UDC 517.9

W. I. Skrypnik (Inst. Math. Nat. Acad. Sci. Ukraine, Kyiv)

COULOMB DYNAMICS NEAR EQUILIBRIUM OF TWO EQUAL NEGATIVE CHARGES IN THE FIELD OF FIXED TWO EQUAL POSITIVE CHARGES

КУЛОНІВСЬКА ДИНАМІКА БІЛЯ РІВНОВАГИ ДВОХ РІВНИХ НЕГАТИВНИХ ЗАРЯДІВ У ПОЛІ ФІКСОВАНИХ ДВОХ РІВНИХ ПОЗИТИВНИХ ЗАРЯДІВ

Periodic and quasiperiodic solutions of the Coulomb equation of motion of two equal negative charges in the field of two fixed and equal positive charges are found with the help of the Lyapunov center theorem.

Знайдено періодичні та квазіперіодичні розв'язки рівнянь руху Кулона двох рівних негативних зарядів у полі фіксованих двох рівних позитивних зарядів із допомогою центральної теореми Ляпунова.

1. Introduction. The Coulomb systems of two and three negative equal charges e_0 in the field of fixed two equal positive charges e' have equilibrium configurations [1]. This fundamental fact allowed us to construct periodic and bounded solutions close to the equilibria of the Coulomb equation of motion for two negative charges restricted to such a line that the positive charges are located symmetrically at a perpendicular to this line [1]. We applied the Siegel [2], Weinstein [3, 4], Moser [5] and center Lyapunov [2, 5–9] theorems which demand a knowledge of the spectra of the matrix U^0 of second derivatives of the potential energy at the equilibrium (for equal masses). The last two theorems, guaranteeing the existence of the periodic solutions in terms of convergent series, restrict the values of $\frac{e_0}{e'}$ through a nonresonance condition. The Weinstein theorem establishes also the existence of the periodic solutions but can be applied only for mechanical systems with a stable equilibrium (U^0 is positive definite and the equilibrium is a minimum of the potential energy). Periodic solutions are also found in planar Coulomb systems of $n-1,\ n>2$ equal negative charges and one and three positive charges [10, 11].

In this paper we find periodic and quasiperiodic solutions of the Coulomb equation of motion for planar and space systems of two equal negative charges in the field of two fixed positive charges. This result is a consequence of an explicit calculation of the eigenvalues of U^0 which in its turn follows from the representation of U^0 as a direct sum of two and three two-dimensional matrices in the planar and space dynamics, respectively.

There are two different cases of the planar dynamics determined by the fact whether the positive charges are outside of the plane with the negative charges or not. This difference is explained by different characters of spectra of U^0 and their canonical matrices, i.e., the matrices which determine linear parts of Hamiltonian vector fields at equilibria (see Appendix). For the external positive charges there is the zero eigenvalue for all values of $e_0,\ e'$ and for some of these values the other eigenvalues are positive. For the internal positive charges there is no such zero eigenvalue and not all the eigenvalues are nonnegative. Besides in the first case the spectrum contains the eigenvalues of the line dynamics. The spectrum of U^0 of the space dynamics contains eigenvalues of both cases which are proportional to $u'=\frac{e_0^2}{4a^3}$, where $2a$ is the equilibrium distance between two negative charges expressed through the distance between two positive charges $2b$. Such the spectrum is an obstruction for the applications of the Lyapunov and Moser theorems.

We circumvent the obstruction of the zero eigenvalue with the help of the Jacobi procedure of an elimination of node from the Celestial Mechanics (Section 18 in [2]). It is known that the zero eigenvalue of a canonical matrix is generated by integrals of motion [2]. The main idea of the procedure of an elimination of node is to produce a canonical transformation which turns the integrals of motions into cyclic variables (a Hamiltonian does not depend on them). Then the linear part of the equation of motion for them will be zero and the linear part of the equation of motion for remaining variables will not contain the zero eigenvalue. The procedure of elimination of node can be formulated in the following theorem (the proof of its first two statements are given in [2]).

Theorem 1.1. *Let $H(x,p)$ be a $2n$-dimensional Hamiltonian, Q be its time independent integral and $w(u,p)$ be a generating function of a canonical transformation such that*

$$v_k = \frac{\partial w}{\partial u_k}, \qquad x_k = \frac{\partial w}{\partial p_k}, \qquad k = 1,\ldots,n, \tag{1.1}$$

$$\frac{\partial w}{\partial u_n} = Q(x,p), \qquad W_{k,j} = \frac{\partial^2 w}{\partial u_k \partial p_j}, \qquad \operatorname{Det} W \neq 0. \tag{1.2}$$

(1) *Then the transformed Hamiltonian $H'(u,v)$ does not depend on u_n. Let also the canonical matrix of H have doubly degenerate zero eigenvalue, the Q-canonical transformation and the Hamiltonian be holomorphic at the neighborhood of the equilibrium. Then* (2) *the canonical matrix of the $2(n-1)$-dimensional Hamiltonian equation*

$$\dot u_j = \frac{\partial H'}{\partial v_j}, \qquad \dot v_j = -\frac{\partial H'}{\partial u_j}, \quad j = 1,\ldots,n-1, \tag{1.3}$$

does not have the zero eigenvalue for the equilibrium value Q_0 of Q and (3) *eigenvalues of the canonical matrices of H and H' are identical.*

A separation of cyclic variables, generated by integrals of motion, in a Hamiltonian equation is also described in [12].

For the planar dynamics with external positive charges and space dynamics the one-dimensional angular momentum Q is an integral of motion. We find w as

$$w = \sum_{j=1}^{n} g_k(u_1,\ldots,u_n)p_k,$$

where $n = 4$ and $n = 6$ correspond to the plane and space dynamics, respectively, introducing special numerations of charges coordinates and momenta $(x_j; p_j) = (x_j^\alpha; p_j^\alpha)$, $j = 1, 2$, $\alpha \leq 3$, for these two cases. The solution of the equation for g_k derived from (1.1), (1.2), which guarantees all the conditions for the Q-canonical transformation, is taken by us from [2]. As a result a solutions of the Coulomb equation for these two systems are given by

$$x_j^\alpha(t) = \sum_{k=1}^{n-1} u_k(t)[\gamma_{k,j,\alpha} + \gamma'_{k,j,\alpha}\cos(u_n(t)) + \gamma''_{k,j,\alpha}\sin(u_n(t))], \tag{1.4}$$

where γ, γ', γ'' are constants and $u_k(t)$, $k = 1,\ldots,n$, are solutions of the equation with the Hamiltonian H' .

Let $u_{(n-1)}, v_{(n-1)}$ be a periodic solution of (1.3) with a period τ. Then

$$\left(\frac{\partial H'}{\partial v_n}\right)(u_{(n-1)}, v_{(n-1)}, Q_0) = H'_{n-1}(t)$$

is also a periodic function such that

$$H'_{n-1}(t) = H^0_{n-1}(t) + \xi, \qquad \int\limits_t^{t+\tau} H^0_{n-1}(s)ds = 0,$$

where ξ is a constant. This implies that

$$u_n(t) = \int\limits_0^t H'_{n-1}(s)ds = \int\limits_0^t H^0_{n-1}(s)ds + \xi t = u_0(t) + \xi t,$$

where u_0 is periodic with the period τ. The last equality and (1.4) determine quasiperiodic (doubly periodic) functions $x_j^\alpha(t)$ if $\gamma' \neq 0$ or $\gamma'' \neq 0$. We will show that their Euclidean norms are periodic functions.

Our paper is organized as follows. In the second, third and fourth sections we formulate theorems concerning existence of periodic and quasiperiodic solutions of the Coulomb equations of motion for planar with external positive charges, for planar with internal positive charges and the space dynamics, respectively. In Appendix we prove the second and third statements of Theorem 1.1.

2. Two external fixed charges. Let $x_j = (x_j^1, x_j^2) \in \mathbb{R}^2$, $j = 1, 2$, be coordinates of two equal point negative charges e_0 and $|x|_j^2 = (x_j^1)^2 + (x_j^2)^2$. Let also two fixed positive charges e' have the coordinates $(0, 0, \pm b) \in \mathbb{R}^3$. Then the potential Coulomb energy of the negative charges is given by

$$U(x_{(2)}) = e_0^2|x_1 - x_2|^{-1} - 2e_0e'\left[\left(\sqrt{|x_1|^2 + b^2}\right)^{-1} + \left(\sqrt{|x_2|^2 + b^2}\right)^{-1}\right], \quad x_{(2)} = (x_1, x_2). \tag{2.1}$$

The equation of motion is represented by

$$m\frac{d^2x_j}{dt^2} = -\frac{\partial U(x_{(2)})}{\partial x_j}, \quad j = 1, 2. \tag{2.2}$$

The first partial derivatives of U look like

$$\frac{\partial}{\partial x_1^\alpha}U(x_{(2)}) = -e_0^2\frac{x_1^\alpha - x_2^\alpha}{|x_1 - x_2|^3} + 2e_0e'\frac{x_1^\alpha}{\left(\sqrt{|x_1|^2 + b^2}\right)^3},$$

$$\frac{\partial}{\partial x_2^\beta}U(x_{(2)}) = -e_0^2\frac{x_2^\beta - x_1^\beta}{|x_1 - x_2|^3} + 2e_0e'\frac{x_2^\beta}{\left(\sqrt{|x_2|^2 + b^2}\right)^3}.$$

The equalities $x_1^1 = a$, $x_2^1 = -a$, $x_1^2 = 0$, $x_2^2 = 0$ determine the equilibrium for which these first derivatives are zeroes if

$$\frac{e_0}{(2a)^3} = \frac{e'}{(\sqrt{a^2 + b^2})^3}, \qquad \left(\frac{e_0}{e'}\right)^{\frac{1}{3}}\frac{1}{2a} = \frac{1}{\sqrt{a^2 + b^2}}.$$

That is

$$a=(4-\eta')^{-\frac{1}{2}}\sqrt{\eta'}b=(3-\eta)^{-\frac{1}{2}}\sqrt{\eta}b, \qquad \eta'=\left(\frac{e_0}{e'}\right)^{\frac{2}{3}}<4, \qquad \eta=\frac{3}{4}\eta'<3,$$

$$U(x^0_{(2)})=U(a,0;-a,0)=\frac{e_0^2}{2a}-\frac{2e_0e'}{a}\left(\frac{e_0}{e'}\right)^{\frac{1}{3}}=\frac{e_0^2}{2a}\left(1-4\left(\frac{e'}{e_0}\right)^{\frac{2}{3}}\right)=\frac{e_0^2}{2a}(1-3\eta^{-1}). \tag{2.3}$$

The second partial derivatives of the potential energy are given by

$$\frac{\partial U(x_{(2)})}{\partial x_1^\alpha \partial x_2^\beta}=\frac{\partial U(x_{(2)})}{\partial x_2^\beta \partial x_1^\alpha}=e_0^2\left[\frac{\delta_{\alpha,\beta}}{|x_1-x_2|^3}-3\frac{(x_1^\alpha-x_2^\alpha)(x_1^\beta-x_2^\beta)}{|x_1-x_2|^5}\right], \quad \alpha,\beta=1,2,$$

and

$$\frac{\partial^2 U(x_{(2)})}{\partial x_j^\beta \partial x_j^\alpha}=-\frac{e_0^2\delta_{\alpha,\beta}}{|x_1-x_2|^3}+\frac{2e_0e'\delta_{\alpha,\beta}}{\left(\sqrt{|x_j|^2+b^2}\right)^3}-\frac{6e_0e'x_j^\alpha x_j^\beta}{\left(\sqrt{|x|_j^2+b^2}\right)^5}+3e_0^2\frac{(x_1^\alpha-x_2^\alpha)(x_1^\beta-x_2^\beta)}{|x_1-x_2|^5}.$$

Let U^0 be the (4×4)-matrix of the second partial derivatives at the equilibrium and

$$u'=\frac{e_0^2}{4a^3}, \qquad u'_*=u_*-\frac{3u'}{2}, \qquad u_*=\frac{3u'}{4}\left(\frac{e_0}{e'}\right)^{\frac{2}{3}}=u'\eta,$$

then its matrix elements look like

$$U^0_{1,\alpha;1,\beta}=U^0_{2,\alpha;2,\beta}=\delta_{\alpha,\beta}\left(\frac{e_0^2}{(2a)^3}-\frac{6e_0e'a^2}{\left(\sqrt{a^2+b^2}\right)^5}\delta_{\alpha,1}+3\frac{e_0^2}{(2a)^3}\delta_{\alpha,1}\right)=\delta_{\alpha,\beta}\left(\frac{u'}{2}-\delta_{\alpha,1}u'_*\right),$$

$$U^0_{1,\alpha;2,\beta}=U^0_{2,\alpha;1,\beta}=\frac{u'}{2}\delta_{\alpha,\beta}(1-3\delta_{\alpha,1}).$$

In order to determine the spectrum of U^0 we put

$$(1,1)=1, \qquad (2,1)=2, \qquad (1,2)=3, \qquad (2,2)=4.$$

As a result

$$U^0=\begin{pmatrix}\frac{u'}{2}-u'_* & -u' \\ -u' & \frac{u'}{2}-u'_*\end{pmatrix}\oplus\begin{pmatrix}\frac{u'}{2} & \frac{u'}{2} \\ \frac{u'}{2} & \frac{u'}{2}\end{pmatrix}.$$

The characteristic polynomial of U^0 looks like

$$p(\lambda)=\mathrm{Det}(-U^0+\lambda I)=\left(\left(\frac{u'}{2}-u'_*-\lambda\right)^2-u'^2\right)\left(\left(\frac{u'}{2}-\lambda\right)^2-\frac{u'^2}{4}\right).$$

The roots ζ_j of this polynomial are given by

$$\zeta_1=-\frac{u'}{2}-u_*=u'-u_*, \qquad \zeta_2=\frac{3u'}{2}-u_*=3u'-u_*, \qquad \zeta_3=u'>0, \qquad \zeta_4=0$$

or

$$\zeta_1 = u'(1-\eta), \qquad \zeta_2 = u'(3-\eta), \qquad \zeta_3 = u', \qquad \zeta_4 = 0.$$

The two first eigenvalues are negative or positive if $3u' < u_*$, $u' > u_*$, respectively, or $\eta > 3$, $\eta < 1$, respectively. The first case is excluded due to (2.3). If $3u' > u_*$, $u' < u_*$, then the second and third roots are positive and the first root is negative. In these cases one can apply the Lyapunov after the elimination of a node.

Now let's describe the Q-canonical transformation announced in Theorem 1.1. Let

$$\begin{aligned} &x_1^1 = x_1, \qquad x_1^2 = x_2, \qquad x_2^1 = x_3, \qquad x_2^2 = x_4, \\ &p_1^1 = p_1, \qquad p_1^2 = p_2, \qquad p_2^1 = p_3, \qquad p_2^2 = p_4. \end{aligned} \tag{2.4}$$

Then the angular moment (the integral of motion corresponding to rotation) is given (x, p are the same as ξ, η in Section 18 of [2])

$$Q = -\sum_{j=1}^{2}(x_j^1 p_j^2 - x_j^2 p_j^1) = \sum_{j=1}^{2}(x_{2j}p_{2j-1} - x_{2j-1}p_{2j}).$$

Further the upper index will show a power. (1.2) gives

$$\sum_{j=1}^{4} \frac{\partial g_k}{\partial u_4} p_k = \sum_{j=1}^{2}(g_{2j}p_{2j-1} - g_{2j-1}p_{2j})$$

or

$$\frac{\partial g_{2k-1}}{\partial u_4} = g_{2k}, \qquad \frac{\partial g_{2k}}{\partial u_4} = -g_{2k-1}, \tag{2.5}$$

where $k = 1, 2$. The simplest solution is given by

$$g_1 = u_1 c, \quad g_2 = -u_1 s, \quad g_3 = u_2 c + u_3 s, \quad g_4 = -u_2 s + u_3 c, \quad c = \cos u_4, \quad s = \sin u_4.$$

This solution is generated by the nonsingular canonical transformation in the neighborhood of point $u_1 = -u_2 = a$, $u_3 = u_4 = 0$, $v_j = 0$, which determines the new equilibrium, since

$$\operatorname{Det} W = \operatorname{Det} G = -u_1, \qquad G_{j,k} = \frac{\partial g_j}{\partial u_k}.$$

To prove this we decompose the determinant of G in the elements of the first row

$$G = \begin{pmatrix} c & 0 & 0 & -u_1 s \\ -s & 0 & 0 & -u_1 c \\ 0 & c & s & -u_2 s + u_3 c \\ 0 & -s & c & -u_2 c - u_3 s \end{pmatrix},$$

$$\operatorname{Det} G = c \operatorname{Det} \begin{pmatrix} 0 & 0 & -u_1 c \\ c & s & -u_2 s + u_3 c \\ -s & c & -u_2 c - u_3 s \end{pmatrix} + u_1 s \begin{pmatrix} -s & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} =$$

$$= -u_1c^2(c^2+s^2) - u_1s^2(c^2+s^2) = -u_1.$$

As a result

$$x_1 = u_1c, \qquad x_2 = -u_1s, \qquad x_3 = u_2c + u_3s, \qquad x_4 = -u_2s + u_3c, \tag{2.6}$$

$$v_1 = cp_1 - sp_2, \qquad v_2 = cp_3 - sp_4, \qquad v_3 = sp_3 + cp_4. \tag{2.7}$$

Let

$$v_0 = sp_1 + cp_2,$$

then

$$p_1 = cv_1 + sv_0, \qquad p_2 = cv_0 - sv_1, \qquad p_3 = cv_2 + sv_3, \qquad p_4 = cv_3 - sv_2$$

and

$$Q = v_4 = u_3v_2 - u_2v_3 - u_1v_0, \qquad v_0 = u_1^{-1}(u_3v_2 - u_2v_3 - v_4).$$

These equalities allow one to derive the invertible matrix which determines the linear part of the Q-canonical transformation in the neighborhood of the equilibrium and calculate the new Hamiltonian. They imply

$$v_2^2 + v_3^2 = p_3^2 + p_4^2, \qquad v_0^2 + v_1^2 = p_1^2 + p_2^2,$$

$$x_1^2 + x_2^2 = u_1^2, \qquad x_3^2 + x_4^2 = u_2^2 + u_3^2,$$

$$(x_1 - x_3)^2 + (x_2 - x_4)^2 = ((u_2 - u_1)c + u_3s)^2 + ((u_2 - u_1)s - u_3c)^2 = (u_2 - u_1)^2 + u_3^2.$$

The new Hamiltonian in the new variables is given by (w does not depend on t)

$$H' = (2m)^{-1}\sum_{j=1}^{3} v_j^2 + (2mu_1^2)^{-1}(u_3v_2 - u_2v_3 - v_4)^2 + U'(u_{(3)}), \tag{2.8}$$

$$U'(u_{(3)}) = e_0^2((u_2 - u_1)^2 + u_3^2)^{-\frac{1}{2}} - 2e_0e'\Big[(u_1^2 + b^2)^{-\frac{1}{2}} + (u_2^2 + u_3^2 + b^2)^{-\frac{1}{2}}\Big].$$

It is known that the eigenvalues of the canonical matrix of H coincide with $\pm\sqrt{m^{-1}\zeta_j}$ [1]. That is its zero eigenvalue is doubly degenerate and we can apply Theorem 1.1. To apply the center Lyapunov theorem for (1.3) one has to exclude resonances for U^0, i.e., the equalities $\dfrac{\zeta_j}{\zeta_s} = k^2,\ k \in \mathbb{Z}^+$, and make the translation $u_1 \to u_1 - a,\ u_2 \to u_2 + a$.

Let $\zeta_1 > 0$. The equalities $\dfrac{\zeta_j}{\zeta_1} \neq k^2,\ k \in \mathbb{Z}^+$, imply

$$\eta = \frac{3}{4}\left(\frac{e_0}{e'}\right)^{\frac{2}{3}} \neq \frac{k^2 - 3}{k^2 - 1}, \frac{k^2 - 1}{k^2}.$$

Since $\zeta_1 < \zeta_3 \leq \zeta_2$ the equalities $\dfrac{\zeta_1}{\zeta_s} = k^2,\ k \in \mathbb{Z}^+,\ s \neq 1$, are not true. If $\zeta_2 > \zeta_3$, i.e., $2 - \eta > 0$, then the resonance for $s = 2$ is not true and the resonance for $s = 3$ coincides with

$$\frac{\zeta_2}{\zeta_3} = k^2, \qquad 3 - \eta = k^2$$

and is not true also. The condition $\zeta_3 > \zeta_2$, i.e., $2 - \eta < 0$, contradicts $\zeta_1 > 0$.

The following conclusion is true:

I. If $\eta < 1$ and

$$\eta \neq \frac{k^2-3}{k^2-1}, \frac{k^2-1}{k^2}, \quad k \in \mathbb{Z}^+,$$

then the resonance in ζ_1 is absent;

II. If $\eta < 1$, then the resonances in ζ_2, ζ_3 are absent.

Let $\zeta_1 < 0$, i.e., $\eta > 1$. Once again there are no resonances in ζ_2, ζ_3 if $\zeta_2 > \zeta_3$. If $\zeta_3 > \zeta_2$, then the nonresonance condition in ζ_2 is given by

$$\frac{\zeta_3}{\zeta_2} \neq k^2, \qquad \eta \neq \frac{3k^2-1}{k^2}, \quad k \in \mathbb{Z}^+.$$

If this condition is not true, then there is no resonance only in ζ_3. The condition $\zeta_2 < 0$ is excluded since $\eta < 3$.

Hence we proved the following theorem.

Theorem 2.1. *Let* $0 < \eta = \dfrac{3}{4}\left(\dfrac{e_0}{e'}\right)^{\frac{2}{3}} < 3$. *Then the equation* (1.3) *for* $n = 4$, $v_4 = 0$ *and* H' *given by* (2.8), *which corresponds to the Coulomb equation* (2.2) *with* $x_j \in \mathbb{R}^2$ *and potential energy* (2.1), *possesses one, two and three periodic solutions related to the following three cases*:

(1) $\eta > 2$,

(2) $0 < \eta < 2$, $\eta > 2$, $\eta \neq \dfrac{3k^2-1}{k^2}$,

(3) $\eta < 1$, $\eta \neq \dfrac{k^2-3}{k^2-1}$, $\dfrac{k^2-1}{k^2}$.

These solutions and their periods $\tau_j(c_j)$ *are holomorphic functions at the origin in the parameters* c_j, $j = 1, 2, 3$. *The first, second and third cases are characterized by* τ_1, τ_1, τ_2 *and* τ_1, τ_2, τ_3, *respectively, where*

$$\tau_1^{-1}(0) = (2\pi)^{-1}\sqrt{m^{-1}u'}, \qquad \tau_2^{-1}(0) = (2\pi)^{-1}\sqrt{m^{-1}(3-\eta)u'},$$
$$\tau_3^{-1}(0) = (2\pi)^{-1}\sqrt{m^{-1}(1-\eta)u'}.$$

The associated quasiperiodic solutions of (2.2) *are given by* (1.4) *with* $n = 4$, *where* γ, γ', γ'' *correspond to* (2.4) *and* (2.6), *and are such that* $|x_j|^2$, $j = 1, 2$, *are periodic functions.*

3. Two internal fixed charges. In this section we consider the planar system of two identical charges in the field of two positive fixed charges located at the points b_1, b_2 with the potential energy

$$U(x_{(2)}) = e_0^2|x_1 - x_2|^{-1} - e_0 e' \sum_{j,k=1,2} |x_j - b_k|^{-1}, \tag{3.1}$$

where

$$|x|^2 = (x_j^1)^2 + (x_j^2)^2, \qquad x_j = (x_j^1, x_j^2) \in \mathbb{R}^2,$$
$$b_j = (b_j^1, b_j^2) \in \mathbb{R}^2, \qquad b_j^1 = 0, \qquad b_1^2 = -b_2^2 = b.$$

The partial derivatives of this potential energy are found from

$$\frac{\partial}{\partial x_1^\alpha} U(x_{(2)}) = -e_0^2 \frac{x_1^\alpha - x_2^\alpha}{|x_1 - x_2|^3} + e_0 e' \sum_{k=1}^{2} \frac{x_1^\alpha - b_k^\alpha}{|x_1 - b_k|^3},$$

$$\frac{\partial}{\partial x_2^\beta} U(x_{(2)}) = -e_0^2 \frac{x_2^\beta - x_1^\beta}{|x_1 - x_2|^3} + e_0 e' \sum_{k=1}^{2} \frac{x_2^\beta - b_k^\beta}{|x_2 - b_k|^3}.$$

The equilibrium is determined by $x_1^{01} = a,\ x_2^{01} = -a,\ x_1^{02} = 0,\ x_2^{02} = 0$ equating to zero the right-hand sides of these equalities. This gives the equilibrium relation between $e_0,\ e',\ a,\ b$ the same as in the previous section.

The second derivatives of the potential energy (3.1) are given by

$$\frac{\partial U(x_{(2)})}{\partial x_1^\alpha \partial x_2^\beta} = \frac{\partial U(x_{(2)})}{\partial x_2^\beta \partial x_1^\alpha} = e_0^2 \left[\frac{\delta_{\alpha,\beta}}{|x_1 - x_2|^3} - 3 \frac{(x_1^\alpha - x_2^\alpha)(x_1^\beta - x_2^\beta)}{|x_1 - x_2|^5} \right], \quad \alpha, \beta = 1, 2, \ldots,$$

and

$$\frac{\partial^2 U(x_{(2)})}{\partial x_j^\beta \partial x_j^\alpha} = -\frac{e_0^2 \delta_{\alpha,\beta}}{|x_1 - x_2|^3} +$$

$$+ e_0 e' \sum_{k=1}^{2} \left[\frac{\delta_{\alpha,\beta}}{|x_j - b_k|^3} - 3 \frac{(x_j^\alpha - b_k^\alpha)(x_j^\beta - b_k^\beta)}{|x_j - b_k|^5} \right] + 3 e_0^2 \frac{(x_1^\alpha - x_2^\alpha)(x_1^\beta - x_2^\beta)}{|x_1 - x_2|^5}.$$

Let $u,\ u',\ u_*,\ u'_*$, be the same as in the previous section,

$$u''_* = \frac{6 e_0 e' b^2}{\left(\sqrt{a^2 + b^2}\right)^5} = \frac{6 e_0 e'}{(2a)^5} \left(\frac{e_0}{e'}\right)^{\frac{5}{3}} \frac{3 - \eta}{\eta} a^2 = u'(3 - \eta)$$

and

$$T_j(\alpha, \beta) = \sum_{k=1}^{2} \frac{(x_j^\alpha - b_k^\alpha)(x_j^\beta - b_k^\beta)}{|x_j - b_k|^5}.$$

Let $T_j^0(\alpha, \beta)$ be the equilibrium value of $T_j^0(\alpha, \beta)$. Then $(|x_j^0 - b_k|^2 = a^2 + b^2)$

$$T_j^0(\alpha, \beta) = 2(a^2 + b^2)^{-\frac{5}{2}} \delta_{\alpha,\beta} (a^2 \delta_{\alpha,1} + b^2 \delta_{\alpha,2}).$$

Indeed, let

$$\tilde{T}_j^0(\alpha, \beta) = \sum_{k=1}^{2} (x_j^{0\alpha} - b_k^\alpha)(x_j^{0\beta} - b_k^\beta),$$

then

$$\tilde{T}_1^0(1, 2) = -((a - b_1^1) b_1^2 + (a - b_2^1) b_2^2) = -(ab - ab) = 0,$$

$$\tilde{T}_2^0(1, 2) = -((-a - b_1^1) b_1^2 + (-a - b_2^1) b_2^2) = ab - ab = 0,$$

$$\tilde{T}_1^0(1, 1) = (a - b_1^1)(a - b_1^1) + (a - b_2^1)(a - b_2^1) = 2a^2,$$

$$\tilde{T}_2^0(1, 1) = (-a - b_1^1)(-a - b_1^1) + (-a - b_2^1)(-a - b_2^1) = 2a^2,$$

$$\tilde{T}_1^0(2,2) = \tilde{T}_2^0(2,2) = (b_1^2)^2 + (b_2^2)^2 = 2b^2.$$

As a result

$$U^0_{1,\alpha;1,\beta} = U^0_{2,\alpha;2,\beta} = \delta_{\alpha,\beta}\left(\frac{e_0^2}{(2a)^3} - \frac{6e_0e'}{(\sqrt{a^2+b^2})^5}(a^2\delta_{\alpha,1} + b^2\delta_{\alpha,2}) + 3\frac{e_0^2}{(2a)^3}\delta_{\alpha,1}\right) =$$

$$= \delta_{\alpha,\beta}\left(\frac{u'}{2} - \delta_{\alpha,1}u'_* - \delta_{\alpha,2}u''_*\right),$$

$$U^0_{1,\alpha;2,\beta} = U^0_{2,\alpha;1,\beta} = \frac{u'}{2}\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}).$$

Let us re-numerate indexes as in the previous section. Then

$$U^0 = \begin{pmatrix} \frac{u'}{2} - u'_* & -u' \\ -u' & \frac{u'}{2} - u'_* \end{pmatrix} \oplus \begin{pmatrix} \frac{u'}{2} - u''_* & \frac{u'}{2} \\ \frac{u'}{2} & \frac{u'}{2} - u''_* \end{pmatrix}.$$

The characteristic polynomial of U^0 looks like

$$p(\lambda) = \mathrm{Det}(-U^0 + \lambda I) = \left(\left(\frac{u'}{2} - u'_* - \lambda\right)^2 - u'^2\right)\left(\left(\frac{u'}{2} - u''_* - \lambda\right)^2 - \frac{u'^2}{4}\right).$$

The roots ζ_j of this polynomial are given by

$$\zeta_1 = -\frac{u'}{2} - u'_*, \qquad \zeta_2 = \frac{3u'}{2} - u'_*, \qquad \zeta_3 = u' - u''_*, \qquad \zeta_4 = -u''_* = -u'(3-\eta)$$

or

$$\zeta_1 = u' - u_* = u'(1-\eta), \qquad \zeta_2 = 3u' - u_* = u'(3-\eta),$$
$$\zeta_3 = u'(\eta - 2), \qquad \zeta_4 = -u'(3-\eta) < 0.$$

At the interval $\eta \in (1,2)$ only one eigenvalue ζ_2 is positive. At the interval $(0,1)$ $\zeta_2 > \zeta_1 > 0$ and $\zeta_3 < 0$. At the interval $\left(2, \frac{5}{2}\right)$ $\zeta_2 > \zeta_3 > 0$ but at the interval $\left(\frac{5}{2}, 3\right)$ $0 < \zeta_2 < \zeta_3$.

Hence the following theorem follows from the center Lyapunov theorem.

Theorem 3.1. *Let* $\eta \neq 1,2$ *belong to* $(0,3)\backslash\left[\frac{5}{2},3\right]$ *or* $\left(\frac{5}{2},3\right)$. *Then the planar Coulomb equation* (2.2) *with the potential energy з* (3.1) *possesses a periodic solution which depends on a real parameter* c. *This solution and its period* $\tau(c)$ *are real analytical functions at the origin in this parameter and* $\tau(0) = 2\pi\sqrt{\frac{m}{\zeta_2}}$ *or* $\tau(0) = 2\pi\sqrt{\frac{m}{\zeta_3}}$.

4. Space dynamics. For two identical negative charges in $\mathbb{R}^3$ in the field of two positive fixed charges located at the points b_1, b_2 the Coulomb potential energy is given by

$$U(x_{(2)}) = e_0^2|x_1 - x_2|^{-1} - e_0e' \sum_{j,k=1,2} |x_j - b_k|^{-1}, \tag{4.1}$$

where

$$|x|^2 = (x_j^1)^2 + (x_j^2)^2 + (x_j^3)^2, \qquad x_j = (x_j^1, x_j^2, x_j^3) \in \mathbb{R}^3,$$
$$b_j = (b_j^1, b_j^2, b_j^3) \in \mathbb{R}^3, \qquad b_j^1 = b_j^3 = 0, \qquad b_1^2 = -b_2^2 = b.$$

The partial derivatives of the potential energy can be taken from the previous section with the extended condition $\alpha, \beta = 1, 2, 3$. The equilibrium is determined by

$$x_1^{01} = a, \qquad x_2^{01} = -a, \qquad x_1^{0\alpha} = 0, \qquad x_2^{0\alpha} = 0, \quad \alpha = 2, 3,$$

and the equilibrium relation has to be taken from the previous section. It is not difficult to check that the matrix U^0 of the second partial derivatives of the potential energy (4.1) at the equilibrium looks like

$$U^0_{1,\alpha;1,\beta} = U^0_{2,\alpha;2,\beta} = \delta_{\alpha,\beta}\left(\frac{u'}{2} - \delta_{\alpha,1}u'_* - \delta_{\alpha,2}u''_*\right), \quad \alpha, \beta = 1, 2, 3,$$
$$U^0_{1,\alpha;2,\beta} = U^0_{2,\alpha;1,\beta} = \frac{u'}{2}\delta_{\alpha,\beta}(1 - 3\delta_{\alpha,1}).$$

Let's re-numerate the indexes in the following way:

$$(1,1) = 1, \qquad (2,1) = 2, \qquad (3,1) = 3, \qquad (1,2) = 4, \qquad (2,2) = 5, \qquad (2,3) = 6.$$

This implies that

$$U^0 = \begin{pmatrix} \frac{u'}{2} - u'_* & -u' \\ -u' & \frac{u'}{2} - u'_* \end{pmatrix} \oplus \begin{pmatrix} \frac{u'}{2} - u''_* & \frac{u'}{2} \\ \frac{u'}{2} & \frac{u'}{2} - u''_* \end{pmatrix} \oplus \begin{pmatrix} \frac{u'}{2} & \frac{u'}{2} \\ \frac{u'}{2} & \frac{u'}{2} \end{pmatrix}.$$

The characteristic polynomial of U^0 is given by

$$p(\lambda) = \mathrm{Det}(-U^0 + \lambda I) =$$
$$= \left(\left(\frac{u'}{2} - u'_* - \lambda\right)^2 - u'^2\right)\left(\left(\frac{u'}{2} - u''_* - \lambda\right)^2 - \frac{u'^2}{4}\right)\left(\left(\frac{u'}{2} - \lambda\right)^2 - \left(\frac{u'}{2}\right)^2\right).$$

Its roots look like

$$\zeta_1 = u' - u_*, \qquad \zeta_2 = 3u' - u_*, \qquad \zeta_3 = u' - u''_*, \qquad \zeta_4 = -u''_*, \qquad \zeta_5 = u', \qquad \zeta_6 = 0,$$

that is

$$\zeta_1 = u'(1 - \eta), \qquad \zeta_2 = u'(3 - \eta), \qquad \zeta_3 = u'(\eta - 2),$$
$$\zeta_4 = -u'(3 - \eta), \qquad \zeta_5 = u', \qquad \zeta_6 = 0.$$

Now let's describe the Q-canonical transformation. For that purpose it is more convenient to use the following numeration of variables:

$$x_1^2 = x_1, \qquad x_2^2 = x_2, \qquad x_1^1 = x_3, \qquad x_1^3 = x_4, \qquad x_2^1 = x_5, \qquad x_2^3 = x_6, \tag{4.2}$$

and the same for momenta. Then the angular moment (the integral of motion corresponding to rotation in the (1.3)-plane) is given by

$$Q = -\sum_{j=1}^{2}(x_j^1 p_j^3 - x_j^3 p_j^1) = \sum_{j=2}^{3}(x_{2j}p_{2j-1} - x_{2j-1}p_{2j}).$$

The generating function $w(u_{(6)}, p_{(6)})$ is given by

$$w = \sum_{j=3}^{6} g_k(u_3, \ldots u_6)p_k + u_1 p_1 + u_2 p_2.$$

(1.2) gives

$$\sum_{j=3}^{6} \frac{\partial g_k}{\partial u_6} p_k = \sum_{j=2}^{3}(g_{2j}p_{2j-1} - g_{2j-1}p_{2j})$$

which results in (2.5) with $k = 2, 3$.

Repeating arguments from the second section we derive the expressions for g_k, x_k, p_k, u_k, v_k, $k > 2$, translating lower indexes of the variables by 2. In particular

$$\begin{gathered} x_3^2 + x_4^2 = u_3^2, \qquad x_5^2 + x_6^2 = u_4^2 + u_5^2, \\ x_1 = u_1, \quad x_2 = u_2, \quad x_3 = u_3 c, \quad x_4 = -u_3 s, \quad x_5 = u_4 c + u_5 s, \quad x_6 = -u_4 s + u_5 c, \end{gathered} \tag{4.3}$$

where $c = \cos u_6$, $s = \sin u_6$.

The new equilibrium is determined by the equalities $u_3 = -u_4 = a$, $u_j = 0$, $j \neq 3{,}4$, $v_j = 0$. It is not difficult to check that the (6×6)-matrix W is the direct sum of the unit (2×2)-matrix and the (4×4)-matrix W from the second section. This shows that $\operatorname{Det} W = -u_3$. The new Hamiltonian i given by

$$\begin{gathered} H' = (2m)^{-1}\sum_{j=1}^{5} v_j^2 + (2mu_3^2)^{-1}(u_5 v_4 - u_4 v_5 - v_6)^2 + U'(u_{(5)}), \quad v_6 = Q, \\ U'(u_{(5)}) = e_0^2((u_2 - u_1)^2 + (u_4 - u_3)^2 + u_5^2)^{-\frac{1}{2}} - e_0 e'[(u_3^2 + (u_1 - b)^2)^{-\frac{1}{2}} + \\ +(u_3^2 + (u_1 + b)^2)^{-\frac{1}{2}} + (u_4^2 + u_5^2 + (u_2 - b)^2)^{-\frac{1}{2}} + \left[(u_4^2 + u_5^2 + (u_2 + b)^2)^{-\frac{1}{2}}\right]. \end{gathered} \tag{4.4}$$

Now let us find periodic solutions of the equation of motion (1.3) and the associated solutions of (2.2) taking into account that the zero eigenvalue of the canonical matrix of H is doubly degenerate. For that we have to exclude resonances between the eigenvalues ζ_j of U^0.

At the intervals $\eta \in (0, 2)$, $\eta \in (2, 3)$, $\zeta_2 > \zeta_5$, $\zeta_5 > \zeta_2$, respectively. At the interval $\left(2, \frac{5}{2}\right)$ $\zeta_5 > \zeta_2 > \zeta_3 > 0$ and at the interval $\left(\frac{5}{2}, 3\right)$ $0 < \zeta_2 < \zeta_3 < \zeta_5$. That is there are no resonances in ζ_2, ζ_5 (see Section 2) and ζ_5 at $\eta \in (0, 2)$ and $\eta \in (2, 3)$, respectively, since ζ_3, ζ_4 are negative at $\eta \in (0, 2)$. This condition also shows that at the interval $\eta \in (0, 1)$ the nonresonance condition may be taken from the Theorem 2.1. Now we can prove the following theorem which follows from the Lyapunov center theorem for (1.3).

Theorem 4.1. *Let* $0 < \eta = \frac{3}{4}\left(\frac{e_0}{e'}\right)^{\frac{2}{3}} < 3$. *Then the Coulomb equation of motion* (1.3) *for* $n = 6$, $v_6 = 0$ *and* H' *given by* (4.4), *which corresponds to* (2.2) *with* $x_j \in \mathbb{R}^3$ *and potential energy* (4.1), *possesses one, two and three periodic solutions related to the following three cases*:

(1) $\eta > 2$,

(2) $0 < \eta < 2$,

(3) $\eta < 1$, $\eta \neq \frac{k^2-3}{k^2-1}, \frac{k^2-1}{k^2}$.

These solutions and their periods $\tau_j(c_j)$ *are holomorphic functions at the origin in the parameters* c_j, $j = 1, 2, 3$. *The first, second and third cases are characterized by* τ_1, τ_1, τ_2 *and* τ_1, τ_2, τ_3, *respectively, where* $\tau_1^{-1}(0) = (2\pi)^{-1}\sqrt{m^{-1}u'}$, $\tau_2^{-1}(0) = (2\pi)^{-1}\sqrt{m^{-1}(3-\eta)u'}$, $\tau_3^{-1}(0) = (2\pi)^{-1}\sqrt{m^{-1}(1-\eta)u'}$. *The associated quasiperiodic solutions of* (2.2) *are given by* (1.4) *with* $n = 6$, *where* γ, γ', γ'' *correspond to* (4.2), (4.3), *and are such that* $|x_j|^2$, $j = 1, 2$, *are periodic functions.*

5. Appendix. Here we prove the third and second statements of Theorem 1.1. We deal with the Hamiltonian H in $\mathbb{R}^{2n}$ which is a holomorphic function at its equilibrium q^0

$$H(q) = \frac{1}{2}(h^0(q-q^0),(q-q^0)) + \ldots, \qquad q = (x;p), \quad x \in \mathbb{R}^n, \quad p \in \mathbb{R}^n,$$

where (.,.) is the Euclidean scalar product in $\mathbb{R}^{2n}$, h^0 is a symmetric matrix and the three dots imply higher power terms in q_j in the Taylor expansion. The canonical matrix Jh^0 is found from the linear part of the equation of motion $\dot{q} = J\partial H$, where

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

I is the unit $(n \times n)$-matrix and ∂ is the vector of first partial derivatives. The direct and inverse canonical transformations of $q_{(n)} = (x,p)_{(n)}$ into $q'_{(n)} = (x',p')_{(n)}$ are given by

$$q'_j - q'^0_j = \sum_{k=1}^{2n} M^{-1}_{j,k}(q_k - q^0_k) + \ldots, \qquad q_j - q^0_j = \sum_{k=1}^{2n} M_{j,k}(q'_k - q'^0_k) + \ldots,$$

where $M_{j,k}$ is an invertible matrix of the linear symplectic transformation in $\mathbb{R}^{2n}$ and q'^0 is the new equilibrium. If $M_{*j,k} = M_{k,j}$, then (see Sections 2 and 15 in [2])

$$H'(q') = \frac{1}{2}(h'^0(q'-q'^0), q'-q'^0) + \ldots,$$

where

$$h'^0 = M_* h^0 M, \qquad M_* J M = J, \quad J = -J^{-1}$$

and J determines the symplectic structure in $\mathbb{R}^{2n}$. This yields

$$-\mathrm{Det}(\lambda I - Jh^0) = \mathrm{Det}(\lambda J + h^0) = (\mathrm{Det}\, M)^{-2}\,\mathrm{Det}(\lambda J + h'^0).$$

That is the characteristic polynomial of the transformed Hamiltonian has the same roots as the initial Hamiltonian.

Statement 2 follows from $\dot{v}_n = 0$, the fact that the characteristic polynomial of a canonical matrix is an even function in the spectral parameter λ (see Section 15 in [2]) and the fact that the characteristic polynomial of Jh^0 is proportional to λ^2.

References

1. *Skrypnik W.* Periodic and bounded solutions of the Coulomb equation of motion of two and three point charges with equilibrium on line // Ukr. Math. J. – 2014. – **66**, № 5. – P. 668 – 682.
2. *Siegel C., Moser J.* Lectures on celestial mechanics. – Berlin etc.: Springer-Verlag, 1971.
3. *Weinstein A.* Normal modes for non-linear Hamiltonian systems // Inv. Math. – 1973. – **98**. – P. 47 – 57.
4. *Moser J.* Periodic orbits near an equilibrium and a theorem by A. Weinstein // Communs Pure and Appl. Math. – 1976. – **29**. – P. 727 – 747.
5. *Moser J.* On the generalization of a theorem of A. Liapunoff // Communs Pure and Appl. Math. – 1958. – **11**. – P. 257 – 271.
6. *Lyapunov A.* General problem of stability of motion. – Moscow, 1950. – 471 p. (Engl. transl.: Int. J. Contr. – 1992. – **55**, № 3. – P. 521 – 790).
7. *Berger M. S.* Nonlinearity and functional analysis. Lectures on nonlinear problems in mathematical analysis. – New York etc.: Acad. Press, 1977.
8. *Marsden J., McCracken M.* The Hopf bifurcation and its applications. – New York: Springer-Verlag, 1976.
9. *Nemytsky V., Stepanov V.* Qualitative theory of differential equations. – Moscow; Leningrad, 1947.
10. *Skrypnik W.* On exact solutions of Coulomb equation of motion of planar charges // J. Geom. and Phys. – 2015. – **98**. – P. 285 – 291.
11. *Skrypnik W.* On regular polygon solutions of Coulomb equation of motion of $n + 2$ charges n of which are planar // J. Math. Phys. – 2004. – **57**, № 10 / doi: 10. 1063/1,4947421.
12. *Arnold V., Kozlov V., Neishtadt A.* Mathematical aspects of the classical and celestial mechanics. – Moscow, 2002.

Received 20.06.14,
after revision — 24.05.16