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ON THE DERIVED NORM OF A FINITE GROUP*

ПРО ПОХІДНУ НОРМУ СКІНЧЕННОЇ ГРУПИ

Given a finite group G, we define a subgroup $CS(G)$ as the intersection of the normalizers of all subgroups of the derived subgroup of G. Let $CS_0 = 1$. We define $CS_{i+1}(G)/CS_i(G) = CS(G/CS_i(G))$ for $i \geq 1$. By $CS_\infty(G)$ we denote the terminal term of the upper series. It is proved that the derived subgroup G' is nilpotent if and only if $G = CS_\infty(G)$. In particular, we obtain the following result: if all elements of odd prime order of G are in $CS(G)$, then G is solvable.

Для заданої групи G її підгрупу $CS(G)$ визначено як перетин нормалізаторів усіх підгруп похідної підгрупи G. Нехай $CS_0 = 1$. Визначимо $CS_{i+1}(G)/CS_i(G) = CS(G/CS_i(G))$ для $i \geq 1$. Позначимо останній член верхнього ряду через $CS_\infty(G)$. Доведено, що похідна підгрупа G є нільпотентною тоді і тільки тоді, коли $G = CS_\infty(G)$. Зокрема, ми отримали такий результат: якщо всі елементи непарного простого порядку в G належать $CS(G)$, то G є розв'язною.

1. Introduction. Let G be a finite group (all groups considered in this paper are finite). The notation and terminology used in this paper are standard, as in [9, 10, 12]. It is known that if the derived subgroup G' of G normalizes each subgroup of G, then G is nilpotent [2]. Naturally, as a dual problem, one can ask that what can be said about the finite groups G satisfying the following condition: G normalizes all subgroups of the derived subgroup G' of G? Note that R. Baer and H. Wielandt in 1934 and 1958, respectively, introduced the following concepts: $N(G)$ denote the intersection of the normalizers of all subgroups of G and $\omega(G)$ denote the intersection of the normalizers of all subnormal subgroups of G. Those concepts were investigated by many authors, for example, see [1, 2, 4 – 6, 8, 11, 13, 14, 18 – 26]. In the note we give an answer to the above question. In fact, we shall study this question in a more general way. First of all, we give the following definition.

Definition 1.1. *Let G be a finite group. $CS(G)$ to be the intersection of the normalizers of all subgroups of the derived subgroup of G. That is,*

$$CS(G) = \bigcap_{H \leq G'} N_G(H).$$

Obviously, $CS(G)$ is a characteristic subgroup of G.

Definition 1.2. *For a finite group G, there exists a series of normal subgroups*

$$1 = CS_0(G) \leq CS_1(G) \leq CS_2(G) \leq \ldots \leq CS_n(G) \leq \ldots$$

satisfying $CS_{i+1}(G)/CS_i(G) = CS(G/CS_i(G))$ for $i = 0, 1, 2, \ldots$ and $CS_n(G) = CS_{n+1}(G)$ for some integer $n \geq 1$. Write $CS_\infty(G)$ for the terminal term of the upper series.

Remark. Fortunately, we find Group Theorists Chernikov and Subbotin proposed the above problem in [15 – 17], when we investigate that what can be said about the finite group G satisfying $G = CS(G)$ and $G = CS_\infty(G)$. So we think the above idea and problem are belong to Chernikov and Subbotin.

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In the memory of Group Theorist Chernikov, we give the following definition.

Definition 1.3. *A finite group G is called a CS-group (Chernikov – Subbotin-group) if $G = CS(G)$, that is, all subgroups of derived subgroup of G are normal in G.*

Throughout the paper, we denote by $\mathcal{F}_{dn}$ the class of finite groups G with G' nilpotent. It is well-known that $\mathcal{F}_{dn}$ is a saturated formation containing all supersoluble groups. The group $G = [N]H$ means a semidirect product of a normal subgroup N and a complement subgroup H. Moreover, H_G is the normal core of the subgroup H in G.

2. Preliminaries. First, we give two examples on $CS(G)$ as remarks. Example 2.1 indicates that the subgroup $CS(G)$ may be nonnilpotent. Example 2.2 shows that $CS(G) < G$ is possible when G is a nilpotent group.

***Example* 2.1.** Assume that $G = S_3$. Then $CS(G) = G$ is nonnilpotent.

***Example* 2.2.** As $\mathrm{Aut}\,(Q_8) \cong S_4$ and $D_8 \leq S_4$, we have the semidirect product $G = [Q_8]D_8$. Then G is a 2-group of order 2^6 and $CS(G) < G$.

Proof. Indeed, the derived subgroup A of D_8 is a subgroup of order 2, the action of A on Q_8 is faithful, so nonnormal in G. Thus $CS(G) < G$.

The case that $CS(G) = 1$ is possible for a solvable group G. For instance, the symmetry group S_4 of four letters satisfies $CS(S_4) = 1$.

The following basic properties of the subgroup $CS(G)$ are required in this paper.

Proposition 2.1. *If $M \leq G$, then $M \bigcap CS(G) \leq CS(M)$.*

Proof. Clearly, $CS(G) = \bigcap_{H \leq G'} N_G(H) \leq \bigcap_{H \leq M'} N_G(H)$. So $M \bigcap CS(G) \leq M \bigcap_{H \leq M'} N_G(H) = \bigcap_{H \leq M'} N_M(H) = CS(M)$.

Proposition 2.2. *Let $N \trianglelefteq G$. Then $CS(G)N/N \leq CS(G/N)$.*

Proof. Clear.

Proposition 2.3. *Let $G = A \times B$ and $(|A|, |B|) = 1$. Then $CS(G) = CS(A) \times CS(B)$.*

Proof. Let H be any subgroup of G' and let π be the set of primes dividing the order of A. Then A is a normal Hall π-subgroup of G and B is a normal Hall π'-subgroup of G. So $H \cap A'$ is a normal Hall π-subgroup of H and $H \cap B'$ is a normal Hall π'-subgroup of H. Since $G' = A' \times B'$, we have

$$H = (H \cap A') \times (H \cap B').$$

Thus

$$N_G(H) = N_G((H \cap A')(H \cap B')) =$$
$$= N_G((H \cap A')) \cap N_G((H \cap B')) =$$
$$= (N_A(H \cap A') \times B) \cap (A \times N_B(H \cap B')) =$$
$$= N_A((H \cap A')) \times N_B((H \cap B')).$$

Now the result follows.

Lemma 2.1 (R. Baer [3, p. 159], Corollary 2)**.** *The following properties of the group G are equivalent*:

(i) $G \in \mathcal{F}_{dn}$;

(ii) *every homomorphic image of G induces in each of its minimal normal subgroups is a cyclic group of automorphisms*;

(iii) *if* M *is a maximal subgroup of* G, *then* M/M_G *is cyclic*;

(iv) *if* M *is a maximal subgroup of* G, *then* M/M_G *is Abelian*;

(v) $(G/\Phi(G)) \in \mathcal{F}_{dn}$.

3. CS-groups. The following facts are clear from Definition 1.3.

Proposition 3.1. (i) *The subgroups of a* CS*-group are* CS*-groups.*

(ii) *The quotient groups of a* CS*-group are* CS*-groups.*

Theorem 3.1. *If* G *is a finite nilpotent group, then* G *is a* CS*-group if and only if all Sylow subgroups of* G *are* CS*-groups.*

The following Theorems 3.2 and 3.3 are belonged to Subbotin [15].

Theorem 3.2 (Subbotin [15])**.** *If* G *is a* CS*-group, then the following statements are true*:

(i) G *is supersolvable*;

(ii) G' *is Abelian.*

For the readers convenience, we give a new simple proof.

Proof. (i) Is obvious. We only need to prove (ii). Let x be in G'. Since by hypothesis, every subgroup of G' is normal in G, the cyclic group $\langle x\rangle$ is normal, and thus $G/C_G(x)$ is embedded in $\mathrm{Aut}(\langle x\rangle)$, which is Abelian. Then G' centralizes x, and hence every element of G' is central in G'.

Theorem 3.3 (Subbotin [15])**.** *If* G *is a nonnilpotent group, then* G *is a* CS*-group if and only if the following conditions hold*:

1) G *can be represented as the semidirect product* $G = [G^{\mathcal{N}}]B$, *where* $G^{\mathcal{N}}$ *is the nilpotent residual of* G *and Abelian, each subgroup of which is normal in* G, *and* B *is a nilpotent CS-group;*

2) *the derived subgroup* B' *of* B *is normal in* G *and its order is coprime with the order of* $G^{\mathcal{N}}$.

4. $CS_\infty(G)$ and $\mathcal{F}_{dn}$-groups. As consequence of Theorem 3.2, we have the following proposition.

Proposition 4.1. *For any finite group* G, *the subgroup* $CS_\infty(G)$ *is solvable.*

We can now characterize $\mathcal{F}_{dn}$-groups.

Proposition 4.2. *Let* G *be a finite group. Then the following statements are equivalent*:

(i) G *is an* $\mathcal{F}_{dn}$*-group*;

(ii) $G/CS(G)$ *is an* $\mathcal{F}_{dn}$*-group.*

Proof. (i) $\Rightarrow$ (ii). Clear.

(ii)$\Rightarrow$ (i). We use induction on the order of G. If $CS(G) = 1$, then nothing needs to be shown. Suppose that $CS(G) > 1$. Thus we can find a minimal normal subgroup N of G such that $N \leq CS(G)$. By Proposition 4.1, $CS(G)$ is solvable, so N is an elementary Abelian p-group for some prime p.

Assume that $N \leq \Phi(G)$, the Frattini subgroup of G, then by Proposition 2.2, $CS(G)/N \leq$ $\leq CS(G/N)$. It follows that $(G/N)/CS(G/N)$ is in $\mathcal{F}_{dn}$ because $G/CS(G) \in \mathcal{F}_{dn}$. We thus have that G/N satisfies the condition of the theorem. By induction, $(G/N)' = G'N/N$ is nilpotent. As $N \leq \Phi(G)$, it follows by (Huppert, 1967, III, Satz 3.5) that $G'N$ is nilpotent and hence G', which gives $G \in \mathcal{F}_{dn}$, as desired.

Assume now that $N \nsubseteq \Phi(G)$, then there is a maximal subgroup M of G such that $G = NM$ with $N \bigcap M = 1$. By Proposition 2.1, $M \cap CS(G) \leq CS(M)$. Thus, by hypothesis that $G/CS(G) \in \mathcal{F}_{dn}$, and as $G/CS(G) \cong M/(CS(G)\cap M)$, we have $M/CS(M) \in \mathcal{F}_{dn}$. Hence M satisfies the condition. By induction, M' is nilpotent. Now, $N \leq CS(G)$ and $CS(G)$ normalizes all subgroups of G'. Thus N normalizes M', it follows that $G' \leq N \times M'$. Since M' is nilpotent, we conclude that G' is nilpotent, as desired.

Theorem 4.1. *Let* G *be a finite group. Then the following statements are equivalent*:

(i) $G \in \mathcal{F}_{dn}$;

(ii) $G/CS_{\infty}(G)$ *is an* $\mathcal{F}_{dn}$*-group*;

(iii) $G = CS_{\infty}(G)$.

Proof. (i)$\Longrightarrow$(ii). Clear.

(ii)$\Longrightarrow$(iii). We first show the following simple fact: If $X > 1$ is an $\mathcal{F}_{dn}$-group, then $CS(X) > 1$. In fact, we may let $X' \neq 1$. Then we have $CS(X) = \bigcap_{H \leq X'} N_X(H) \geq Z(X') > 1$. The fact follows. Using this fact and noting that $CS(G/CS_{\infty}(G)) = \bar{C}S_{\infty}(G)$, we deduce $G = CS_{\infty}(G)$.

(iii)$\Longrightarrow$(i). As $CS_{\infty}(G/CS(G)) = CS_{\infty}(G)/CS(G)$, by induction, $(G/CS(G)) \in \mathcal{F}_{dn}$. It follows from Proposition 4.2 that $G \in \mathcal{F}_{dn}$.

5. Applications. Gaschütz and N. Itô proved that if all minimal subgroups of a group G are normal (which are called PN-groups), then G is soluble and the Fitting length of G is at most 3 (see [10, p. 436] (Satz 5.7) or [7]). In this section, the following dual problem is considered: study the finite groups all of whose minimal subgroups normalize every subgroup of the derived subgroup of G.

Theorem 5.1. *Let* G *be a* p*-solvable group. Suppose that all elements of* G *of order* p *are in* $CS(G)$*. If* $p = 2$*, in addition, all elements of* G *of order* 4 *are in* $CS(G)$*. Then the* $l_p(G) \leq 1$.

Proof. We use induction on $|G|$. Clearly, $G/O_{p'}(G)$ satisfies the hypothesis and $l_p(G/O_{p'}(G) = = l_p(G)$. We may assume that $O_{p'}(G) = 1$.

Let P be a Sylow p-subgroup of $CS(G)$. By Theorem 4.1, $CS(G)'$ is nilpotent. Thus $O_{p'}(G) = 1$ implies $CS(G)'$ is a p-group, and hence P is normal in G. Also $F_p(G) = O_{p',p}(G) = O_p(G)$. As G is p-solvable by the condition, by [12, p. 269] (Theorem 9.3.1), we know

$$C_G(O_p(G)) \leq O_p(G).$$

We now claim that G is q-nilpotent for any prime $q \neq p$. Otherwise, there exists a prime q such that G is non-q-nilpotent. Then there exists a subgroup K with the following properties: K is non-q-nilpotent but all proper subgroups of K are q-nilpotent. By a theorem of Itô [12, p. 296] (Theorem 10.3.3), $K = [Q]R$, where Q is a normal q-subgroup, $\exp(Q) = p$ or 4, and R is a cyclic r-subgroup, the prime $r \neq q$. Consider the subgroup

$$M = O_p(G)Q.$$

Let $p > 2$. By above, $\Omega_1(G_p) \leq P \leq O_p(G)$, so $\Omega_1(G_p) = \Omega_1(O_p(G))$. Then $\Omega_1(O_p(G)) \trianglelefteq G$. Since $Q = K' \leq G'$, it follows that $\Omega_1(O_p(G))$ normalizes Q and $[Q, \Omega_1(O_p(G))] = 1$. By [10, p. 437] (5.12), we get $[Q, O_p(G)] = 1$. Thus $Q \leq C_G(O_p(G))$. As $C_G(O_p(G)) \leq O_p(G)$ and Q is a p'-group, Q must be 1, a contradiction. Similar for the case when $p = 2$.

Now let $G_{q'}$ denote the normal q-complement of G for every prime $q \neq p$. Then $G_p \leq G_{q'}$ and G_p is the intersection of all $G_{q'}$, hence $G_p \trianglelefteq G$, of course, $l_p(G) = 1$.

Theorem 5.2. *Let* G *be a finite group. If all elements of odd prime order of* G *are in* $CS(G)$*, then* G *is solvable.*

Proof. By induction, every proper subgroup of G is solvable, so we can assume $G' = G$, and thus every subgroup of G is normalized by all elements of odd prime order. We can assume that G is not a 2-group, so let p be an odd prime divisor of $|G|$. I argue that G has a normal p-complement, and this will be a contradiction since $G' = G$. By a theorem of Frobenius, it suffices to show that if a subgroup X of order not divisible by p normalizes a p-subgroup P in G, then X centralizes P. To prove that X centralizes P, it suffices to show that X centralizes every element y in P of prime order. But y normalizes every subgroup, so $[y, X] \leq X$. Also $[y, X] \leq [P, X] \leq P$, and thus $[y, X]$ is trivial, as wanted.

Theorem 5.3. *Let G be a finite group. If all elements of prime order or order 4 of G are in $CS(G)$, then G' is nilpotent, in particular, the Fitting length of G is bounded by 2.*

Proof. Let p be any prime dividing $|G|$ and let P be a Sylow p-subgroup of G. As G is solvable, hence p-solvable. According to Theorem 5.1, we have $F_p(G) = O_{p',p}(G) = O_{p'}(G)P$, the maximal normal p-nilpotent subgroup of G. Nextly, by Frattini argument $G = N_G(P)O_{p'}(G)$. On the other hand, by Schur – Zassensaus's theorem [12, p. 253] (Theorem 9.1.2), $N_G(P) = [P]M$, where M is a Hall p'-subgroup of $N_G(P)$ and hence $G = F_p(G)M$.

By hypothesis, $\Omega_1(P)$ and $\Omega_2(P)$ normalize M'. Hence M' centralizes $\Omega_1(P)$ and $\Omega_2(P)$, and thus centralizes P. Since $C_G(P) \leq F_p(G)$ by [12, p. 269] (Theorem 9.3.1),

$$M' \leq F_p(G).$$

Now $G = F_p(G)M$, it follows that $G' \leq F_p(G)$. Therefore, G' is p-nilpotent. Hence G' is nilpotent.

Theorem 5.4. *Let G be a finite group. If all elements of G of order of prime or 4 are in $CS(G)$, then*

(i) *G is solvable*;

(ii) $l_p(G) \leq 1$ *for every prime p*,

(iii) *G' is nilpotent, in particular, the Fitting length of G is bounded by* 2.

Proof. This follows from Theorems 5.1 – 5.3.

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