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B-COERCIVE CONVOLUTION EQUATIONS IN WEIGHTED FUNCTION SPACES AND APPLICATIONS

B-КОЕРЦИТИВНІ РІВНЯННЯ В ЗГОРТКАХ У ВАГОВИХ ФУНКЦІОНАЛЬНИХ ПРОСТОРАХ ТА ЇХ ЗАСТОСУВАННЯ

We study the B-separability properties of elliptic convolution operators in weighted Besov spaces and establish sharp estimates for the resolvents of the convolution operators. As a result, it is shown that these operators are positive and, in addition, play the role of negative generators of analytic semigroups. Moreover, the maximal B-regularity properties of the Cauchy problem for a parabolic convolution equation are established. Finally, these results are applied to obtain the maximal regularity properties for anisotropic integro-differential equations and the system of infinitely many convolution equations.

Вивчаються властивості B-сепарабельності еліптичних операторів згортки у зважених просторах Бєсова. Встановлено точні оцінки для резольвент операторів згортки. В результаті показано, що ці оператори є додатними, а також від'ємними генераторами аналітичних напівгруп. Крім того, встановлено властивості максимальної B-регулярності задачі Коші для параболічного рівняння у згортках. Ці результати застосовано до отримання властивостей максимальної регулярності для анізотропних інтегро-диференціальних рівнянь та для систем нескінченного числа рівнянь у згортках.

Introduction. In recent years, maximal regularity properties for differential operator equations have been studied extensively, e.g., in [1 – 8, 12, 13, 18, 20, 22, 26, 28, 29]. Moreover, convolution-differential equations (CDEs) have been investigated, e.g., in [10, 16, 19, 21, 22, 27] and the references therein. However, convolution differential-operator equations (CDOEs) is relatively less investigated subject. In [14, 18, 21, 23, 24] parabolic type CDEs with operator coefficient was investigated. In [15, 21] regularity properties of degenerate CDOEs studed in weighted L_p spaces. In conrary to these, the main aim of the present paper is to obtain separability property of the elliptic CDOE

$$(L+\lambda)u = \sum_{|\alpha|\le l} a_\alpha * D^\alpha u + A*u + \lambda u = f(x) \tag{1.1}$$

and the maximal regularity property of the Cauchy problem for the parabolic CDOE

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\le l} a_\alpha * D^\alpha u + A*u = f(t,x), \quad u(0,x)=0$$

in E-valued weighted Besov spaces, where E is a Banach space, $A=A(x)$ is a linear operator in E, $a_\alpha = a_\alpha(x)$ are complex-valued functions and λ is a complex spectral parameter.

By using the Fourier multiplier theorems in weighted Banach valued Besov spaces $B^s_{p,q,\gamma}(R^n;E)$, in Section 2 we derive the coercive estimate of resolvent and particularly and we show that this operator is positive. Namely, we prove that for all $f\in B^s_{p,q,\gamma}(R^n;E)$ there is a unique solution

$$u\in B^{l,s}_{p,q,\gamma}\big(R^n;E(A),E\big)$$

of the problem (1.1) and the following uniformly estimate holds:

$$\sum_{|\alpha|\le l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^\alpha u \right\|_{B^s_{p,q,\gamma}(R^n;E)} +$$

$$+\|A * u\|_{B^s_{p,q,\gamma}(R^n;E)} + |\lambda| \|u\|_{B^s_{p,q,\gamma}(R^n;E)} \le C \|f\|_{B^s_{p,q,\gamma}(R^n;E)}.$$

Particularly, this result implies that if $f \in B^s_{p,q,\gamma}(R^n;E)$, then all terms of the equations (1.1) are also from $B^s_{p,q,\gamma}(R^n;E)$, (i.e., all terms are separated). This important effect allows to obtain some spectral properties of the convolution operator Q.

Moreover, under some assumptions we conclude that the corresponding convolution operator Q has a domain coinciding with the Besov space

$$B^{l,s}_{p,q,\gamma}\big(R^n;E(A),E\big) = B^{l,s}_{p,q,\gamma}(R^n;E) \cap B^s_{p,q,\gamma}\left(R^n;E(A)\right)$$

and there are positive constants C_1 and C_2 such that

$$C_1 \|u\|_{B^{l,s}_{p,q,\gamma}\left(R^n;E(A),E\right)} \le \|Qu\|_{B^s_{p,q,\gamma}(R^n;E)} \le C_2 \|u\|_{B^{l,s}_{p,q,\gamma}\left(R^n;E(A),E\right)}$$

for all $u \in B^{l,s}_{p,q,\gamma}\big(R^n;E(A),E\big)$.

By using the positivity properties of the convolution operator Q and the semigroup theory, in Section 3 we conclude that the above Cauchy problem has a unique solution satisfying the coercive estimate. In Sections 4 and 5, by putting concrete vector spaces instead of E and concrete linear operators instead of A, the maximal regularity properties of convolution differential operators are obtained in vector valued Besov spaces.

1. Notations and background. Let E be a Banach space and $\gamma = \gamma(x)$, $x = (x_1, x_2, \ldots, x_n)$, be a positive measurable weighted function on a measurable subset $\Omega \subset R^n$. Let $L_{p,\gamma}(\Omega;E)$ denote the space of strongly E-valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}(\Omega;E)} = \|f\|_{L_p(E;\gamma)} = \int\limits_\Omega \left(\|f(x)\|^p_E \gamma(x)\, dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega;E)$ will be denoted by $L_p = L_p(\Omega;E)$,

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = \|f\|_{L_\infty(E;\gamma)} = \operatorname*{ess\,sup}_{x\in\Omega} \Big[\gamma(x) \big\|f(x)\big\|_E\Big].$$

The weighted $\gamma(x)$ we will consider satisfy an A_p condition; i.e., $\gamma(x) \in A_p$, $1 < p < \infty$, if there is a positive constant C such that

$$\left(\frac{1}{|Q|} \int\limits_Q \gamma(x) dx \right) \left(\frac{1}{|Q|} \int\limits_Q \gamma^{-\frac{1}{p-1}}(x)\, dx \right)^{p-1} \le C$$

for all compacts $Q \subset R^n$.

Let $\mathbb{C}$ be the set of complex numbers and

$$S_\varphi = \big\{ \lambda; \lambda \in \mathbb{C}, |\arg \lambda| \le \varphi \big\} \cup \{0\}, \quad 0 \le \varphi < \pi.$$

Let E_1 and E_2 be two Banach spaces and $L(E_1, E_2)$ denotes the spaces of bounded linear operators acting from E_1 to E_2. For $E_1 = E_2 = E$ it will be denoted by $L(E)$.

A closed linear operator function $A = A(x)$ is said to be uniformly φ-positive in Banach space E, if $D(A(x))$ is dense in E and does not depend on x and there is a positive constant M so that

$$\left\| \left(A(x) + \lambda I\right)^{-1} \right\|_{L(E)} \leq M\left(1 + |\lambda|\right)^{-1}$$

for every $x \in R^n$ and $\lambda \in S_\varphi, \varphi \in [0, \pi)$, where I is an identity operator in E. Sometimes instead of $A + \lambda I$ we will write $A + \lambda$ and it will be denoted by A_λ. It is known [25] (§ 1.15.1) that there exists fractional powers A^θ of the positive operator A.

Let $E\left(A^\theta\right)$ denote the space $D(A^\theta)$ with graphical norm

$$\|u\|_{E\left(A^\theta\right)} = \left(\|u\|_E^p + \left\|A^\theta u\right\|_E^p\right)^{\frac{1}{p}}, \qquad 1 \leq p < \infty, \quad -\infty < \theta < \infty.$$

Let $S = S(R^n; E)$, or simply $S(E)$ denotes Schwarz class, i.e., the space of E-valued rapidly decreasing smooth functions on R^n equipped with its usual topology generated by seminorms. $S(R^n; \mathbb{C})$ denoted by just S.

Let $S'(R^n; E)$ denote the space of all continuous linear operators, $L\colon S \longrightarrow E$, equipped with the bounded convergence topology. Recall $S(R^n; E)$ is norm dense in $B^s_{p,q,\gamma}(R^n; E)$ when

$$1 \leq p < \infty, \quad 1 \leq q < \infty, \quad \gamma \in A_p.$$

We shall use Fourier analytic definition of weighted Besov spaces in this study. Therefore, we need to consider some subsets $\{J_k\}_{k=0}^\infty$ and $\{I_k\}_{k=0}^\infty$ of R^n. Let $\{J_k\}_{k=0}^\infty$ given by

$$J_0 = \left\{t \in R^n : |t| \leq 1\right\}, \qquad J_k = \left\{t \in R^n : 2^{k-1} \leq |t| \leq 2^k\right\} \quad \text{for} \quad k \in \mathbb{N}.$$

Enlarge each J_k to form a sequence $\{I_k\}_{k=0}^\infty$ of overlapping subsets defined by

$$I_0 = \left\{t \in R^n : |t| \leq 2\right\}, \qquad I_k = \left\{t \in R^n : 2^{k-1} \leq |t| \leq 2^{k+1}\right\} \quad \text{for} \quad k \in \mathbb{N}.$$

Next, we define the unity $\{\varphi_k\}_{k \in \mathbb{N}_0}$ of functions from $S(R^n; R)$, where $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N} = \{1, 2, \ldots\}$ is the of natural numbers. Let $\psi \in S(R, R)$ be nonnegative function with support in $[2^{-1}, 2]$, which satisfies

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \qquad \text{for} \quad s \in R\backslash\{0\}$$

and

$$\varphi_k(t) = \psi\left(2^{-k}|t|\right), \qquad \varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t) \qquad \text{for} \quad k \in \mathbb{N}, \quad t \in R^n.$$

Let $\varphi_k \equiv 0$ if $k < 0$. Later, we will need the following useful properties:

$$\operatorname{supp} \varphi_k \subset \overline{I}_k \qquad \text{for each} \quad k \in \mathbb{N}_0,$$

$$\sum_{k=0}^{\infty} \varphi_k(s) = 1 \qquad \text{for each} \quad s \in R^n,$$

$$J_m \cap \operatorname{supp} \varphi_k = \varnothing \qquad \text{if} \quad |m-k| > 1,$$

$$\varphi_{k-1}(s) + \varphi_k(s) + \varphi_{k+1}(s) = 1 \qquad \text{for each} \quad s \in \operatorname{supp} \varphi_k, \quad k \in \mathbb{N}_0.$$

Let $1 \le p \le q \le \infty$ and $s \in R$. The weighted Besov space is the set of all functions $f \in S'(R;E)$ for which

$$\|f\|_{B^s_{p,q}(E;\gamma)} = \|f\|_{B_{p,q,\gamma}(R^n;E)} = \left\| \left\{ 2^{ks} \big(\check{\varphi}_k * f \big) \right\}_{k=0}^{\infty} \right\|_{l_q(L_{p,\gamma}(R^n;E))} \equiv$$

$$\equiv \begin{cases} \left[\sum_{k=0}^{\infty} 2^{ksq} \, \|\check{\varphi}_k * f\|^q_{L_{p,\gamma}(R^n;E)} \right]^{\frac{1}{q}} & \text{if} \quad q \neq \infty, \\ \sup_{k \in \mathbb{N}_0} \left[2^{ks} \, \|\check{\varphi}_k * f\|_{L_{p,\gamma}(R^n;E)} \right] & \text{if} \quad q = \infty \end{cases}$$

is finite; here p and s are main and smoothness indexes respectively.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \sum_{k=1}^{n} \alpha_k$, where α_k are integers and $D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}$. An E-valued generalized function $D^\alpha f$ is called a generalized derivative in the sense of Schwarz distributions of the function $f \in S(R^n;E)$, equality

$$\langle D^\alpha f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^\alpha \varphi \rangle$$

holds for all $\varphi \in S$.

Let F denote the Fourier transform. Through this section the Fourier transformation of a function f will be denoted by $\widehat{f}$, $Ff = \hat{f}$ and $F^{-1} f = \check{f}$. It is known that

$$F\left(D_x^\alpha f\right) = (i\xi_1)^{\alpha_1} \ldots (i\xi_n)^{\alpha_n} \widehat{f}, \qquad D_\xi^\alpha \left(F\left(f\right)\right) = F\Big[(-ix_1)^{\alpha_1} \ldots (-ix_n)^{\alpha_n} f\Big]$$

for all $f \in S'(R^n;E)$, where $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n$.

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded to E. Let l is a positive integer and $D_k^l = \dfrac{\partial^l}{\partial x_k^l}$. Consider the E-valued function spaces defined by

$$B^{l,s}_{p,q,\gamma}(\Omega; E_0, E) = \Bigg\{ u \colon u \in B^s_{p,q,\gamma}(\Omega; E_0), D_k^l u \in B^s_{p,q,\gamma}(\Omega; E),$$

$$\|u\|_{B^{l,s}_{p,q,\gamma}(\Omega;E_0,E)} = \|u\|_{B^s_{p,q,\gamma}(\Omega;E_0)} + \sum_{k=1}^{n} \left\| D_k^l u \right\|_{B^s_{p,q,\gamma}(\Omega;E)} < \infty \Bigg\}.$$

A function $u \in B^{l,s}_{p,q,\gamma}\left(R^n; E(A), E\right)$ satisfying the equation (1.1) a.e. on R^n is called a solution of (1.1).

The CDOE (1.1) is said to be weighted B-separable (or $B^s_{p,q,\gamma}(R^n;E)$-separable) if for all $f \in B^s_{p,q,\gamma}(R^n;E)$ it has a unique solution $u \in B^{l,s}_{p,q,\gamma}\big(R^n; E(A), E\big)$ and

$$\sum_{|\alpha|\leq l} \|a_\alpha * D^\alpha u\|_{B^s_{p,q,\gamma}(R^n;E)} + \|A * u\|_{B^s_{p,q,\gamma}(R^n;E)} \leq C\|f\|_{B^s_{p,q,\gamma}(R^n;E)}.$$

Let E_1 and E_2 be two Banach spaces. A function $\Psi \in L_\infty\big(R^n; L(E_1, E_2)\big)$ is called a multiplier from $B^s_{p,q,\gamma}(R^n; E_1)$ to $B^s_{p,q,\gamma}(R^n; E_2)$ for $p \in (1,\infty)$ and $q \in [1,\infty]$ if the map $u \to Ku = F^{-1}\Psi(\xi)Fu$, $u \in S(R^n; E_1)$ are well defined and extends to a bounded linear operator

$$K : B^s_{p,q,\gamma}(R^n; E_1) \to B^s_{p,q,\gamma}(R^n; E_2).$$

$M^q_p(E_1, E_2, s)$ denotes the collection of multiplier from $B^s_{p,q}(R^n; E_1)$ to $B^s_{p,q}(R^n; E_2)$. Let

$$\Phi_h = \big\{\Psi_h \in M^q_p(E_1, E_2, s),\ h \in M(h)\big\}.$$

We say that Φ_h is a collection of uniformly bounded multipliers (UBM) if there exists a positive constant M independent on $h \in M(h)$ such that

$$\big\|F^{-1}\Psi_h Fu\big\|_{B^s_{p,q}(R^n;E_2)} \leq M\|u\|_{B^s_{p,q}(R^n;E_1)}$$

for all $h \in M(h)$ and $u \in S(R^n; E_1)$.

Definition 1.1. *Let E be a Banach space and $1 \leq p \leq 2$. Let E so that*

$$\|Ff\|_{L_{p'}(R^n;E)} \leq C\|f\|_{L_p(R^n;E)} \qquad \text{for each} \quad f \in S(R^n, E),$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then the space E is said to be the Fourier type p.

Definition 1.2. *Let $A = A(x)$, $x \in R^n$, be a closed linear operator in E with domain definition $D(A)$ independent of x. Then, the Fourier transformation of $A(x)$ is defined as*

$$\big\langle \hat{A}u, \varphi\big\rangle = \langle Au, \hat{\varphi}\rangle, \qquad u \in D(A), \quad \varphi \in S(R^n).$$

(For details see [3, p. 7].)

Let $h \in R, m \in \mathbb{N}$ and e_k, $k = 1, 2, \ldots, n$, be standart unit vectors of R^n. Let

$$\Delta_k(h)f(x) = f\big(x + he_k\big) - f(x).$$

Definition 1.3. *Let $A = A(x)$ be a closed linear operator in E with the domain definition $D(A)$ independent of x. It is differentiable if*

$$\left(\frac{\partial A}{\partial x_k}\right) u = \lim_{h\to 0} \frac{\|\Delta_k(h)A(x)u\|_E}{h} < \infty, \quad u \in D(A).$$

Let $A = A(x)$, $x \in R^n$, be a closed linear operator in E with domain definition $D(A)$ independent of x and $u \in S'\big(R^n; E(A)\big)$. We define the convolution $A * u$ in the distribution sense (see, e.g., [3]).

The space $C^{(m)}(\Omega; E)$ will denote the spaces of E-valued uniformly bounded, m-times continuously differentiable functions on Ω.

Let us first recall an important fact [11] (Corollary 4.11) that will used in this section.

Theorem 1.1. *Let $p, q \in [1, \infty]$. If $\Psi \in C^l\big(R^n, L(E_1, E_2)\big)$ satisfies for some positive constant K,*

$$\sup_{x \in R^n} \Big\| (1 + |x|)^{|\alpha|} D^\alpha \Psi(x) \Big\|_{L(E_1, E_2)} \le K$$

for each multiindex α with $|\alpha| \le l$, then Ψ is Fourier multiplier from $B^s_{p,q}(R^n, E_1)$ to $B^s_{p,q}(R^n, E_2)$ provided one of the following conditions hold:

(a) E_1 *and* E_2 *are arbitrary Banach spaces and* $l = n + 1$;

(b) E_1 *and* E_2 *are uniformly convex Banach spaces and* $l = n$;

(c) E_1 *and* E_2 *have Fourier type* p *and* $l = \left[\dfrac{n}{p}\right] + 1$.

2. Convolution-elliptic operator equations. In this section we present the separability properties of the CDOE

$$(L + \lambda)u = \sum_{|\alpha| \le l} a_\alpha * D^\alpha u + A * u + \lambda u = f(x),$$

where $A = A(x)$ is a linear operator in a Banach space E, $a_\alpha = a_\alpha(x)$ are complex valued functions and λ is a complex parameter.

We fined the sufficient conditions under which the problem is separable in $B^s_{p,q,\gamma}(R^n; E)$.

***Condition* 2.1.** Let $a_\alpha \in L_\infty(R^n)$ such that the following conditions satisfied:

$$L(\xi) = \sum_{|\alpha| \le l} \hat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}, \qquad \big|L(\xi)\big| \ge C\big|a_0(\xi)\big||\xi|^l,$$

$$|a_0(\xi)| = \max_\alpha \big|\hat{a}_\alpha(\xi)\big|.$$

For proving the main result of this section let at first, show the following lemmas.

Lemma 2.1. *Suppose the Condition* 2.1 *holds. Assume $\hat{A}(\xi)$ is an uniformly φ-positive operator in a Banach space E with $0 < \varphi < \pi - \varphi_1$. Then operator functions*

$$\sigma_0(\xi, \lambda) = \lambda G(\xi, \lambda), \qquad \sigma_1(\xi, \lambda) = \hat{A}(\xi) G(\xi, \lambda),$$

$$\sigma_2(\xi, \lambda) = \sum_{|\alpha| \le l} |\lambda|^{1 - \frac{|\alpha|}{l}} \hat{a}_\alpha(\xi)(i\xi)^\alpha G(\xi, \lambda)$$

are uniformly bounded, where

$$G(\xi, \lambda) = \Big[\hat{A}(\xi) + \lambda + L(\xi)\Big]^{-1}.$$

Proof. Let us note that for the sake of simplicity we shall note change constants in every step. By virtue of [9] (Lemma 2.3), for $L(\xi) \in S_{\varphi_1}$, $\lambda \in S_\varphi$ and $\varphi_1 + \varphi < \pi$ there is a positive constant C so that

$$\big|\lambda + L(\xi)\big| \ge C\Big(|\lambda| + \big|L(\xi)\big|\Big). \tag{2.1}$$

By using the resolvent properties of positive operators and by (2.1) we obtain

$$\left\|\sigma_0(\xi,\lambda)\right\|_{L(E)} \leq M|\lambda|\Big[1+|\lambda|+\big|L(\xi)\big|\Big]^{-1} \leq M_0,$$

$$\left\|\sigma_1(\xi,\lambda)\right\|_{L(E)} = \Big\|I-\big(\lambda+L(\xi)\big)G(\xi,\lambda)\Big\|_{L(E)} \leq$$

$$\leq 1+\big|\lambda+L(\xi)\big|\left\|G(\xi,\lambda)\right\|_{L(E)} \leq 1+M\big|\lambda+L(\xi)\big|\big(1+\big|\lambda+L(\xi)\big|\big)^{-1} \leq M_1.$$

Next, let us consider σ_2. It is clear to see that

$$\left\|\sigma_2(\xi,\lambda)\right\|_{L(E)} = \left\|\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\hat{a}_\alpha(\xi)(i\xi)^\alpha G(\xi,\lambda)\right\|_{L(E)} \leq$$

$$\leq C\sum_{|\alpha|\leq l}|\lambda|\big|\hat{a}_\alpha(\xi)\big|\Big[|\xi||\lambda|^{-\frac{1}{l}}\Big]^{|\alpha|}\left\|G(\xi,\lambda)\right\|_{L(E)} \leq$$

$$\leq C\sum_{|\alpha|\leq l}|\lambda|\big|\hat{a}_\alpha(\xi)\big|\prod_{k=1}^{n}\Big[|\xi_k||\lambda|^{-\frac{1}{l}}\Big]^{\alpha_k}\left\|G(\xi,\lambda)\right\|_{L(E)}.$$

Therefore, $\sigma_2(\xi,\lambda)$ is bounded if

$$\sum_{|\alpha|\leq l}|\lambda|\big|\hat{a}_\alpha(\xi)\big|\prod_{k=1}^{n}\Big[|\xi_k||\lambda|^{-\frac{1}{l}}\Big]^{\alpha_k}\left\|G(\xi,\lambda)\right\|_{L(E)} \leq C.$$

Since $\hat{A}$ is uniformly positive and $L(\xi)\in S_{\varphi_1}$, by using the well known inequalities

$$y_1^{\alpha_1}y_2^{\alpha_2}\ldots y_n^{\alpha_n} \leq C\left(1+\sum_{k=1}^{n}y_k^l\right), \qquad y_k\geq 0, \quad |\alpha|\leq l, \tag{2.2}$$

for $y_k=\Big(|\lambda|^{-\frac{1}{l}}|\xi_k|\Big)^{\alpha_k}$ we obtain

$$\sum_{|\alpha|\leq l}|\lambda|\big|\hat{a}_\alpha(\xi)\big|\prod_{k=1}^{n}\Big[|\xi_k||\lambda|^{-\frac{1}{l}}\Big]^{\alpha_k}\left\|G(\xi,\lambda)\right\|_{L(E)} \leq$$

$$\leq C\sum_{|\alpha|\leq l}|\lambda|\big|\hat{a}_\alpha(\xi)\big|\left[1+\sum_{k=1}^{n}|\xi_k|^l|\lambda|^{-1}\right]\big[1+|\lambda+L(\xi)|\big]^{-1}.$$

Taking into account Condition 2.1 and by (2.2) we get

$$\left\|\sigma_2(\xi,\lambda)\right\|_{L(E)} \leq C\left[|\lambda|+\sum_{k=1}^{n}|\xi_k|^l\right]\big[1+|\lambda|+\big|L(\xi)\big|\big]^{-1} \leq$$

$$\leq C\left[|\lambda|+\sum_{k=1}^{n}|\xi_k|^l\right]\left[1+|\lambda|+\sum_{|\alpha|\leq l}\big|\hat{a}_\alpha(\xi)\big|\,|\xi^\alpha|\right]^{-1} \leq C.$$

Lemma 2.2. *Let all conditions of Lemma* 2.1 *hold. Let* $\hat{a}_\alpha \in C^{(n)}(R^n)$, $\big[D^\beta \hat{A}(\xi)\big]\hat{A}^{-1}(\xi) \in \in C\big(R^n; L(E)\big)$ *for* $|\beta| \leq n+1$ *and*

$$\Big\| \big[D^\beta \hat{A}(\xi)\big]\hat{A}^{-1}(\xi)\Big\|_{L(E)} \leq C_1. \tag{2.3}$$

Then operator functions $D^\beta \sigma_j(\xi,\lambda)$, $j = 0, 1, 2$, *are uniformly bounded.*

Proof. Let us first estimate $\dfrac{\partial \sigma_1}{\partial \xi_i}$. It is easy to see that

$$\big|D^\beta \hat{a}_\alpha(\xi)\big| \leq C_2, \quad |\beta| \leq n+1. \tag{2.4}$$

Really,

$$\left\|\frac{\partial \sigma_1}{\partial \xi_i}\right\|_{L(E)} \leq \|I_1\|_{L(E)} + \big\|I_2\big\|_{L(E)} + \big\|I_3\big\|_{L(E)},$$

where

$$I_1 = \left[\frac{\partial \hat{A}(\xi)}{\partial \xi_i}\right] G(\xi,\lambda), \qquad I_2 = \hat{A}(\xi)\left[\frac{\partial \hat{A}(\xi)}{\partial \xi_i}\right]\big[G(\xi,\lambda)\big]^2$$

and

$$I_3 = \hat{A}(\xi)\left[\frac{\partial L(\xi)}{\partial \xi_i}\right]\big[G(\xi,\lambda)\big]^2.$$

By using (2.3) we get

$$\|I_1\|_{L(E)} = \left\|\left[\frac{\partial \hat{A}(\xi)}{\partial \xi_i}\right]\hat{A}^{-1}(\xi)\hat{A}(\xi)G(\xi,\lambda)\right\|_{L(E)} \leq$$

$$\leq \left\|\left[\frac{\partial \hat{A}(\xi)}{\partial \xi_i}\right]\hat{A}^{-1}(\xi)\right\|_{L(E)} \big\|\sigma_1(\xi,\lambda)\big\|_{L(E)} \leq C.$$

Due to resolvent properties of $\hat{A}$ and by using (2.3) we obtain

$$\big\|I_2\big\|_{L(E)} \leq \left\|\left[\frac{\partial \hat{A}(\xi)}{\partial \xi_i}\right]\hat{A}^{-1}(\xi)\right\|_{L(E)} \big\|\sigma_1(\xi,\lambda)\big\|^2_{L(E)} \leq C.$$

By using (2.3), (2.4) for $\lambda \in S_\varphi$ and $L(\xi) \in S_{\varphi_1}$ with $\varphi_1 + \varphi < \pi$ we have

$$\big\|I_3\big\|_{L(E)} \leq \left|\frac{\partial L(\xi)}{\partial \xi_i}\right| \big\|G(\xi,\lambda)\big\|_{L(E)} \|\sigma_1(\xi,\lambda)\|_{L(E)} \leq$$

$$\leq C\sum_{|\alpha|\leq l}\left[\left|\frac{\partial \hat{a}_\alpha(\xi)}{\partial \xi_i}\right| |\xi^\alpha| + \big|\hat{a}_\alpha(\xi)\big| \left|\xi_1^{\alpha_1}\ldots\xi_i^{\alpha_i-1}\ldots\xi_n^{\alpha_n}\right|\right][1 + |\lambda + L(\xi)|]^{-1} \leq C.$$

Then by using (2.2) we get from above $\|I_3\|_{L(E)} \leq C$. In a similar way the uniformly boundedness of $\sigma_0(\xi, \lambda)$ is proved. Next we shall prove $\frac{\partial}{\partial \xi_i}\sigma_2(\xi, \lambda)$ is uniformly bounded. Similarly,

$$\left\|\frac{\partial}{\partial \xi_i}\sigma_2(\xi, \lambda)\right\|_{L(E)} \leq \|J_1\|_{L(E)} + \|J_2\|_{L(E)} + \|J_3\|_{L(E)} + \|J_4\|_{L(E)},$$

where

$$J_1 = \sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left[\frac{\partial \hat{a}_\alpha(\xi)}{\partial \xi_i}\right](i\xi)^\alpha G(\xi, \lambda),$$

$$J_2 = \sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \hat{a}_\alpha(\xi) i\alpha_i \xi_i^{-1}(i\xi)^\alpha G(\xi, \lambda),$$

$$J_3 = \sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \hat{a}_\alpha(\xi)(i\xi)^\alpha \left[\frac{\partial L(\xi)}{\partial \xi_i}\right]\left[G(\xi, \lambda)\right]^2$$

and

$$J_4 = \sum_{|\alpha| \leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \hat{a}_\alpha(\xi)(i\xi)^\alpha \left[\frac{\partial}{\partial \xi_i}\hat{A}(\xi)\right]\left[G(\xi, \lambda)\right]^2.$$

Let us first show J_1 is uniformly bounded. Since,

$$\|J_1\|_{L(E)} \leq \sum_{|\alpha| \leq l} \left|\frac{\partial \hat{a}_\alpha(\xi)}{\partial \xi_i}\right| \left\||\lambda|^{1-\frac{|\alpha|}{l}}(i\xi)^\alpha G(\xi, \lambda)\right\|_{L(E)}.$$

By resolvent properties of $\hat{A}$, by virtue of (2.1), (2.2) and (2.4) we obtain $\|J_1\|_{L(E)} \leq C$. In a similar way by using (2.1), (2.2) and (2.4) we get

$$\|J_k\|_{L(E)} \leq C, \quad k = 2, 3, 4.$$

Hence, operator functions $\frac{\partial \sigma_j}{\partial \xi_i}$, $j = 0, 1, 2$, are uniformly bounded. Now, it remains to show $D^\beta \sigma_j(\xi, \lambda)$ are uniformly bounded for $|\beta| \leq n + 1$. It is clear to see that

$$\left\|\frac{\partial^2 \sigma_1}{\partial \xi_i^2}\right\|_{L(E)} \leq \|I_4\|_{L(E)} + \|I_5\|_{L(E)} + \|I_6\|_{l(E)},$$

where

$$I_4 = \frac{\partial^2 \hat{A}(\xi)}{\partial \xi_i^2} G(\xi, \lambda) - \left[\frac{\partial \hat{A}(\xi)}{\partial \xi_i}\right]^2 \left[G(\xi, \lambda)\right]^2 - \frac{\partial \hat{A}(\xi)}{\partial \xi_i}\frac{\partial L(\xi)}{\partial \xi_i}\left[G(\xi, \lambda)\right]^2,$$

$$I_5 = \left[\frac{\partial \hat{A}(\xi)}{\partial \xi_i}\right]^2 \left[G(\xi, \lambda)\right]^2 + \frac{\partial \hat{A}(\xi)}{\partial \xi_i}\frac{\partial^2 \hat{A}(\xi)}{\partial \xi_i^2}\left[G(\xi, \lambda)\right]^2 -$$

$$-2\hat{A}(\xi)\left[\frac{\partial\hat{A}(\xi)}{\partial\xi_i}\hat{A}(\xi)\right]\left[\frac{\partial\hat{A}(\xi)}{\partial\xi_i}+\frac{\partial L(\xi)}{\partial\xi_i}\right]\left[G(\xi,\lambda)\right]^3+$$

$$+\hat{A}(\xi)\frac{\partial L(\xi)}{\partial\xi_i}\left[\frac{\partial\hat{A}(\xi)}{\partial\xi_i}+\frac{\partial L(\xi)}{\partial\xi_i}\right]\left[G(\xi,\lambda)\right]^3,$$

$$I_6=\frac{\partial\hat{A}(\xi)}{\partial\xi_i}\frac{\partial L(\xi)}{\partial\xi_i}\left[G(\xi,\lambda)\right]^2+\hat{A}(\xi)\frac{\partial^2 L(\xi)}{\partial\xi_i^2}\left[G(\xi,\lambda)\right]^2.$$

By using resolvent properties of positive operator $\hat{A}(\xi)$ and conditions of lemma we have

$$\|I_4\|_{L(E)}\leq\left\|\frac{\partial^2\hat{A}(\xi)}{\partial\xi_i^2}\hat{A}^{-1}(\xi)\right\|\|\sigma_1\|+\left\|\frac{\partial\hat{A}(\xi)}{\partial\xi_i}\hat{A}^{-1}(\xi)\right\|^2\|\sigma_1\|^2+$$

$$+2\|\sigma_1\|\left\|\frac{\partial\hat{A}(\xi)}{\partial\xi_i}\hat{A}^{-1}(\xi)\right\|\left\|\left[\frac{\partial\hat{A}(\xi)}{\partial\xi_i}+\frac{\partial L}{\partial\xi_i}\right]\hat{A}^{-1}(\xi)\right\|\|\sigma_1\|^2.$$

Then by using (2.1)–(2.4) we obtain from above $\|I_4\|_{L(E)}\leq M$. By using the same arguments we get

$$\|I_k\|_{L(E)}\leq M,\quad k=5,6.$$

From the representations of $\sigma_j(\xi,\lambda)$, $j=0,1,2$, it easy to see that operator functions $D^\beta\sigma_j(\xi,\lambda)$ contain the similar terms as I_k; namely, the functions $D^\beta\sigma_j(\xi,\lambda)$ will be represented as the combinations of principal terms

$$\left[D^m\hat{A}(\xi)+\xi^\sigma D^\gamma\hat{a}_\alpha(\xi)\right]\left[\hat{A}(\xi)+\lambda+L(\xi)\right]^{-|\beta|},\tag{2.5}$$

where $|m|\leq|\beta|$ and $|\sigma|+|\gamma|\leq|\beta|$. So, by using the similar arguments as above we obtain

$$\left\|D^\beta\sigma_j(\xi,\lambda)\right\|\leq C,\quad j=0,1,2.$$

Hence, we get operator functions $D^\beta\sigma_j(\xi,\lambda)$ are uniformly bounded for each multiindex $|\beta|\leq$ $\leq n+1$.

Lemma 2.3. *Let all conditions of Lemma 2.2 are satisfied and*

$$|\xi|^\eta\left|D^\beta\hat{a}_\alpha(\xi)\right|\leq C_1,\qquad|\xi|^\eta\left\|\left[D^\beta\hat{A}(\xi)\right]\hat{A}^{-1}(\xi)\right\|_{L(E)}\leq C_2.\tag{2.6}$$

Then the estimates hold

$$|\xi|^\eta\left\|D^\beta\sigma_j(\xi,\lambda)\right\|_{L(E))}\leq M,\quad j=0,1,2,$$

for all $\eta\leq|\beta|\leq n+1$ and $\xi\in R^n$.

Proof. Since, $D^\beta\sigma_j(\xi,\lambda)$ is in the form of (2.5), by reasoning as in Lemma 2.2, by (2.6) we have

$$\left\||\xi_i|^\eta D^\beta\sigma_j(\xi,\lambda)\right\|_{L(E))}\leq C,\qquad j=0,1,2,\quad i=1,2,\ldots,n,$$

that in its turn implies assertion of Lemma 2.3.

From Lemma 2.3 we obtain the following corollary.

***Corollary* 2.1.** *Let all conditions of Lemma* 2.3 *are satisfied,* $p, q \in [1, \infty)$. *Then operator-functions* $\sigma_j(\xi, \lambda)$, $j = 0, 1, 2$, *are UBM in* $B^s_{p,q,\gamma}(R^n; E)$.

Proof. To prove $\sigma_j(\xi, \lambda)$ are UBM in $B^s_{p,q,\gamma}(R^n; E)$, we need the following estimates:

$$\left\|(1 + |\xi|)^\eta D^\beta \sigma_j(\xi, \lambda)\right\|_{L_\infty(R^n, L(E))} \le K, \quad K > 0,$$

for $\xi \in R^n \backslash 0$, $|\beta| \le n + 1$. From Lemma 2.3 it follows $\sigma_i \in C^{(n)}(R^n \backslash 0, L(E))$ and

$$\left\|D^\beta \sigma_j(\xi, \lambda)\right\|_{L_\infty(L(E)))} \le K_1, \qquad |\xi|^\eta \left\|D^\beta \sigma_j(\xi, \lambda)\right\|_{L_\infty(L(E)))} \le K_2, \quad \eta \le |\beta| \le n + 1.$$

Hence, operator functions $\sigma_i(\xi, \lambda)$ are Fourier multipliers in $B^s_{p,q,\gamma}(R^n, E)$. Let we denote $B^s_{p,q,\gamma}(R^n, E)$ by X.

Now we are ready to state the main result of the present section.

***Condition* 2.2.** Suppose the following conditions be satisfied:

(1) $L(\xi) = \sum_{|\alpha| \le l} \hat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}$, $\left|L(\xi)\right| \ge C\left|a_0(\xi)\right||\xi|^l$, $\left|a_0(\xi)\right| = \max_\alpha \left|\hat{a}_\alpha(\xi)\right|$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in R^n$;

(2) E is a Banach space;

(3) $\hat{A}(\xi)$ is an uniformly φ-positive operator in E, with $0 < \varphi < \pi - \varphi_1$ and

$$\left[D^\beta \hat{A}(\xi)\right] \hat{A}^{-1}(\xi) \in C\left(R^n; L(E)\right),$$

$$\hat{a}_\alpha \in C(R^n), \qquad |\xi|^k \left|D^\beta \hat{a}_\alpha(\xi)\right| \le C_1, \quad k \le |\beta| \le n + 1,$$

$$|\xi|^k \left\|\left[D^\beta \hat{A}(\xi)\right] \hat{A}^{-1}(\xi)\right\|_{L(E)} \le C_2, \quad k \le |\beta| \le n + 1.$$

Theorem 2.1. *Suppose the Condition* 2.2 *is satisfied, then for* $f \in B^s_{p,q,\gamma}(R^n; E)$, $p, q \in [1, \infty)$, *the equation* (1.1) *has a unique solution* $u \in B^{l,s}_{p,q,\gamma}\left(R^n; E(A), E\right)$ *and the following coercive uniform estimate holds*:

$$|\lambda| \|u\|_X + \sum_{|\alpha| \le l} |\lambda|^{1 - \frac{|\alpha|}{l}} \left\|a_\alpha * D^\alpha u\right\|_X + \|A * u\|_X \le C\|f\|_X \tag{2.7}$$

for $\lambda \in S_\varphi$ *with* $|\lambda| \ge \lambda_0 > 0$.

Proof. By applying the Fourier transform to equation (1.1), we obtain

$$\left[\hat{A}(\xi) + L(\xi) + \lambda\right] u\hat{}(\xi) = f\hat{}(\xi).$$

Since $L(\xi) \in S_{\varphi_1}$ for $\xi \in R^n$ and $\hat{A}$ is a positive operator in E, we get $\left[\hat{A}(\xi) + L(\xi) + \lambda\right]^{-1} \in L(E)$. So we obtain that the solution of the equation (1.1) can be represented in the form

$$u(x) = F^{-1}\left[\hat{A}(\xi) + \lambda + L(\xi)\right]^{-1} f\hat{}.$$

There are positive constants C_1 and C_2 such that

$$C_1 |\lambda| \|u\|_X \le \left\|F^{-1}\left[\sigma_0(\xi, \lambda) f\hat{}\right]\right\|_X \le C_2 |\lambda| \|u\|_X,$$

$$C_1 \|A * u\|_X \le \left\|F^{-1}\left[\sigma_1(\xi, \lambda) f\hat{}\right]\right\|_X \le C_2 \|A * u\|_X, \tag{2.8}$$

$$C_1 \sum_{|\alpha| \le l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X \le \left\| F^{-1} \left[\sigma_2(\xi, \lambda) f\hat{} \right] \right\|_X \le$$

$$\le C_2 \sum_{|\alpha| \le l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_X ,$$

where

$$\sigma_0(\xi, \lambda) = \lambda \big[\hat{A}(\xi) + (\lambda + L(\xi) \big]^{-1}, \qquad \sigma_1(\xi, \lambda) = \hat{A}(\xi) \big[\hat{A}(\xi) + \lambda + L(\xi) \big]^{-1},$$

$$\sigma_2(\xi, \lambda) = \sum_{|\alpha| \le l} |\lambda|^{1-\frac{|\alpha|}{l}} \hat{a}_\alpha(\xi) (i\xi)^\alpha \big[\hat{A}(\xi) + \lambda + L(\xi) \big]^{-1}.$$

To show the estimate (2.7) it is enough to prove that operator functions $\sigma_j(\xi, \lambda),\ j = 0, 1, 2,$ are UBM in X. This fact is obtained from the Lemma 2.3. That is we obtain the assertion.

Let Q be the operator in $X = B^s_{p,q,\gamma}(R^n; E)$ generated by problem (1.1) for $\lambda = 0$, i.e.,

$$D(Q) \subset X_0 = B^{l,s}_{p,q,\gamma}\big(R^n; E(A), E\big), \qquad Qu = \sum_{|\alpha| \le l} a_\alpha * D^\alpha u + A * u.$$

***Result* 2.1.** Assume all conditions of Theorem 2.1 hold. Then for all $\lambda \in S_\varphi,\ |\lambda| \ge \lambda_0 > 0,$ there exist the resolvent of operator Q and the following estimate holds:

$$\sum_{|\alpha| \le l} |\lambda|^{1-\frac{|\alpha|}{l}} \Big\| a_\alpha * \left[D^\alpha (Q + \lambda)^{-1} \right] \Big\|_{L(X)} +$$

$$+ |\lambda| \big\| (Q + \lambda)^{-1} \big\|_{L(X)} + \big\| A * (Q + \lambda)^{-1} \big\|_{L(X)} \le C.$$

***Condition* 2.3.** Let

$$D(A) = D(\hat{A}) = D\big(\hat{A}(\xi_0)\big), \quad \xi_0 \in R^n.$$

Moreover, there are positive constants C_1 and C_2 so that for $u \in D(A),\ x \in R^n$,

$$C_1 \big\| \hat{A}(\xi_0) u \big\| \le \|A(x) u\| \le C_2 \big\| \hat{A}(\xi_0) u \big\|.$$

***Remark* 2.1.** The Condition 2.2 is checked for regular elliptic operators with sufficiently smooth coefficients. Really, by setting $E = L_{p_1}(\Omega),\ p_1 \in (1, \infty)$, for bounded domain $\Omega \subset R^m$ with enough smooth boundary $\partial\Omega$ and by putting regular elliptic operators instead of $A(x)$ and $\hat{A}(\xi)$ one can get it. So, by virtue of [17, 19] we obtain that the operators $A(x),\ \hat{A}(\xi)$ are positive and the above estimates hold.

Theorem 2.2. *Assume all conditions of Theorem* 2.1 *and Condition* 2.3 *are satisfied. Then the problem* (1.1) *has a unique solution* $u \in X_0$ *and the coercive uniform estimate holds*

$$\sum_{|\alpha| \le l} |\lambda|^{1-\frac{|\alpha|}{l}} \|D^\alpha u\|_X + \|Au\|_X \le C \|f\|_X$$

for all $f \in X,\ p, q \in [1, \infty)$ *and* $\lambda \in S_\varphi$ *with* $|\lambda| \ge \lambda_0 > 0$.

Proof. By applying the Fourier transform to equation (1.1), we have $\hat{u}(\xi) = D(\xi,\lambda)f\hat{}(\xi)$, where

$$D(\xi,\lambda) = \left[\hat{A}(\xi) + L(\xi) + \lambda)\right]^{-1}.$$

So we obtain that the solution of equation (1.1) can be represented as $u(x) = F^{-1}D(\xi,\lambda)f\hat{}$. Moreover, by Condition 2.2 we get

$$\left\|AF^{-1}D(\xi,\lambda)f\hat{}\right\|_X \leq M\left\|\hat{A}(\xi_0)F^{-1}D(\xi,\lambda)f\hat{}\right\|_X.$$

Hence, by using (2.8) it sufficient to show that the operator functions

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}}\xi^{\alpha}D(\xi,\lambda) \qquad \text{and} \qquad \hat{A}(\xi_0)D(\xi,\lambda)$$

are UBMs in X. Really, in view of (3) part of Condition 2.2 these fact are proved by reasoning as in Lemma 2.3.

***Condition* 2.4.** There are positive constants C_1 and C_2 such that

$$C_1\sum_{k=1}^{n}\left|a_k\xi_k\right|^l \leq \left|L(\xi)\right| \leq C_2\sum_{k=1}^{n}\left|a_k\xi_k\right|^l, \quad \xi \neq 0,$$

and

$$D(A) = D(\hat{A}) = D\big(A(x_0)\big), \qquad \hat{A}(\xi)A^{-1}(x_0) \in L_\infty\big(R^n; B(E)\big), \quad \xi, x_0 \in R^n,$$

$$C_1\left\|A(x_0)u\right\| \leq \left\|A(x)u\right\| \leq C_2\left\|A(x_0)u\right\|, \qquad u \in D(A), \quad x \in R^n.$$

Theorem 2.3. *Let all conditions of Theorem* 2.2 *and Condition* 2.4 *hold. Then for* $u \in X_0$ *there are positive constants* M_1 *and* M_2 *so that*

$$M_1\|u\|_{X_0} \leq \sum_{|\alpha|\leq l}\left\|a_\alpha * D^\alpha u\right\|_X + \|A*u\|_X \leq M_2\|u\|_{X_0}.$$

Proof. The left part of the above inequality is obtained from Theorem 2.2. So, it remains to prove the right-hand side of the estimate. Really, from Condition 2.3 we have

$$\|A*u\|_X \leq M\left\|F^{-1}\hat{A}\hat{u}\right\|_X \leq C\left\|F^{-1}\hat{A}A^{-1}(x_0)A(x_0)\hat{u}\right\|_X \leq$$

$$\leq C\left\|F^{-1}A(x_0)\hat{u}\right\|_X \leq C\|Au\|_X, \quad u \in X_0.$$

Hence, applying the Fourier transform to equation (1.1) and by reasoning as Theorem 2.2, it is sufficient to prove that the function

$$\sum_{|\alpha|\leq l}\hat{a}_\alpha\xi^\alpha\left[\sum_{k=1}^{n}\xi_k^{l_k}\right]^{-1}$$

is a multiplier in X. Then by using Condition 2.3 and proof of Lemma 2.3 we get the desired result.

***Result* 2.2.** Theorem 2.3 implies the estimate

$$C_1\|u\|_{X_0} \leq \|Qu\|_X \leq C_2\|u\|_{X_0}$$

for $u \in X_0$.

***Result* 2.3.** Result 2.1 particularly implies that the operator $Q + a$ is positive in X for $a > 0$, i.e., if $\hat{A}$ is uniformly positive for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, then it is clear to see that the operator $Q + a$ is a generator of an analytic semigroup in X.

3. The Cauchy problem for parabolic CDOE. In this section we derive the maximal regularity properties of parabolic CDOE.

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Let $X = B^s_{p,q,\gamma}(R^n; E)$, $Y = B^s_{p,q,\gamma}(R_+; X)$ and $X_0 = B^{l,s}_{p,q,\gamma}\big(R^n; E_0, E\big)$. $\tilde{B}^{l,1,s}_{p,q,\gamma}\big(R^n_+; E_0, E\big)$ denotes the space of all functions $u \in \tilde{B}^{l,1,s}_{p,q,\gamma}\big(R^n_+; E_0, E\big)$ that possess the generalized derivatives

$$D_t u, D^\alpha_x u \in B^s_{p,q,\gamma}(R_+; X)$$

with the norm

$$\|u\|_{B^{l,1,s}_{p,q,\gamma}\left(R^n_+;E_0,E\right)} = \|u\|_{B^s_{p,q,\gamma}(R_+;X)} + \|D_t u\|_{B^s_{p,q,\gamma}(R_+;X)} + \left\|D^\alpha u\right\|_{B^s_{p,q,\gamma}(R_+;X)}.$$

Consider the Cauchy problem for parabolic CDOE

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\leq l} a_\alpha * D^\alpha u + A * u = f(t,x), \quad u(0,x) = 0, \tag{3.1}$$

where $A = A(x)$ is a linear operator in E, $a_\alpha = a_\alpha(x)$ are complex-valued functions.

Theorem 3.1. *Assume Condition* 2.2 *holds for* $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, $s > 0$. *Suppose* $\gamma \in A_p$ *and* E *is a Banach space satisfies the* B*-weighted multiplier condition. Then for* $f \in Y$ *the problem* (3.1) *has a unique solution* $u \in \tilde{B}^{l,1,s}_{p,q}\big(R^{n+1}_+; E(A), E\big)$ *satisfying the estimate*

$$\left\|\frac{\partial u}{\partial t}\right\|_Y + \sum_{|\alpha|\leq l} \|a_\alpha * D^\alpha u\|_Y + \|A * u\|_Y \leq C\|f\|_Y. \tag{3.2}$$

Proof. It is clear to see that

$$\tilde{B}^{l,1,s}_{p,q}\big(R^{n+1}_+; E(A), E\big) = B^{1,s}_{p,q}\big(R_+; D(Q), X\big).$$

Fom the Result 2.1 that the operator Q is positive in X for $\varphi > \frac{\pi}{2}$. Then it is known that the operator Q generated an analytic semigroup in X. It is easy to see that the problem (3.1) can be expressed as the following problem:

$$\frac{\partial u}{\partial t} + Qu(t) = f(t), \qquad u(0) = 0, \quad t \in R_+. \tag{3.3}$$

In view of resolvent properties of Q, $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, by virtue of [3] (Proposition 8.10), [15, 19] and by Result 2.2 we obtain that, for $f \in Y$ the equation (3.3) has a unique solution $u \in$ $\in B^{1,s}_{p,q}\big(R_+; D(Q), X\big)$ satisfying

$$\left\|\frac{\partial u}{\partial t}\right\|_X + \|Qu\|_X \leq C\|f\|_X. \tag{3.4}$$

By Theorem 2.1 and estimate (3.4) we obtain (3.2).

4. Degenerate convolution elliptic equations. Consider the E-valued weighted Besov spaces $B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)$ defined as

$$B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)=\left\{u;\ u\in B^{s}_{p,q,\gamma}(\Omega;E_0),\ D^{[l]}_{x_k}u\in B^{s}_{p,q,\gamma}(\Omega;E)\right\},$$

$$\|u\|_{B^{[l],s}_{p,q,\gamma}(\Omega;E_0,E)}=\|u\|_{B^{s}_{p,q,\gamma}(\Omega;E_0)}+\sum_{k=1}^{n}\left\|D^{[l]}_{x_k}u\right\|_{B^{;s}_{p,q,\gamma}(\Omega;E)}<\infty.$$

Let us consider the following degenerate elliptic CDOE:

$$(L+\lambda)u=\sum_{|\alpha|\le l}a_\alpha * D^{[\alpha]}u+A*u+\lambda u=f. \tag{4.1}$$

We shall show that for all $f\in B^{s}_{p,q,\gamma}(R^n;E)$ and sufficiently large $|\lambda|,\ \lambda\in S_\varphi$ the equation (4.1) has a unique solution $u\in B^{[l],s}_{p,q,\gamma}\big(R^n;E(A),E\big)$ and the coercive uniform estimate

$$\sum_{|\alpha|\le l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha * D^{[\alpha]}u\right\|_{B^{s}_{p,q,\gamma}(R^n;E)}+\|A*u\|_{B^{s}_{p,q,\gamma}(R^n;E)}+|\lambda|\|u\|_{B^{s}_{p,q,\gamma}(R^n;E)}\le$$

$$\le C\|f\|_{B^{s}_{p,q,\gamma}(R^n;E)} \tag{4.2}$$

holds.

Recall that $\tilde\gamma_k(x),\ k=1,2,\ldots,n,$ are positive measurable functions in R and

$$D^{[\alpha]}=D^{[\alpha_1]}_1D^{[\alpha_2]}_2\ldots D^{[\alpha_n]}_n,\qquad D^{[a_k]}_k=\left(\tilde\gamma_k(x_k)\frac{\partial}{\partial x_k}\right)^{a_k},\qquad \tilde\gamma=\big(\tilde\gamma_1,\tilde\gamma_2,\ldots,\tilde\gamma_n\big).$$

It is clear that under the substitution

$$z_k=\int_0^{x_k}\tilde\gamma_k^{-1}(y)\,dy \tag{4.3}$$

spaces $B^{s}_{p,q}(R^n;E)$ and $B^{[l],s}_{p,q}\big(R^n;E(A),E\big)$ are mapped isomorphically onto the weighted spaces $B^{s}_{p,q,\gamma}(R^n;E),\ B^{l,s}_{p,q,\gamma}\big(R^n;E(A),E\big)$ respectively, where

$$\gamma=\prod_{k=1}^{n}\tilde\gamma_k\big(x_k(z_k)\big).$$

Moreover, under substitution (4.3) the degenerate problem (4.1) is redused to the following nondegenerate problem:

$$(L+\lambda)u=\sum_{|\alpha|\le l}a_\alpha * D^{\alpha}u+A*u+\lambda u=f \tag{4.4}$$

considered in weighted space $B^{s}_{p,q,\gamma}(R^n;E)$, where A is a linear operator in Banach space E and a_α are complex numbers.

Now, we consider Cauchy problem the degenerate parabolic convolution equation

$$\frac{\partial u}{\partial t} + \sum_{|\alpha|\le l} a_\alpha * D^{[\alpha]} u + A * u = f(t,x),$$
$$u(0,x) = 0, \qquad t \in R_+, \quad x \in R. \tag{4.5}$$

In a similar way, under the substitution (4.3) the degenerate Cauchy problem (4.5) considered in $B^s_{p,q}(R^n; E)$ is transformed into undegenerate Cauchy problem (3.1) considered in the weighted space $B^s_{p,q,\gamma}(R^n; E)$.

Let H be the operator generated by problem (4.1), i.e.,

$$D(H) = B^{[l],s}_{p,q,\gamma}\big(R^n; E(A), E\big), \qquad Hu = \sum_{|\alpha|\le l} a_\alpha * D^{[\alpha]} u + A * u,$$

and we denote $B^s_{p,q,\gamma}(R^n; E)$ by X.

From Theorems 2.1 and 3.1 we obtain the following results.

***Result* 4.1.** Under conditions of Theorem 2.1 and (4.3) the equation (4.1) has a unique solution $u(x)$ that belongs to space $B^{[l],s}_{p,q,\gamma}(R^n; E(A), E)$ and the coercive uniform estimate

$$\sum_{|\alpha|\le l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^{[\alpha]} u \right\|_X + \|A * u\|_X + |\lambda| \|u\|_X \le C \|f\|_X$$

holds for all $f \in B^s_{p,q,\gamma}(R^n; E)$ and for sufficiently large $\lambda \in S_\varphi$.

Moreover, for $\lambda \in S_\varphi$ there exist the resolvent of operator H and has the estimate

$$\sum_{|\alpha|\le l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * D^{[\alpha]} (H+\lambda)^{-1} \right\|_{L(X)} +$$
$$+ \left\| A * (H+\lambda)^{-1} \right\|_{L(X)} + \left\| \lambda (H+\lambda)^{-1} \right\|_{L(X)} \le C.$$

***Result* 4.2.** For all $f \in B^s_{p,q,\gamma}(R_+; X)$ there is unique solution $u(t,x)$ of problem (4.1) satisfying the following coercive estimate:

$$\left\| \frac{\partial u}{\partial t} \right\|_Y + \sum_{|\alpha|\le l} \left\| a_\alpha * D^{[\alpha]} u \right\|_Y + \|A * u\|_Y \le C \|f\|_Y.$$

5. Boundary-value problems for CDEs. In this section the boundary-value problems (BVPs) for the anisotropic type integro-differential equations is studied. The maximal regularity properties of this problem in weighted mixed $B^s_{\mathbf{p},q,\gamma}$ norm is obtained. In this direction it can be mention, e.g., the works [3, 10, 11, 16, 19]. Let $\tilde{\Omega} = R^n \times \Omega$, where $\Omega \subset R^\mu$ is an open connected set with compact C^{2m}-boundary $\partial\Omega$. Consider the BVP for CDE

$$(L+\lambda)u = \sum_{|\alpha|\le l} a_\alpha * D^\alpha u + \sum_{|\alpha|\le 2m} \big(b_\alpha c_\alpha D^\alpha_y\big) * u + \lambda u = f(x,y), \tag{5.1}$$
$$x \in R^n, \quad y \in \Omega \subset R^\mu,$$

$$B_j u = \sum_{|\beta|\le m_j} b_{j\beta}(y) D^\beta_y u(x,y) = 0, \quad y \in \partial\Omega, \tag{5.2}$$

where

$$D_j = -i\frac{\partial}{\partial y_j}, \qquad y = (y_1, \ldots, y_\mu), \qquad a_\alpha = a_\alpha(x), \qquad b_\alpha = b_\alpha(x), \qquad c_\alpha = c_\alpha(y),$$

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n), \qquad u = u(x,y), \quad j = 1, 2, \ldots, m.$$

Let $\tilde{\Omega} = R^n \times \Omega$, $\mathbf{p} = (p_1, p)$, and $\gamma(x) = |x|^\alpha$, $L_{\mathbf{p},\gamma}(\tilde{\Omega})$ will be denote the space of all $\mathbf{p}$-summable scalar-valued functions with weighted mixed norm (see, e.g., [7], § 1), i.e., the space of all measurable functions f defined on $\tilde{\Omega}$, for which

$$\|f\|_{L_{\mathbf{p},\gamma}(\tilde{\Omega})} = \left(\int\limits_{R^n} \left(\int\limits_{\Omega} |f(x,y)|^{p_1} \gamma(x) dx \right)^{\frac{p}{p_1}} dy \right)^{\frac{1}{p}} < \infty.$$

Analogously $B^s_{\mathbf{p},q,\gamma}(\tilde{\Omega})$ denotes the Besov space with corresponding weighted mixed norm [7] (§ 18) and let

$$\tilde{B}^s_{\mathbf{p},q,\gamma}(\tilde{\Omega}) = B^s_{p,q,\gamma}\left(R^n; B^s_{p_1,q,\gamma}(\Omega)\right),$$

$$\tilde{B}^{l,2m,s}_{\mathbf{p},q,\gamma}(\tilde{\Omega}) = B^{l,s}_{p,q,\gamma}\Big(R^n; B^{2m,s}_{p_1,q,\gamma}(\Omega), B^s_{p_1,q,\gamma}(\Omega)\Big).$$

Let Q denote the operator generated by BVP (5.1), (5.2).

In general, $l \neq 2m$ so equation (5.1) is anisotropic. For $l = 2m$ we get isotropic equation.

Theorem 5.1. *Let the following conditions be satisfied*:

(1) $c_\alpha \in C(\bar{\Omega})$ *for each* $|\alpha| = 2m$ *and* $c_\alpha \in L_\infty(\Omega) + L_{r_k}(\Omega)$ *for each* $|\alpha| = k < 2m$ *with* $r_k \geq p_1$, $p_1 \in (1, \infty)$ *and* $2m - k > \dfrac{l}{r_k}$, $-1 < \alpha < p - 1$, $k = 1, 2, \ldots, n$;

(2) $b_{j\beta} \in C^{2m - m_j}(\partial\Omega)$ *for each* j, β, $m_j < 2m$, p, $q \in (1, \infty)$;

(3) *for* $y \in \bar{\Omega}$, $\xi \in R^\mu$, $\lambda \in S_{\varphi_0}$, $\varphi_0 \in \left(0, \dfrac{\pi}{2}\right)$, $|\xi| + |\lambda| \neq 0$, *let* $\lambda + \sum_{|\alpha|=2m} c_\alpha(y)\xi^\alpha \neq 0$;

(4) *for each* $y_0 \in \partial\Omega$ *local BVP in local coordinates corresponding to* y_0

$$\lambda + \sum_{|\alpha|=2m} c_\alpha(y_0) D^\alpha g(y) = 0,$$

$$B_{j0} g = \sum_{|\beta|=m_j} b_{j\beta}(y_0) D^\beta g(y) = h_j, \quad j = 1, 2, \ldots, m,$$

has a unique solution $g(y) \in C_0(R_+)$ *for all* $h = (h_1, h_2, \ldots, h_m) \in R^m$ *and for* $\xi' \in R^{\mu-1}$ *with* $|\xi'| + |\lambda| \neq 0$;

(5) *the* (1) *part of Condition* 2.2 *satisfied*; $\hat{a}_\alpha, \hat{b}_\alpha \in C^{(n)}(R^n)$ *and there are positive constants* C_1 *and* C_2, *so that*

$$|\xi|^k \big|D^\beta \hat{a}_\alpha(\xi)\big| \leq C_1, \qquad |\xi|^k \big|D^\beta \hat{b}_\alpha(\xi)\big| \leq C_2$$

for all $k \leq |\beta| \leq n + 1$ *and* $\xi \in R^n \backslash \{0\}$.

Then for all $f \in \tilde{B}^{s}_{\mathbf{p},q,\gamma}(\tilde{\Omega})$ *problem* (5.1), (5.2) *has a unique solution* $u \in \tilde{B}^{l,2m,s}_{\mathbf{p},q,\gamma}\big(\tilde{\Omega}\big)$ *and the following coercive uniform estimate holds*:

$$\sum_{|\alpha|\le l} |\lambda|^{1-\frac{|\alpha|}{l}} \, \|a_\alpha * D^\alpha u\|_{\tilde{B}^{s}_{\mathbf{p},q,\gamma}\left(\tilde{\Omega}\right)} + |\lambda| \|u\|_{\tilde{B}^{s}_{\mathbf{p},q,\gamma}(\tilde{\Omega})} +$$

$$+ \sum_{|\alpha|\le 2m} \|b_\alpha c_\alpha D^\alpha * u\|_{\tilde{B}^{s}_{\mathbf{p},q,\gamma}\left(\tilde{\Omega}\right)} \le C\|f\|_{\tilde{B}^{s}_{\mathbf{p},q,\gamma}\left(\tilde{\Omega}\right)}$$

for $\lambda \in S_\varphi$ *and* $\varphi \in [0,\pi)$.

Proof. Let $X = B^{s}_{p_1,q,\gamma}(\Omega)$. Consider the operator A defined by the following equalities:

$$D(A) = B^{2m,s}_{p_1,q}\left(\Omega; B_j u = 0\right), \qquad A(x)u = \sum_{|\alpha|\le 2m} b_\alpha(x) c_\alpha(y) D^\alpha u(y). \tag{5.3}$$

The problem (5.1), (5.2) can be rewritten in the form of (1.1), where $u(x) = u(x,\cdot)$, $f(x) = f(x,\cdot)$ are functions with values in $X = B^{s}_{p_1,q,\gamma}(\Omega)$. It is easy to see that $\hat{A}(\xi)$ and $D^\beta \hat{A}(\xi)$ are operators in X defined by

$$\begin{gathered} D(\hat{A}) = D(D^\beta \hat{A}) = B^{2m,s}_{p_1,q}(\Omega; B_j u = 0), \\ \hat{A}(\xi)u = \sum_{|\alpha|\le 2m} \hat{b}_\alpha(\xi) c_\alpha(y) D^\alpha u(y), \quad |\beta| \le n, \\ D^\beta_\xi \hat{A}(\xi)u = \sum_{|\alpha|\le 2m} D^\beta_\xi \hat{b}_\alpha(\xi) c_\alpha(y) D^\alpha u(y). \end{gathered} \tag{5.4}$$

In view of conditions (1)–(5) and by virtue of [17, 19] the operators $\hat{A}(\xi) + \lambda$ and $D^\beta \hat{A}(\xi) + \lambda$ for sufficiently large $\lambda > 0$ are uniformly positive in X. Moreover, following problems for $f \in X$ and for arg $\lambda \in S_{\varphi_0}$, $|\lambda| \to \infty$:

$$\begin{gathered} \lambda u(y) + \sum_{|\alpha|\le 2m} \hat{b}_\alpha(\xi) c_\alpha(y) D^\alpha u(y) = f(y), \\ B_j u = \sum_{|\beta|\le m_j} b_{j\beta}(y) D^\beta u(y) = 0, \quad j = 1,2,\ldots,m, \end{gathered} \tag{5.5}$$

$$\begin{gathered} \lambda u(y) + \sum_{\alpha\le 2m} D^\beta \hat{b}_\alpha(\xi) c_\alpha(y) D^\alpha u(y) = f(y), \\ B_j u = \sum_{|\beta|\le m_j} b_{j\beta}(y) D^\beta u(y) = 0, \quad j = 1,2,\ldots,m, \end{gathered} \tag{5.6}$$

has unique solutions belong to $B^{2m,s}_{p_1,q,\gamma}(\Omega)$ and the coercive estimates hold

$$\|u\|_{B^{2m,s}_{p_1,q,\gamma}(\Omega)} \le C\big\|(\hat{A}(\xi) + \lambda)u\big\|_X, \qquad \|u\|_{B^{2m,s}_{p_1,q,\gamma}(\Omega)} \le C\big\|(D^\beta \hat{A}(\xi) + \lambda)u\big\|_X \tag{5.7}$$

for solutions of problems (5.5) and (5.6) respectively. Then by (5.4) in view of (5) condition and by virtue of embedding theorems [7] (§ 18.4) and [21] we obtain

$$\left\|(\hat{A}(\xi)+\lambda)u\right\|_X \leq C\|u\|_{B^{2m,s}_{p_1,q,\gamma}(\Omega)} \leq C\left\|(\hat{A}(\xi)+\lambda)u\right\|_X,$$

$$\left\|(D^\beta\hat{A}(\xi)+\lambda)u\right\|_X \leq C\|u\|_{B^{2m,s}_{p_1,q,\gamma}(\Omega)} \leq C\left\|(D^\beta\hat{A}(\xi)+\lambda)u\right\|_X. \tag{5.8}$$

Morever by using condition (5), for $u \in B^{2m,s}_{p_1,q,\gamma}(\Omega)$ we have

$$|\xi|^k\left\|(D^\beta\hat{A}(\xi)+\lambda)u\right\|_X \leq C\left\|(\hat{A}(\xi)+\lambda)u\right\|_X,$$

i.e., all conditions of Theorem 2.1 hold and we obtain the assertion.

6. The system of infinite many integro-differential equations. Consider the following infinity system of convolution equation:

$$\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u_m + \sum_{j=1}^{\infty} d_j * u_j(x) + \lambda u = f_m(x), \qquad x \in R^n, \quad m = 1,2,\ldots,\infty. \tag{6.1}$$

***Condition* 6.1.** There are positive constants C_1 and C_2 so that for $\left\{d_j(x)\right\}_1^\infty \in l_r$ for all $x \in R^n$ and some $x_0 \in R^n$,

$$C_1\left|d_j(x_0)\right| \leq |d_j(x)| \leq C_2\left|d_j(x_0)\right|.$$

Let

$$D(x) = \{d_m(x)\}, \quad d_m > 0, \quad u = \{u_m\}, \qquad D * u = \{d_m * u_m\}, \quad m = 1,2,\ldots,\infty,$$

$$l_r(D) = \left\{u\colon u \in l_r, \|u\|_{l_r(D)} = \|D * u\|_{l_r} = \left(\sum_{m=1}^{\infty}|d_m(x_0) * u_m|^r\right)^{\frac{1}{r}} < \infty\right\}, \quad 1 < r < \infty.$$

Let Q be a differential operator in $X = B^s_{p,q,\gamma}\left(R^n;l_r\right)$ generated by problem (6.1) and $B = L\left(B^s_{p,q,\gamma}(R^n;l_r)\right)$. Here $\gamma(x) = |x|^\alpha$, $-1 < \alpha < p-1$.

Theorem 6.1. *Suppose the first part of Conditions* 2.1 *and* 6.1 *hold*, $\hat{a}_\alpha, \hat{d}_m \in C^{(n)}(R^n)$ *and there are positive constants* C_1 *and* C_2 *so that*

$$|\xi|^k\left|D^\beta\hat{a}_\alpha(\xi)\right| \leq C_1, \qquad |\xi|^k\left|D^\beta\hat{d}_m(\xi)\right| \leq C_2$$

for all $k \leq |\beta| \leq n+1$ *and* $\xi \in R^n\backslash\{0\}$.

Then:

(a) *for all* $f(x) = \{f_m(x)\}_1^\infty \in B^s_{p,q,\gamma}\left(R^n;l_r(D)\right)$ *for* $\lambda \in S_\varphi$, $\varphi \in [0,\pi)$, *problem* (6.1) *has a unique solution* $u = \{u_m(x)\}_1^\infty$ *that belongs to space* $B^{l,s}_{p,q,\gamma}\left(R^n;l_r(D),l_r\right)$ *and the coercive uniform estimate*

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\|a_\alpha * D^\alpha u\|_X + \|D * u\|_X + |\lambda|\|u\|_X \leq C\|f\|_X \tag{6.2}$$

holds for the solution of the problem (6.1);

(b) *for* $\lambda \in S_\varphi$ *there exists a resolvent* $(Q+\lambda)^{-1}$ *of operator* Q *and*

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \left\| a_\alpha * \left[D^\alpha (Q+\lambda)^{-1}\right]\right\|_B +$$

$$+\left\|D*(Q+\lambda)^{-1}\right\|_B + \left\|\lambda (Q+\lambda)^{-1}\right\|_B \leq C. \tag{6.3}$$

Proof. Really, let $E = l_r$, A be infinite matrices, such that

$$A = \left[d_m(x)\delta_{jm}\right], \quad m,j = 1,2,\ldots,\infty.$$

Then

$$\hat{A}(\xi) = \left[\hat{d}_m(\xi)\delta_{jm}\right], \qquad D^\beta \hat{A}(\xi) = \left[D^\beta \hat{d}_m(\xi)\delta_{jm}\right], \quad m,j = 1,2,\ldots,\infty.$$

It is clear to see that $\hat{A}$ and $D^\beta \hat{A}(\xi)$ are uniformly positive in l_r. Therefore, by virtue of Theorem 2.1 and Result 2.1 we obtain that the problem (6.1) for all $f \in X$ and $\lambda \in S_\varphi$ has a unique solution $u \in B^{l,s}_{p,q,\gamma}(R^n; l_r(D), l_r)$ and estimates (6.2), (6.3) are satisfied.

***Remark* 6.1.** There are a lot of positive operators in concrete Banach spaces. Therefore, putting concrete Banach spaces instead of E and concrete positive differential, pseudodifferential operators, or finite, infinite matrices, etc. instead of operator A on (1.1) or (3.1) we can obtain the maximal regularity properties of different class of convolution equations and Cauchy problems for parabolic CDOEs or system of equations by virtue of Theorems 2.1 and 3.1.

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