lambda)$ defined in Section 1 is a nontrivial solution of Eq. (1.1) satisfying conditions (1.2), (1.4), (1.5) and (1.6). Putting $w(x, \lambda)$ into (1.3), we get the characteristic equation

$$F(\lambda) \equiv w'(\pi, \lambda) + \lambda w(\pi, \lambda) = 0. \tag{2.1}$$

By Theorem 1.1, the set of eigenvalues of boundary-value problem (1.1) – (1.7) coincides with the set of real roots of Eq. (2.1). Let

$$q_1 = \frac{1}{p_1}\int_0^{r_1} |q(\tau)| d\tau, \qquad q_2 = \frac{1}{p_2}\int_{r_1}^{r_2} |q(\tau)|\, d\tau \qquad \text{and} \qquad q_3 = \frac{1}{p_3}\int_{r_2}^{\pi} |q(\tau)|\, d\tau.$$

Lemma 2.1. (1) *Let* $\lambda \geq 4q_1^2$. *Then for the solution* $w_1(x, \lambda)$ *of Eq.* (1.11), *the following inequality holds*:

$$|w_1(x, \lambda)| \leq 2, \quad x \in [0, r_1]. \tag{2.2}$$

(2) *Let* $\lambda \geq \max\left\{4q_1^2, 4q_2^2\right\}$. *Then for the solution* $w_2(x, \lambda)$ *of Eq.* (1.12), *the following inequality holds*:

$$|w_2(x, \lambda)| \leq 4\left(\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right), \quad x \in [r_1, r_2]. \tag{2.3}$$

(3) *Let* $\lambda \geq \max\left\{4q_1^2, 4q_2^2, 4q_3^2\right\}$. *Then for the solution* $w_2(x, \lambda)$ *of Eq.* (1.13), *the following inequality holds*:

$$|w_3(x, \lambda)| \leq \frac{8\theta_1 p_2 + 4\theta_2 p_3 \eta_1}{\eta_1 p_2 \eta_2}\left(\left|\frac{\gamma_1}{\delta_1}\right| + \left|\frac{p_2\gamma_2}{p_1\delta_2}\right|\right) + \frac{\theta_2 p_3}{\eta_2}\left|\frac{4\gamma_1\delta_2 q_1 + \gamma_2 p_2 \delta_1}{2p_2\delta_1\delta_2 q_1}\right|, \quad x \in [r_2, \pi]. \tag{2.4}$$

Proof. Let $B_{1\lambda} = \max_{[0,r_1]} |w_1(x,\lambda)|$. Then from (1.11), for any $\lambda > 0$, the following inequality holds:

$$B_{1\lambda} \le 1 + \frac{1}{s} B_{1\lambda} q_1.$$

If $s \ge 2q_1$ we get (2.2). Differentiating (1.11) with respect to x, we have

$$w_1'(x,\lambda) = -\frac{s}{p_1} \sin \frac{s}{p_1} x - \frac{1}{p_1^2} \int_0^x q(\tau) \cos \frac{s}{p_1} (x-\tau)\, w_1(\tau - \Delta(\tau), \lambda)\, d\tau. \qquad (2.5)$$

Taking into account (2.5) and (2.2), for $s \ge 2q_1$, the following inequality holds:

$$\frac{|w_1'(x,\lambda)|}{s} \le \frac{2}{p_1}. \qquad (2.6)$$

Let $B_{2\lambda} = \max_{[r_1,r_2]} |w_2(x,\lambda)|$. Then from (1.12), (2.2) and (2.6), for $s \ge 2q_1$, the following inequality holds:

$$B_{2\lambda} \le 4 \left\{ \left| \frac{\gamma_1}{\delta_1} \right| + \left| \frac{p_2 \gamma_2}{p_1 \delta_2} \right| \right\}.$$

Hence if $\lambda \ge \max\left\{4q_1^2, 4q_2^2\right\}$ we get (2.3).

Differentiating (1.12) with respect to x, we obtain

$$w_2'(x,\lambda) = -\frac{s\gamma_1}{p_2\delta_1} w_1(r_1,\lambda) \sin \frac{s}{p_2}(x-r_1) + \frac{\gamma_2 w_1'(r_1,\lambda)}{\delta_2} \cos \frac{s}{p_2}(x-r_1) -$$

$$-\frac{1}{p_2^2} \int_{r_1}^x q(\tau) \cos \frac{s}{p_2}(x-\tau) w_2(\tau - \Delta(\tau), \lambda)\, d\tau. \qquad (2.7)$$

By virtue of (2.7) and (2.3), for $s \ge 2q_2$, the following inequality holds true:

$$\frac{|w_2'(x,\lambda)|}{s} \le \frac{2\gamma_1}{p_2\delta_1} + \frac{\gamma_2}{2\delta_2 q_1} + \frac{2}{p_2} \left\{ \left| \frac{\gamma_1}{\delta_1} \right| + \left| \frac{p_2\gamma_2}{p_1\delta_2} \right| \right\}. \qquad (2.8)$$

Let $B_{3\lambda} = \max_{[r_2,\pi]} |w_3(x,\lambda)|$. Then from (1.13), (2.2), (2.3) and (2.8), for $s \ge 2q_3$, the following inequality holds:

$$B_{3\lambda} \le \frac{8\theta_1 p_2 + 4\theta_2 p_3 \eta_1}{\eta_1 p_2 \eta_2} \left(\left| \frac{\gamma_1}{\delta_1} \right| + \left| \frac{p_2\gamma_2}{p_1\delta_2} \right| \right) + \frac{\theta_2 p_3}{\eta_2} \left| \frac{4\gamma_1\delta_2 q_1 + \gamma_2 p_2 \delta_1}{2p_2\delta_1\delta_2 q_1} \right|.$$

Hence, if $\lambda \ge \max\left\{4q_1^2, 4q_2^2, 4q_3^2\right\}$, then we arrive at Eq. (2.4).

Theorem 2.1. *Problem* (1.1)–(1.7) *has an infinite set of positive eigenvalues.*

Proof. Differentiating (1.13) with respect to x, we have

$$w_3'(x,\lambda) = -\frac{s\theta_1}{p_3\eta_1} w_2(r_2,\lambda) \sin \frac{s}{p_3}(x-r_2) + \frac{\theta_2 w_2'(r_2,\lambda)}{\eta_2} \cos \frac{s}{p_3}(x-r_2) -$$

$$-\frac{1}{p_3^2} \int_{r_2}^x q(\tau) \cos \frac{s}{p_3}(x-\tau) w_3(\tau - \Delta(\tau), \lambda) d\tau. \qquad (2.9)$$

From (1.11) – (1.13), (2.1), (2.5), (2.7) and (2.9), we get

$$-\frac{s\theta_1}{p_3\eta_1}\left[\frac{\gamma_1}{\delta_1}\left(\cos\frac{sr_1}{p_1}-\frac{1}{sp_1}\int\limits_0^{r_1}q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}(r_2-r_1)+\right.$$

$$+\frac{\gamma_2p_2}{s\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1}q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)-$$

$$\left.-\frac{1}{sp_2}\int\limits_{r_1}^{r_2}q(\tau)\sin\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right]\sin\frac{s}{p_3}(\pi-r_2)+$$

$$+\frac{\theta_2}{\eta_2}\left[-\frac{s\gamma_1}{p_2\delta_1}\left(\cos\frac{sr_1}{p_1}-\frac{1}{sp_1}\int\limits_0^{r_1}q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)+\right.$$

$$+\frac{\gamma_2}{\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1}q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}(r_2-r_1)-$$

$$\left.-\frac{1}{p_2^2}\int\limits_{r_1}^{r_2}q(\tau)\cos\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\frac{s}{p_3}(\pi-r_2)-$$

$$-\frac{1}{p_3^2}\int\limits_{r_2}^{\pi}q(\tau)\cos\frac{s}{p_3}(\pi-\tau)\,w_3\left(\tau-\Delta\left(\tau\right),\lambda\right)d\tau+$$

$$+\lambda\left\{\frac{\theta_1}{\eta_1}\left[\frac{\gamma_1}{\delta_1}\left(\cos\frac{sr_1}{p_1}-\frac{1}{sp_1}\int\limits_0^{r_1}q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}(r_2-r_1)+\right.\right.$$

$$+\frac{\gamma_2p_2}{s\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1}q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)-$$

$$\left.-\frac{1}{sp_2}\int\limits_{r_1}^{r_2}q(\tau)\sin\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\right]\cos\frac{s}{p_3}(\pi-r_2)+$$

$$+\frac{\theta_2p_3}{s\eta_2}\left[-\frac{s\gamma_1}{p_2\delta_1}\left(\cos\frac{sr_1}{p_1}-\frac{1}{sp_1}\int\limits_0^{r_1}q(\tau)\sin\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\sin\frac{s}{p_2}(r_2-r_1)+\right.$$

$$+\frac{\gamma_2}{\delta_2}\left(-\frac{s}{p_1}\sin\frac{sr_1}{p_1}-\frac{1}{p_1^2}\int\limits_0^{r_1}q(\tau)\cos\frac{s}{p_1}(r_1-\tau)w_1(\tau-\Delta(\tau),\lambda)d\tau\right)\cos\frac{s}{p_2}(r_2-r_1)-$$

$$-\frac{1}{p_2^2}\int_{r_1}^{r_2} q(\tau)\cos\frac{s}{p_2}(r_2-\tau)w_2(\tau-\Delta(\tau),\lambda)d\tau\Bigg]\sin\frac{s}{p_3}(\pi-r_2)-$$

$$-\frac{1}{sp_3}\int_{r_2}^{\pi} q(\tau)\sin\frac{s}{p_3}(\pi-\tau)\,w_3(\tau-\Delta(\tau),\lambda)\,d\tau\Bigg\}=0. \tag{2.10}$$

Let λ be sufficiently large. Then, by (2.2) – (2.4), Eq. (2.10) may be rewritten in the form

$$s\cos s\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)+O(1)=0. \tag{2.11}$$

Obviously, for large s Eq. (2.11) has an infinite set of roots. Thus, we arrive at the desired result.

3. Asymptotic formulas for eigenvalues and eigenfunctions. Now we begin to study asymptotic properties of eigenvalues and eigenfunctions. In the following we shall assume that s is sufficiently large. From (1.11) and (2.2), we get

$$w_1(x,\lambda)=O(1). \tag{3.1}$$

From expressions of (1.12) and (2.3), we see that

$$w_2(x,\lambda)=O(1). \tag{3.2}$$

By virtue of (1.13) and (2.4), we procure the following equation:

$$w_3(x,\lambda)=O(1). \tag{3.3}$$

The existence and continuity of the derivatives $w'_{1s}(x,\lambda)$ for $0\le x\le r_1$, $|\lambda|<\infty$, $w'_{2s}(x,\lambda)$ for $r_1\le x\le r_2$, $|\lambda|<\infty$ and $w'_{3s}(x,\lambda)$ for $r_2\le x\le\pi, |\lambda|<\infty$ follows from Theorem 1.4.1 in [2]:

$$\begin{aligned} w'_{1s}(x,\lambda)&=O(1), \quad x\in[0,r_1],\\ w'_{2s}(x,\lambda)&=O(1), \quad x\in[r_1,r_2],\\ w'_{3s}(x,\lambda)&=O(1), \quad x\in[r_2,\pi]. \end{aligned} \tag{3.4}$$

Theorem 3.1. *Let n be a natural number. For each sufficiently large n, there is exactly one eigenvalue of problem* (1.1) – (1.7) *near* $\dfrac{(n+1/2)^2\pi^2}{\left(r_1/p_1+(r_2-r_1)/p_2+(\pi-r_2)/p_3\right)^2}$.

Proof. We consider the expression which is denoted by $O(1)$ in Eq. (2.11). If formulas (3.1) – (3.4) are taken into consideration, it can be shown by differentiation with respect to s that for large s this expression has bounded derivative. We shall show that, for large n, only one root of (2.11) lies near to each $\dfrac{(n+1/2)^2\pi^2}{\left(r_1/p_1+(r_2-r_1)/p_2+(\pi-r_2)/p_3\right)^2}$. Let us consider the function

$$\phi(s)=s\cos s\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)+O(1).$$

Its derivative, which has the form

$$\phi'(s) = \cos s\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right) -$$
$$-s\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right)\sin s\left(\frac{r_1}{p_1} + \frac{r_2 - r_1}{p_2} + \frac{\pi - r_2}{p_3}\right) + O(1),$$

does not vanish for s close to sufficiently large n. Thus our assertion follows by Rolle's theorem.

Let n be sufficiently large. In what follows we shall denote by $\lambda_n = s_n^2$ the eigenvalue of problem (1.1) – (1.7) situated near $\dfrac{(n + 1/2)^2\,\pi^2}{\left(r_1/p_1 + (r_2 - r_1)/p_2 + (\pi - r_2)/p_3\right)^2}$. We set

$$s_n = \frac{\left(n + \dfrac{1}{2}\right)\pi}{\left(\dfrac{r_1}{p_1} + \dfrac{r_2 - r_1}{p_2} + \dfrac{\pi - r_2}{p_3}\right)} + \delta_n.$$

From (2.11) $\delta_n = O\left(\dfrac{1}{n}\right)$. Consequently, we procure

$$s_n = \frac{\left(n + \dfrac{1}{2}\right)\pi}{\left(\dfrac{r_1}{p_1} + \dfrac{r_2 - r_1}{p_2} + \dfrac{\pi - r_2}{p_3}\right)} + O\left(\frac{1}{n}\right). \tag{3.5}$$

Formula (3.5) make it possible to obtain asymptotic expressions for eigenfunction of problem (1.1) – (1.7). By (1.11), (2.5) and (3.1), we have

$$w_1(x,\lambda) = \cos\frac{sx}{p_1} + O\left(\frac{1}{s}\right), \tag{3.6}$$

$$w_1'(x,\lambda) = -\frac{s}{p_1}\sin\frac{sx}{p_1} + O\left(1\right). \tag{3.7}$$

By means of (1.12), (3.2), (3.6) and (3.7), we acquire

$$w_2(x,\lambda) = \frac{\gamma_1}{\delta_1}\cos\frac{s}{p_2}\left(\frac{r_1\left(p_2 - p_1\right)}{p_1} + x\right) + O\left(\frac{1}{s}\right), \tag{3.8}$$

$$w_2'(x,\lambda) = -\frac{s\gamma_1}{\delta_1 p_2}\sin\frac{s}{p_2}\left(\frac{r_1\left(p_2 - p_1\right)}{p_1} + x\right) + O\left(1\right). \tag{3.9}$$

In view of (1.13), (3.3), (3.8) and (3.9), we attain the following:

$$w_3(x,\lambda) = \frac{\theta_1\gamma_1}{\eta_1\delta_1}\cos\frac{s}{p_3}\left(\frac{p_3\left(r_1\left(p_2 - p_1\right) + p_1 r_2\right) - r_2 p_1 p_2}{p_1 p_2} + x\right) + O\left(\frac{1}{s}\right). \tag{3.10}$$

Putting (3.5) into (3.6), (3.8) and (3.10), we readily derive

$$u_{1n}(x)=\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi x}{p_1\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)}\right)+O\left(\frac{1}{n}\right),$$

$$u_{2n}(x)=\frac{\gamma_1}{\delta_1}\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{p_2\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)}\left(\frac{r_1\left(p_2-p_1\right)}{p_1}+x\right)\right)+O\left(\frac{1}{n}\right),$$

$$u_{3n}(x)=\frac{\theta_1\gamma_1}{\eta_1\delta_1}\times$$

$$\times\cos\left(\frac{\left(n+\frac{1}{2}\right)\pi}{p_3\left(\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3}\right)}\left(\frac{p_3(r_1\left(p_2-p_1\right)+p_1r_2)-r_2p_1p_2}{p_1p_2}+x\right)\right)+O\left(\frac{1}{n}\right).$$

Hence the eigenfunctions $u_n(x)$ have the following asymptotic representation:

$$u_n(x)=\begin{cases}u_{1n}(x)=w_1\left(x,\lambda_n\right), & x\in[0,r_1),\\ u_{2n}(x)=w_2\left(x,\lambda_n\right), & x\in\left(r_1,r_2\right),\\ u_{3n}(x)=w_3\left(x,\lambda_n\right), & x\in\left(r_2,\pi\right].\end{cases}$$

Under some additional conditions the more exact asymptotic formulas which depend upon the retardation may be obtained. Let us assume that the following conditions are fulfilled:

(a) the derivatives $q'(x)$ and $\Delta''(x)$ exist and are bounded in $[0,r_1)\cup(r_1,r_2)\cup(r_2,\pi]$ and have finite limits

$$q'(r_1\pm 0)=\lim_{x\to r_1\pm 0}q'(x),\qquad q'(r_2\pm 0)=\lim_{x\to r_2\pm 0}q'(x),\qquad \Delta''(r_1\pm 0)=\lim_{x\to r_1\pm 0}\Delta''(x)$$

and

$$\Delta''(r_2\pm 0)=\lim_{x\to r_2\pm 0}\Delta''(x),$$

respectively;

(b) $\Delta'(x)\leq 1$ in $[0,r_1)\cup(r_1,r_2)\cup(r_2,\pi]$, $\Delta(0)=0, \lim_{x\to r_1+0}\Delta(x)=0$ and $\lim_{x\to r_2+0}\Delta(x)=$ $=0$.

By using (b), we have

$$\begin{aligned}&x-\Delta(x)\geq 0, \quad \text{if} \quad x\in[0,r_1),\\ &x-\Delta(x)\geq r_1, \quad \text{if} \quad x\in(r_1,r_2),\\ &x-\Delta(x)\geq r_2, \quad \text{if} \quad x\in(r_2,\pi].\end{aligned}\tag{3.11}$$

From (3.6), (3.8), (3.10) and (3.11), we have

$$w_1(\tau-\Delta(\tau),\lambda)=\cos\frac{s(\tau-\Delta(\tau))}{p_1}+O\left(\frac{1}{s}\right),$$
$$w_2(\tau-\Delta(\tau),\lambda)=\frac{\gamma_1}{\delta_1}\frac{\gamma_1}{\delta_1}\cos\frac{s}{p_2}\left(\frac{r_1(p_2-p_1)}{p_1}+\tau-\Delta(\tau)\right)+O\left(\frac{1}{s}\right),\tag{3.12}$$
$$w_3(\tau-\Delta(\tau),\lambda)=\frac{\theta_1\gamma_1}{\eta_1\delta_1}\cos\frac{s}{p_3}\left(\frac{p_3(r_1(p_2-p_1)+p_1r_2)-r_2p_1p_2}{p_1p_2}+\tau-\Delta(\tau)\right)+O\left(\frac{1}{s}\right).$$

Under the conditions (a) and (b), the following formulas:

$$O\left(\frac{1}{s}\right)=\begin{cases}\displaystyle\int_0^{r_1}\frac{q(\tau)}{2}\sin\frac{s}{p_1}(2\tau-\Delta(\tau))\,d\tau,\\ \displaystyle\int_0^{r_1}\frac{q(\tau)}{2}\cos\frac{s}{p_1}(2\tau-\Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_1}^{r_2}\frac{q(\tau)}{2}\sin\frac{s}{p_2}(2\tau-\Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_1}^{r_2}\frac{q(\tau)}{2}\cos\frac{s}{p_2}(2\tau-\Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_2}^{\pi}\frac{q(\tau)}{2}\sin\frac{s}{p_3}(2\tau-\Delta(\tau))\,d\tau,\\ \displaystyle\int_{r_2}^{\pi}\frac{q(\tau)}{2}\cos\frac{s}{p_3}(2\tau-\Delta(\tau))\,d\tau\end{cases}\tag{3.13}$$

can be proved by the same technique in Lemma 3.3.3 in [2]. Using the abbreviations

$$A(x)=\int_0^x\frac{q(\tau)}{2}\sin\frac{s\Delta(\tau)}{p_1}d\tau,\qquad B(x)=\int_0^x\frac{q(\tau)}{2}\cos\frac{s\Delta(\tau)}{p_1}d\tau,$$

$$C(x)=\int_{r_1}^x\frac{q(\tau)}{2}\sin\frac{s\Delta(\tau)}{p_2}d\tau,\qquad D(x)=\int_{r_1}^x\frac{q(\tau)}{2}\cos\frac{s\Delta(\tau)}{p_2}d\tau,$$

$$E(x)=\int_{r_2}^x\frac{q(\tau)}{2}\sin\frac{s\Delta(\tau)}{p_3}d\tau,\qquad F(x)=\int_{r_2}^x\frac{q(\tau)}{2}\cos\frac{s\Delta(\tau)}{p_3}d\tau,$$

$$Z_p^r=\frac{r_1}{p_1}+\frac{r_2-r_1}{p_2}+\frac{\pi-r_2}{p_3},\qquad \Delta_p^r=\frac{1}{p_3}+\frac{B(r_1)}{p_1}+\frac{D(r_2)}{p_2}+\frac{F(\pi)}{p_3}$$

and putting expressions (3.13) into (2.10), and then using $s_n=\dfrac{(n+1/2)\pi}{Z_p^r}+\delta_n$ we get $\delta_n=$ $=-\dfrac{\Delta_p^r}{(n+1/2)\pi}+O\left(\dfrac{1}{n^2}\right)$ and finally

$$s_n=\frac{\left(n+\frac{1}{2}\right)\pi}{Z_p^r}-\frac{\Delta_p^r}{\left(n+\frac{1}{2}\right)\pi}+O\left(\frac{1}{n^2}\right).\tag{3.14}$$

Thus, we proved the following theorem.

Theorem 3.2. *If conditions* (a) *and* (b) *are satisfied, then the positive eigenvalues* $\lambda_n = s_n^2$ *of problem* (1.1)–(1.7) *have* (3.14) *asymptotic representation for* $n \to \infty$.

We now may obtain a more accurate asymptotic formula for the eigenfunctions. From (1.11) and (3.12)

$$w_1(x,\lambda) = \cos\frac{sx}{p_1}\left[1 + \frac{A(x)}{sp_1}\right] - \frac{B(x)\sin\dfrac{sx}{p_1}}{sp_1} + O\left(\frac{1}{s^2}\right). \tag{3.15}$$

Replacing s by s_n and using (3.14) we have

$$u_{1n}(x) = \cos\frac{\left(n+\dfrac{1}{2}\right)\pi x}{p_1 Z_p^r}\left[1 + \frac{A(x)Z_p^r}{\left(n+\dfrac{1}{2}\right)\pi p_1}\right] + \left[\frac{x\Delta_p^r}{\left(n+\dfrac{1}{2}\right)\pi p_1}\right]\sin\frac{\left(n+\dfrac{1}{2}\right)\pi x}{p_1 Z_p^r} + O\left(\frac{1}{n^2}\right). \tag{3.16}$$

From (1.12), (2.5), (3.12), (3.13) and (3.15) we obtain

$$w_2(x,\lambda) = \frac{\gamma_1}{\delta_1}\left\{\left[1 + \frac{1}{s}\left(\frac{A(r_1)}{p_1} + \frac{C(x)}{p_2}\right)\right]\cos\left(\frac{s}{p_2}\left(\frac{r_1(p_2-p_1)}{2p_1} + x\right)\right) - \right.$$
$$\left. - \frac{(D(x)/p_2 + B(r_1)/p_1)}{s}\sin\frac{s}{p_2}\left(\frac{r_1(p_2-p_1)}{2p_1} + x\right)\right\} + O\left(\frac{1}{s^2}\right). \tag{3.17}$$

Now, replacing s by s_n and using (3.14), we get

$$u_{2n}(x) = \frac{\gamma_1}{\delta_1}\left\{\left[1 + \frac{Z_p^r\left(\dfrac{A(r_1)}{p_1} + \dfrac{C(x)}{p_2}\right)}{\left(n+\dfrac{1}{2}\right)\pi}\right]\cos\left(\frac{\left(n+\dfrac{1}{2}\right)\pi}{Z_p^r p_2}\left(\frac{r_1(p_2-p_1)}{2p_1} + x\right)\right) + \right.$$
$$+ \frac{Z_p^r\Delta_p^r\left(\dfrac{D(x)}{p_2} + \dfrac{B(r_1)}{p_1}\right)\left(\dfrac{r_1(p_2-p_1)}{2p_1} + x\right)}{p_2\left(n+\dfrac{1}{2}\right)^2\pi^2}\times$$
$$\left.\times\sin\left(\frac{\left(n+\dfrac{1}{2}\right)\pi}{Z_p^r p_2}\left(\frac{r_1(p_2-p_1)}{2p_1} + x\right)\right)\right\} + O\left(\frac{1}{n^2}\right). \tag{3.18}$$

From (1.13), (2.7), (3.12), (3.13) and (3.17) we have

$$w_3(x,\lambda) = \frac{\theta_1\gamma_1}{\eta_1\delta_1}\left\{\left[1 + \frac{\left(\dfrac{A(r_1)}{p_1} + \dfrac{C(r_2)}{p_2} + \dfrac{E(x)}{p_3}\right)}{s}\right]\times\right.$$

$$\times \cos\left(\frac{s}{p_3}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)\right)-$$

$$-\frac{1}{s}\left(\frac{B(r_1)}{p_1}+\frac{D(r_2)}{p_2}+\frac{F(x)}{p_3}\right)\times$$

$$\left.\times \sin\left(\frac{s}{p_3}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)\right)\right\}+O\left(\frac{1}{s^2}\right).$$

Now, replacing s by s_n and using (3.14), we obtain

$$u_{3n}(x)=\frac{\theta_1\gamma_1}{\eta_1\delta_1}\left\{\left[1+\frac{Z_p^r\left(\dfrac{A(r_1)}{p_1}+\dfrac{C(r_2)}{p_2}+\dfrac{E(x)}{p_3}\right)}{\left(n+\dfrac{1}{2}\right)\pi}\right]\times\right.$$

$$\times \cos\left(\frac{\left(n+\dfrac{1}{2}\right)\pi}{Z_p^rp_3}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)\right)+$$

$$+\frac{Z_p^r\Delta_p^r\left(\dfrac{B(r_1)}{p_1}+\dfrac{D(r_2)}{p_2}+\dfrac{F(x)}{p_3}\right)}{p_3\left(n+\dfrac{1}{2}\right)^2\pi^2}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)\times$$

$$\left.\times \sin\left(\frac{\left(n+\dfrac{1}{2}\right)\pi}{Z_p^rp_3}\left(\frac{p_3\left(r_1\left(p_2-p_1\right)+p_1r_2\right)-r_2p_1p_2}{p_1p_2}+x\right)\right)\right\}+O\left(\frac{1}{n^2}\right). \qquad (3.19)$$

Thus, we have proven the following theorem.

Theorem 3.3. *If conditions* (a) *and* (b) *are satisfied*, *then the eigenfunctions* $u_n(x)$ *of problem* (1.1) – (1.7) *have the following asymptotic representation for* $n\to\infty$:

$$u_n(x)=\begin{cases}u_{1n}(x), & x\in[0,r_1),\\ u_{2n}(x), & x\in(r_1,r_2),\\ u_{3n}(x), & x\in(r_2,\pi],\end{cases}$$

where $u_{1n}(x)$, $u_{2n}(x)$ *and* $u_{3n}(x)$ *defined as in* (3.16), (3.18) *and* (3.19), *respectively.*

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