UDC 512.5

C. Li (School Math. and Statistics, Jiangsu Normal Univ., Xuzhou, China),
X. Zhang (Yancheng Inst. Technology, China),
X. Yi (School Sci., Zhejiang Sci.-Techn. Univ., Hangzhou, China)

A NOTE ON SΦ-SUPPLEMENTED SUBGROUPS*

ПРО SΦ-ДОПОВНЕНІ ПІДГРУПИ

We give new and brief proofs of the results obtained by X. Li and T. Zhao in [SΦ-supplemented subgroups of finite groups // Ukr. Math. J. – 2012. – **64**, № 1. – P. 102–109].

Наведено нові та короткі доведення результатів, що були отримані X. Лі та T. Жао в [SΦ-supplemented subgroups of finite groups // Ukr. Math. J. – 2012. – **64**, № 1. – P. 102 – 109].

All groups considered in this note will be finite. In [2], X. Li and T. Zhao called that a subgroup H of a group G is SΦ-supplemented in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq \Phi(H)$, where $\Phi(H)$ is the Frattini subgroup of H. They investigated the influence of SΦ-supplemented subgroups on the p-nilpotency, supersolvability and formation. However, some conditions of many theorems are unnecessary and may been removed. Moreover, the proofs are complicated. In the note, we prove some new results which can deduce their theorems and consequently simplify their proofs.

1. Preliminaries.

Lemma 1.1. *Let p be a fixed prime dividing the order of a group G and P a Sylow p-subgroup of G. Suppose that all maximal subgroups of P are* SΦ*-supplemented in G. Then, either G is a group whose Sylow p-subgroups are of order p, or G is a p-nilpotent group.*

Proof. Suppose that $|P| > p$. Let M be a maximal subgroup of P. By the hypothesis, M is SΦ-supplemented in G. Then there exists a subnormal subgroup T of G such that $MT = G$ and $M \cap T \leq \Phi(M)$. Obviously, $T < G$. Since $|G : T|$ is a power of p and $T \lhd \lhd G$, we have $O^p(G) \leq T$. Consequently, $O^p(G) < G$ and so $O^p(G) \cap P < P$. We may choose a maximal subgroup P_1 of P such that $O^p(G) \cap P \leq P_1 < P$. Then $O^p(G) \cap P = O^p(G) \cap P_1$. Since P_1 is SΦ-supplemented in G, there is also a subnormal K such that $O^p(G) \leq K$ and $P_1 \cap K \leq \Phi(P_1)$ as above arguments. It follows that $O^p(G) \cap P = O^p(G) \cap P_1 \leq K \cap P_1 \leq \Phi(P_1) \leq \Phi(P)$. By Tate's theorem (see [1], IV, 4.7), $O^p(G)$ is p-nilpotent. It follows that G is p-nilpotent.

Lemma 1.2. *Let p be a fixed prime dividing the order of a group G and P a Sylow p-subgroup of G. Suppose that P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is* SΦ*-supplemented in G. Then G is p-nilpotent.*

Proof. Suppose that $|P : D| = p$. Then, since $p \leq |D|$, we have that $p^2 \leq |P|$. By Lemma 1.1, the group G is p-nilpotent. Hence we may assume that $|P : D| > p$. Let $H \leq P$ such that $|H| = |D|$. By the hypothesis, H is SΦ-supplemented in G. Then there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T \leq \Phi(H)$. Obviously, $T < G$. Hence G has a proper normal K such that $T \leq K$. Since G/K is a p-group, G has a normal maximal subgroup M such that $HM = G$ and $|G : M| = p$. It is easy to see that M satisfies the hypotheses of the theorem. By induction, M is p-nilpotent, and so G is p-nilpotent.

* This paper was supported by the Natural Science Foundation of China (No. 11401264 and 11571145) and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

Lemma 1.3. *Let E be a normal subgroup of a group G. Suppose that for every prime p dividing $|E|$, there exists a Sylow p-subgroup P of E such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is* SΦ*-supplemented in G. Then every G-chief factor of E is cyclic.*

Proof. By [2] (Lemma 2.1(1)), every subgroup H of P with $|H| = |D|$ is SΦ-supplemented in E. Applying Lemma 1.2, E is p-nilpotent. Let $E_{p'}$ be the normal p-complement of E. Obviously, $E_{p'}$ is normal in G. First, assume that $E_{p'} \neq 1$. It is easy to see that the hypotheses of the theorem hold for $(G/E_{p'}, E/E_{p'})$ and $(G, E_{p'})$. By induction, $E/E_{p'} \leq Z_{\mathcal{U}}(G/E_{p'})$ and $E_{p'} \leq Z_{\mathcal{U}}(G)$. Hence, $E \leq Z_{\mathcal{U}}(G)$. Now, assume that $P = E$. In view of [2] (Lemma 2.4), we may assume $|P : D| > p$. Let $H \leq P$ such that $|H| = |D|$. By the hypothesis, H is SΦ-supplemented in G. Then there exists a subnormal subgroup T of G such that $HT = G$ and $H \cap T \leq \Phi(H)$. It is easy to see that G has a normal maximal subgroup M such that $HM = G$ and $|G : M| = p$. It is easy to see that $P \cap M$ satisfies the hypotheses of the theorem. By induction, $P \cap M \leq Z_{\mathcal{U}}(G)$. Note that $|P/P \cap M| = p$, we have $P \leq Z_{\mathcal{U}}(G)$.

2. Brief proofs of results in [2].

Theorem 2.1 ([2], Theorem 3.1)**.** *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ such that $(|G|, p-1) = 1$. If every maximal subgroup of P is* SΦ*-supplemented in G, then G is p-nilpotent.*

Proof. By Lemma 1.1, $|P| = p$ or G is p-nilpotent. If $|P| = p$, then G is also p-nilpotent by [4] (Lemma 2.6).

From Lemma 1.2, we arrive at the following Theorems 2.2 and 2.3.

Theorem 2.2 ([2], Theorem 3.2)**.** *Let G be a group and P a Sylow p-subgroup of G, where p is a prime dividing $|G|$ such that $(|G|, p-1) = 1$. Let D be a subgroup of P such that $1 < |D| < |P|$. If every subgroup H of P with $|H| = |D|$ is* SΦ*-supplemented in G, then G is p-nilpotent.*

Theorem 2.3 ([2], Theorem 3.3)**.** *Let p be a prime dividing $|G|$ and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and there exists a subgroup D of P such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is* SΦ*-supplemented in G, then G is p-nilpotent.*

Remark. From Lemma 1.2, the conditions that $(|G|, p-1) = 1$ in Theorem 2.2 and $N_G(P)$ is p-nilpotent in Theorem 2.3 are unnecessary.

Theorem 2.4 ([2], Theorem 3.4)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and E a normal subgroup of a group G such that $G/E \in \mathcal{F}$. If for every prime p dividing $|E|$, there exists a Sylow p-subgroup P of E such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is* SΦ*-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Since $E \leq Z_{\mathcal{U}}(G)$ by Lemma 1.3 and $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$, we have $E \leq Z_{\mathcal{F}}(G)$ and so $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$. It follows that $G \in \mathcal{F}$.

Theorem 2.5 ([2], Theorem 3.5)**.** *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and E a normal subgroup of a group G such that $G/E \in \mathcal{F}$. If for every prime p dividing $|F^*(E)|$, there exists a Sylow p-subgroup P of $F^*(E)$ such that P has a subgroup D satisfying $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is* SΦ*-supplemented in G, then $G \in \mathcal{F}$.*

Proof. Since $F^*(E) \leq Z_{\mathcal{U}}(G)$ by Lemma 1.3 and $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$, we have $F^*(E) \leq Z_{\mathcal{F}}(G)$. By [3] (Theorem B), $E \leq Z_{\mathcal{F}}(G)$. Since $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$, we have $G \in \mathcal{F}$.

References

1. *Huppert B.* Endiche Gruppen I. – Berlin: Springer-Verlag, 1968.
2. *Li X., Zhao T.* SΦ-supplemented subgroups of finite groups // Ukr. Math. J. – 2012. – **64**, № 1. – P. 102 – 109.
3. *Skiba A. N.* On two questions of L. A. Shemetkov concerning hypercyclically embedded subgroups of finite groups // J. Group Theory. – 2010. – **13**. – P. 841 – 850.
4. *Wei H., Wang Y.* c^*-Supplemented subgroups and p-nilpotency of finite groups // Ukr. Math. J. – 2007. – **59**, № 8. – P. 1011 – 1019.

Received 27.09.13