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COMMON FIXED POINT THEOREMS FOR HYBRID GENERALIZED (F,φ)-CONTRACTIONS UNDER COMMON LIMIT RANGE PROPERTY WITH APPLICATIONS

СПІЛЬНІ ТЕОРЕМИ ПРО НЕРУХОМУ ТОЧКУ ДЛЯ ГІБРИДНИХ УЗАГАЛЬНЕНИХ (F,φ)-СТИСКАНЬ З ВЛАСТИВІСТЮ СПІЛЬНОГО ГРАНИЧНОГО ДІАПАЗОНУ З ЗАСТОСУВАННЯМИ

We consider a relatively new hybrid generalized F-contraction involving a pair of mappings and use this contraction to prove a common fixed-point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying generalized (F,φ)-contraction condition with the common limit range property in complete metric spaces. A similar result involving a hybrid pair of mappings satisfying the rational-type Hardy – Rogers (F,φ)-contractive condition is also proved. We generalize and improve several results available from the existing literature. As applications of our results, we prove two theorems for the existence of solutions of certain system of functional equations encountered in dynamic programming and the Volterra integral inclusion. Moreover, we provide an illustrative example.

Розглянуто відносно нове узагальнене гібридне F-стискання, що включає пару відображень. Це стисканя застосовано при доведенні спільної теормеми про нерухому точку для випадково співпадаючих ідемпотентних матриць, що задовольняють узагальнену умову (F,φ)-стискання при влативості спільного граничного діапазону в повних метричних просторах. Також доведено подібний результат для гібридних пар відображень, що задовольняють умову Гарді – Роджерса про (F,φ)-стискання раціонального типу. Узагальнено та покращено деякі відомі літературні результати. Як застосування наших результатів, доведено дві теореми про існування розв'язків деякої системи функціональних рівнянь, що зустрічаються в динамічному програмуванні, та інтегрального включення Вольтерра. Крім того, наведено ілюстративний приклад.

1. Introduction and preliminaries. Let (X,d) be a metric space. Then, following the Nadler [28], we adopt the following notations:

$CL(X)=\{A\colon A$ is a nonempty closed subset of $X\}$.

$CB(X)=\{A\colon A$ is a nonempty closed and bounded subset of $X\}$.

For nonempty closed and bounded subsets A,B of X and $x\in X$,

$$d(x,A)=\inf\{d(x,a)\colon a\in A\}$$

and

$$\mathcal{H}(A,B)=\max\big\{\sup\big\{d(a,B)\colon a\in A\big\},\sup\big\{d(b,A)\colon b\in B\big\}\big\}.$$

Recall that $CB(X)$ is a metric space with the metric $\mathcal{H}$ which is known as the Hausdorff – Pompeiu metric on $CB(X)$.

In 1969, Nadler [28] proved that every multivalued contraction mapping defined on a complete metric space has a fixed point. In proving this result, Nadler used the idea of Hausdorff metric to establish the multivalued version of Banach Contraction Principle which runs as follows:

Theorem 1. *Let (X,d) be a complete metric space and $\mathcal{T}$ a mapping from X into $CB(X)$ such that for all $x,y\in X$,*

$$\mathcal{H}(\mathcal{T}x,\mathcal{T}y)\le\lambda d(x,y),$$

where $\lambda\in[0,1)$. Then $\mathcal{T}$ has a fixed point, i.e., there exists a point $x\in X$ such that $x\in\mathcal{T}x$.

Hybrid fixed point theory involving pairs of single-valued and multivalued mappings is a relatively new development in nonlinear analysis (see e.g., [11, 12, 15, 24, 29, 45] and references therein). The much discussed concepts of commutativity and weak commutativity were extended to hybrid pair of mappings on metric spaces by Kaneko [20, 21]. In 1989, Singh et al. [40] extended the notion of compatible mappings and obtained some coincidence and common fixed point theorems for nonlinear hybrid contractions. It was observed that under compatibility the fixed point results usually require continuity of one of the underlying mappings. Afterwards, Pathak [30] generalized the concept of compatibility by defining weak compatibility for hybrid pairs of mappings (including single valued case as well) and utilized the same to prove common fixed point theorems. Naturally, compatible mappings are weakly compatible but not conversely.

In 2002, Aamri and El-Moutawakil [1] introduced the property (E.A.) for single-valued mappings. Later, Kamran [19] extended the notion of (E.A.) property to hybrid pairs of mappings. In 2011, Sintunavarat and Kumam [44] introduced the notion of common limit range (CLR) property for single-valued mappings and showed its superiority over the property (E.A.). Motivated by this fact, Imdad et al. [14] established common limit range property for a hybrid pair of mappings and proved some fixed point results in symmetric (semimetric) spaces. For more details on hybrid contraction conditions, one can consult [2, 7, 10, 13, 16, 18, 22, 29, 34, 35, 41 – 43].

The following definitions and results are standard in the theory of hybrid pair of mappings.

Definition 1. *Let $f: X \to X$ and $T: X \to CB(X)$ be a single-valued and multivalued mapping respectively. Then*:

A point $x \in X$ is a fixed point of f (resp. T) if $x = fx$ (resp. $x \in Tx$). The set of all fixed points of f (resp. T) is denoted by $F(f)$ (resp. $F(T)$).

A point $x \in X$ is a coincidence point of f and T if $fx \in Tx$. The set of all coincidence points of f and T is denoted by $\mathcal{C}(f,T)$.

A point $x \in X$ is a common fixed point of f and T if $x = fx \in Tx$. The set of all common fixed points of f and T is denoted by $F(f,T)$.

T is a closed multivalued mapping if the graph of T, i.e., $G(T) = \{(x,y) : x \in X, y \in Tx\}$ is a closed subset of $X \times X$.

We also recall the following terminology often used in the considerations of a hybrid pairs of mappings.

Definition 2. *Let (X,d) be a metric space with $f: X \to X$ and $T: X \to CB(X)$. Then a hybrid pair of mappings (f,T) is said to be*:

commuting on X [20] *if $fTx \subseteq Tfx$ $\forall x \in X$*;

weakly commuting on X [21] *if $\mathcal{H}(fTx, Tfx) \le d(fx, Tx)$ $\forall x \in X$*;

compatible [40] *if $fTx \in CB(X)$ $\forall x \in X$ and $\lim_{n\to\infty} \mathcal{H}(Tfx_n, fTx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that*

$$\lim_{n\to\infty} Tx_n \to A \in CB(X) \quad \text{and} \quad \lim_{n\to\infty} fx_n \to t \in A;$$

noncompatible [22] *if exists at least one sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} Tx_n \to A \in CB(X) \quad \text{and} \quad \lim_{n\to\infty} fx_n \to t \in A \quad \text{but} \quad \lim_{n\to\infty} \mathcal{H}(Tfx_n, fTx_n)$$

is either non-zero or nonexistent;

weakly compatible [17] *if $Tfx = fTx$ for each $x \in \mathcal{C}(f,T)$*;

coincidentally idempotent [13] *if for every* $v \in \mathcal{C}(f,T),\ ffv = fv$, *i.e.,* f *is idempotent at the coincidence points of* f *and* T;

occasionally coincidentally idempotent [36] *if* $ffv = fv$ *for some* $v \in \mathcal{C}(f,T)$;

enjoy the property (E.A.) [19] *if exists a sequence* $\{x_n\}$ *in* X *such that*

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n,$$

for some $t \in X$ *and* $A \in CB(X)$;

enjoy common limit range property with respect to the mapping f (*in short* CLR_f *property*) [14] *if exists a sequence* $\{x_n\}$ *in* X *such that*

$$\lim_{n\to\infty} fx_n = fu \in A = \lim_{n\to\infty} Tx_n,$$

for some $u \in X$ *and* $A \in CB(X)$.

The following example demonstrates the interplay of the occasionally coincidentally idempotent property with other notions described in the preceding definition.

***Example* 1** ([18], Example 1)**.** Let $X = \{1,2,3\}$ (with the standard metric),

$$f\colon \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \text{and} \quad T\colon \begin{pmatrix} 1 & 2 & 3 \\ \{1\} & \{1,3\} & \{1,3\} \end{pmatrix}.$$

Then, it is straight forward to observe the following:

$\mathcal{C}(f,T) = \{1,2\}$ and $F(f,T) = \{1\}$,

(f,T) is not commuting and not weakly commuting,

(f,T) is not compatible,

(f,T) is not weakly compatible,

(f,T) is not coincidentally idempotent since $ff2 = f3 = 2 \neq 3 = f2$,

(f,T) is occasionally coincidentally idempotent since $ff1 = 1 = f1$,

Obviously, in this case (f,T) is also noncompatible, but simple modifications of this example can show that the occasionally coincidentally idempotent property is independent of this notion, too.

The following example (taken from [18]) demonstrates the relationship between the property (E.A.) and common limit range property.

***Example* 2** ([18], Example 2 and 3)**.** Let $X = [0,2]$ be a metric space equipped with the usual metric $d(x,y) = |x-y|$. Define $f, g\colon X \to X$ and $T\colon X \to CB(X)$ as follows:

$$fx = \begin{cases} 2-x, & \text{if } 0 \le x < 1, \\ \dfrac{9}{5}, & \text{if } 1 \le x \le 2, \end{cases} \quad gx = \begin{cases} 2-x, & \text{if } 0 \le x \le 1, \\ \dfrac{9}{5}, & \text{if } 1 < x \le 2, \end{cases} \quad Tx = \begin{cases} \left[\dfrac{1}{2}, \dfrac{3}{2}\right], & \text{if } 0 \le x \le 1, \\ \left[\dfrac{1}{4}, \dfrac{1}{2}\right], & \text{if } 1 < x \le 2. \end{cases}$$

One can verify that the pair (f,T) enjoys the property (E.A.), but not the CLR_f property. On the other hand, the pair (g,T) satisfies the CLR_g property.

***Remark* 1.** If a pair (f,T) satisfies the property (E.A) along with the closedness of $f(X)$, then the pair also satisfies the CLR_f property.

Throughout this paper, we denote by $\mathbb{R}$ the set of all real numbers, by $\mathbb{R}^+$ the set of all positive real numbers and by $\mathbb{N}$ the set of all positive integers. In what follows, $\mathcal{F}$ denote the family of all functions $F\colon \mathbb{R}^+ \to \mathbb{R}$ that satisfy the following conditions:

(F_1) F is continuous and strictly increasing;

(F_2) for each sequence $\{\beta_n\}$ of positive numbers, $\lim_{n\to\infty}\beta_n = 0 \iff \lim_{n\to\infty} F(\beta_n) = -\infty$;

(F_3) there exists $k \in (0,1)$ such that $\lim_{\beta\to 0^+}\beta^k F(\beta) = 0$.

Some examples of functions $F \in \mathcal{F}$ are $F(t) = \ln t$, $F(t) = t + \ln t$, $F(t) = -1/\sqrt{t}$, $F(t) = \ln(t^2 + t)$, see [47].

Definition 3 [47]. *Let (X,d) be a metric space. A self-mapping T on X is called an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(d(Tx,Ty)) \leq F(d(x,y)), \tag{1}$$

for all $x,y \in X$ with $d(Tx,Ty) > 0$.

Example **3** [47]. Let $F\colon \mathbb{R}^+ \to \mathbb{R}$ be a mapping given by $F(x) = \ln x$. It is clear that F satisfies (F_1) – (F_3) for any $k \in (0,1)$. Under this setting, (1) reduces to

$$d(Tx,Ty) \leq e^{-\tau} d(x,y) \quad \text{for all} \quad x,y \in X, \quad Tx \neq Ty.$$

Notice that for $x,y \in X$ such that $Tx = Ty$, the previous inequality also holds and hence T is a contraction.

In what follows, for a metric space (X,d) and a multivalued mapping $T\colon X \to CL(X)$, we denote

$$M(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}\left[d(x,Ty) + d(y,Tx)\right]\right\}.$$

Definition 4 [39]. *Let (X,d) be a metric space. A multivalued mapping $T\colon X \to CL(X)$ is called an F-contraction if there exist $F \in \mathcal{F}$ and $\tau \in \mathbb{R}^+$ such that for all $x,y \in X$ with $y \in Tx$, exists $z \in Ty$,*

$$\tau + F(d(y,z)) \leq F\left(M(x,y)\right), \quad \text{whenever} \quad d(y,z) > 0. \tag{2}$$

Example **4** [39]. Let $F\colon \mathbb{R}^+ \to \mathbb{R}$ be mapping given by $F(x) = \ln x$. Then for each multivalued mapping $T\colon X \to CL(X)$ satisfying (2), we have

$$d(y,z) \leq e^{-\tau} M(x,y) \quad \text{for all} \quad x,y \in X, \quad z \in Ty, \quad y \neq z.$$

It is clear that for $z,y \in X$ such that $y = z$ the previous inequality also holds.

Some fixed point results for single-valued (resp. multivalued) F-contractions were obtained in [3, 23, 47] (resp. [39]).

Our aim in this paper is to prove a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying generalized (F,φ)-contraction condition under CLR property in complete metric spaces. A similar result for a variant of rational type Hardy – Rogers generalized (F,φ)-contractive condition is also derived. Here, it can be pointed out that Sgroi and Vetro [39] introduced and studied such conditions for multivalued mappings while the similar conditions were earlier introduced an studied by Wardowski [47] for single-valued mappings. Our results generalize and improve several known results of the existing literature. Finally, we utilize our results to prove the existence of solutions of certain system of functional equations arising in dynamic programming, as well as Volterra integral inclusion besides providing an illustrative example.

2. The Main Results. This section is divided into two parts. In the first subsection, we prove a common fixed point theorem for a hybrid pair of occasionally coincidentally idempotent mappings satisfying a generalized (F,φ)-contractions condition via CLR property in complete metric spaces, while in the second one we obtain results for hybrid pairs which satisfy a rational Hardy – Rogers type (F,φ)-contractive condition.

Definition 5. *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. Then hybrid pair (f,T) is said to be a generalized (F,φ)-contraction, if there exist an increasing, upper semicontinuous mapping from the right-hand side*

$$\Phi=\big\{\varphi\colon[0,\infty)\to[0,\infty)\mid \limsup_{s\to t^+}\varphi(s)<\varphi(t),\ \ \varphi(t)<t\forall\ t>0\big\},$$

$F\in\mathcal{F}$ and $\tau\in\mathbb{R}^+$ such that

$$\tau+F(\mathcal{H}^p(Tx,Ty))\le$$
$$\le F\left(\varphi\left(\max\left\{\begin{array}{c} d^p(fx,Tx),d^p(fy,Ty),d^p(fy,fx),\dfrac{1}{2}\left[d^p(fx,Ty)+d^p(fy,Tx)\right],\\ \dfrac{d^p(fx,Tx)d^p(fy,Ty)}{1+d^p(fy,fx)},\dfrac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(fy,fx)},\dfrac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(Tx,Ty)}\end{array}\right\}\right)\right) \tag{3}$$

for all $x,y\in X$, $p\ge1$ with $\mathcal{H}(Tx,Ty)>0$.

Definition 6. *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. Then hybrid pair (f,T) is said to be a rational Hardy–Rogers type (F,φ)-contraction, if there exist an increasing, upper semicontinuous mapping from the right-hand side*

$$\Phi=\{\varphi\colon[0,\infty)\to[0,\infty)\mid \limsup_{s\to t^+}\varphi(s)<\varphi(t),\ \varphi(t)<t,\forall\quad t>0\},$$

$F\in\mathcal{F}$ and $\tau\in\mathbb{R}^+$ such that

$$\tau+F(\mathcal{H}^p(Tx,Ty))\le$$
$$\le F\left(\varphi\left(\begin{array}{c}\alpha d^p(fx,fy)+\dfrac{\beta\left[1+d^p(fx,Tx)\right]d^p(fy,Ty)}{1+d^p(fx,fy)}+\gamma\left[d^p(fx,Tx)+d^p(fy,Ty)\right]+\\ +\delta\left[d^p(fx,Ty)+d^p(fy,Tx)\right]\end{array}\right)\right) \tag{4}$$

for all $x,y\in X$ with $Tx\ne Ty$, where $p\ge1,\ \alpha,\beta,\gamma,\delta\ge0,\ \alpha+\beta+2\gamma+2\delta\le1$.

Now we propose our first main result as follows:

Theorem 2. *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. If the hybrid pair (f,T) satisfies generalized (F,φ)-contraction condition* (3), *and also enjoys the CLR_f property, then the mappings f and T have a coincidence point.*

Moreover, if the hybrid pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

Proof. Since the pair (f,T) enjoys the CLR_f property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}fx_n=fu\in A=\lim_{n\to\infty}Tx_n,$$

for some $u\in X$ and $A\in CB(X)$. We assert that $fu\in Tu$. If not, then using condition (3), we have

$$\tau+F\left(\mathcal{H}^p(Tx_n,Tu)\right)\le$$
$$\le F\left(\varphi\left(\max\left\{\begin{array}{c} d^p(fx_n,Tx_n),d^p(fu,Tu),d^p(fu,fx_n),\dfrac{1}{2}\left[d^p(fx_n,Tu)+d^p(fu,Tx_n)\right],\\ \dfrac{d^p(fx_n,Tx_n)d^p(fu,Tu)}{1+d^p(fu,fx_n)},\dfrac{d^p(fx_n,Tu)d^p(fu,Tx_n)}{1+d^p(fu,fx_n)},\dfrac{d^p(fx_n,Tu)d^p(fu,Tx_n)}{1+d^p(Tx_n,Tu)}\end{array}\right\}\right)\right).$$

Passing to the limit as $n \to \infty$, we get

$$\tau + F(\mathcal{H}^p(A,Tu)) \le$$

$$\le F\left(\varphi\left(\max\left\{\begin{matrix} d^p(fu,A), d^p(fu,Tu), 0, \dfrac{1}{2}\left[d^p(fu,Tu)+d^p(fu,A)\right], \\ \dfrac{d^p(fu,A)d^p(fu,Tu)}{1+d^p(fu,fu)}, \dfrac{d^p(fu,Tu)d^p(fu,A)}{1+d^p(fu,fu)}, \dfrac{d^p(fu,Tu)d^p(fu,A)}{1+d^p(A,Tu)} \end{matrix}\right\}\right)\right)$$

$$= F\left(\varphi\left(\max\left\{\begin{matrix} d^p(fu,A), d^p(fu,Tu), 0, \dfrac{1}{2}\left[d^p(fu,Tu)+d^p(fu,A)\right], \\ d^p(fu,A)d^p(fu,Tu), \dfrac{d^p(fu,A)d^p(fu,Tu)}{1+d^p(A,Tu)} \end{matrix}\right\}\right)\right).$$

Using $fu \in A, \tau > 0$, (F_1) and property of Φ, we obtain

$$\mathcal{H}^p(A,Tu) \le \varphi\left(\max\left\{0, d^p(fu,Tu), 0, \frac{1}{2}[d^p(fu,Tu)+0], 0, 0\right\}\right) =$$

$$= \varphi\left(d^p(fu,Tu)\right) < d^p(fu,Tu).$$

Since $fu \in A$ the above inequality implies

$$d(fu,Tu) \le \mathcal{H}(A,Tu) < d(fu,Tu),$$

a contradiction. Hence $fu \in Tu$ which shows that the pair (f,T) has a coincidence point (i.e., $\mathcal{C}(f,T) \neq \varnothing$).

Now, assume that the hybrid pair (f,T) is occasionally coincidentally idempotent. Then for some $v \in \mathcal{C}(f,T)$, we have $ffv = fv \in Tv$. Our claim is that $Tv = Tfv$. If not, then using condition (3), we get

$$\tau + F(\mathcal{H}^p(Tfv,Tv)) \le$$

$$\le F\left(\varphi\left(\max\left\{\begin{matrix} d^p(ffv,Tfv), d^p(fv,Tv), d^p(fv,ffv), \dfrac{1}{2}\left[d^p(fv,Tfv)+d^p(ffv,Tv)\right], \\ \dfrac{d^p(ffv,Tfv)d^p(fv,Tv)}{1+d^p(fv,ffv)}, \dfrac{d^p(fv,Tfv)d^p(ffv,Tv)}{1+d^p(fv,ffv)}, \dfrac{d^p(fv,Tfv)d^p(ffv,Tv)}{1+d^p(Tfv,Tv)} \end{matrix}\right\}\right)\right)$$

$$= F\left(\varphi\left(\max\left\{\begin{matrix} d^p(fv,Tfv), d^p(fv,Tv), 0, \dfrac{1}{2}\left[d^p(fv,Tfv)+d^p(fv,Tv)\right], \\ d^p(fv,Tfv)d^p(fv,Tv), d^p(fv,Tfv)d^p(fv,Tv), \dfrac{d^p(fv,Tfv)d^p(fv,Tv)}{1+d^p(Tfv,Tv)} \end{matrix}\right\}\right)\right).$$

Since $fv \in Tv$, the above inequality implies

$$\tau + F(\mathcal{H}^p(Tfv,Tv)) \le F\left(\varphi\left(\max\left\{d^p(fv,Tfv), 0, 0, \frac{1}{2}d^p(fv,Tfv), 0, 0, 0\right\}\right)\right) =$$

$$= F\left(\varphi\left(d^p(Tfv,fv)\right)\right).$$

Using (F_1) and property of Φ, we obtain

$$d^p(Tfv,fv) < d^p(Tfv,fv),$$

which is a contradiction. Thus we have $fv = ffv \in Tv = Tfv$ which shows that fv is a common fixed point of the mappings f and T.

Theorem 2 is proved.

In view of Remark 1, we have the following natural result:

***Corollary* 1.** *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. If the hybrid pair (f,T) satisfies generalized (F,φ)-contraction condition* (3), *and enjoys the property* (E.A.) *along with the closedness of $f(X)$, then the mappings f and T have a coincidence point.*

Moreover, if the hybrid pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

Notice that, a noncompatible hybrid pair always satisfies the property (E.A.). Hence, we get the following corollary:

***Corollary* 2.** *Let f be a self mapping on a metric space (X,d), T a mapping from X into $CB(X)$ satisfying generalized (F,φ)-contraction condition* (3). *If the hybrid pair (f,T) is noncompatible and $f(X)$ a closed subset of X, then the mappings f and T have a coincidence point.*

Moreover, if the pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

If $F\colon\mathbb{R}^+\to\mathbb{R}$ is defoned by $F(t)=\ln t$ and denoting $e^{-\tau}=k$, then we have the following corollary:

***Corollary* 3.** *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. Assume that there exist $k\in(0,1)$, $\varphi\in\Phi$ such that*

$$\mathcal{H}^p(Tx,Ty)\leq k\varphi\left(\max\left\{\begin{array}{c} d^p(fx,Tx),d^p(fy,Ty),d^p(fy,fx),\dfrac{1}{2}\left[d^p(fx,Ty)+d^p(fy,Tx)\right],\\ \dfrac{d^p(fx,Tx)d^p(fy,Ty)}{1+d^p(fy,fx)},\dfrac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(fy,fx)},\dfrac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(Tx,Ty)}\end{array}\right\}\right)$$

for all $x,y\in X$ with $\mathcal{H}(Tx,Ty)>0,\ p\geq 1$, and the hybrid pair (f,T) enjoys the CLR_f. Then the mappings f and T have a coincidence point.

Moreover, if the hybrid pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

Since every members of $\mathcal{F}$ and Φ are increasing, we can deduce the following far more natural results from Theorem 2:

***Corollary* 4.** *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. Assume that there exist $F\in\mathcal{F}$, $\varphi\in\Phi$ and $\tau\in\mathbb{R}^+$ such that*

$$\tau+F(\mathcal{H}^p(Tx,Ty))\leq F\left(\varphi\left(\max\left\{d^p(fx,Tx),d^p(fy,Ty),d^p(fy,fx),\right.\right.\right.$$
$$\left.\left.\left.\frac{1}{2}\left[d^p(fx,Ty)+d^p(fy,Tx)\right]\right\}\right)\right)$$

for all $x,y\in X$ with $\mathcal{H}(Tx,Ty)>0,\ p\geq 1$, and the hybrid pair (f,T) enjoys the CLR_f. Then the mappings f and T have a coincidence point.

Moreover, if the hybrid pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

***Remark* 2.** Corollary 4 is an improved version of Theorem 11 due to Kadelburg et al. [18].

***Corollary* 5.** *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. Assume that there exist $F\in\mathcal{F}$, $\varphi\in\Phi$ and $\tau\in\mathbb{R}^+$ such that*

$$\tau+F(\mathcal{H}^p(Tx,Ty))\leq F\left(\varphi\left(\max\left\{\frac{d^p(fx,Tx)d^p(fy,Ty)}{1+d^p(fy,fx)},\frac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(fy,fx)},\right.\right.\right.$$
$$\left.\left.\left.\frac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(Tx,Ty)}\right\}\right)\right)$$

for all $x,y\in X$ with $\mathcal{H}(Tx,Ty)>0$, $p\geq1$, and the hybrid pair (f,T) enjoys the CLR_f. Then the mappings f and T have a coincidence point.

Moreover, if the hybrid pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

Now, we present our second main result as follows:

Theorem 3. *Let (X,d) be a metric space, $f\colon X\to X$ and $T\colon X\to CB(X)$. If the hybrid pair (f,T) satisfies a rational Hardy – Rogers (F,φ)-contraction condition* (4) *and also enjoys the CLR_f property, then the mappings f and T have a coincidence point.*

Moreover, if the hybrid pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

Proof. As the pair (f,T) shares the CLR_f property, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty}fx_n=fu\in A=\lim_{n\to\infty}Tx_n,$$

for some $u\in X$ and $A\in CB(X)$. We assert that $fu\in Tu$. If not, then using condition (4), we have

$$\tau+F(\mathcal{H}^p(Tx_n,Tu))\leq F\left(\varphi\left(\begin{array}{c}\alpha\,d^p(fx_n,fu)+\dfrac{\beta[1+d^p(fx_n,Tx_n)]d^p(fu,Tu)}{1+d^p(fx_n,fu)}+\\+\gamma[d^p(fx_n,Tx_n)+d^p(fu,Tu)]+\delta[d^p(fx_n,Tu)+d^p(fu,Tx_n)]\end{array}\right)\right).$$

Passing to the limit as $n\to\infty$ in the above inequality, we obtain

$$\tau+F(\mathcal{H}^p(A,Tu))\leq F(\varphi(\beta+\gamma+\delta)d^p(fu,Tu)).$$

Using $\tau>0$ and (F_1) and property of Φ, it follows that

$$d^p(fu,Tu)\leq d^p(A,Tu)<(\beta+\gamma+\delta)d^p(fu,Tu),$$

a contradiction, as $\beta+\gamma+\delta\leq1$. Hence, $fu\in Tu$ which shows that the hybrid pair (f,T) has a coincidence point (i.e., $\mathcal{C}(f,T)\neq\varnothing$).

Now, if the mappings f and T are occasionally coincidentally idempotent, then there exists $v\in\mathcal{C}(f,T)$ such that $ffv=fv\in Tv$. Our claim is that fu is the common fixed point of f and T. It is sufficient to show that $Tv=Tfv$. If not, then using condition (4), we get

$$\tau+F(\mathcal{H}^p(Tfv,Tv))\leq$$
$$\leq F\left(\varphi\left(\begin{array}{c}\alpha\,d^p(ffv,fv)+\dfrac{\beta[1+d^p(ffv,Tfv)]d^p(fv,Tv)}{1+d^p(ffv,fv)}+\\+\gamma[d^p(ffv,Tfv)+d^p(fv,Tv)]+\delta[d^p(ffv,Tv)+d^p(fv,Tfv)]\end{array}\right)\right)=$$

$$= F\left(\varphi\begin{pmatrix}\beta[1+d^p(fv,Tfv)]d^p(fv,Tv)+\gamma[d^p(fv,Tfv)+d^p(fv,Tv)]+\\ +\delta[d^p(fv,Tv)+d^p(fv,Tfv)]\end{pmatrix}\right).$$

Since $fv \in Tv$, the above inequality implies

$$\tau + F\left(d^p(Tfv,Tv)\right) \leq F\left(\varphi(\gamma+\delta)d^p(fv,Tfv)\right).$$

Using (F_1) and property of Φ, we can have

$$d^p(Tfv,fv) < (\gamma+\delta)d^p(fv,Tfv),$$

a contradiction, as $\gamma+\delta \leq 1$. Thus, $fv = ffv \in Tv = Tfv$ which shows that fv is a common fixed point of the mappings f and T.

Theorem 3 is proved.

If $F\colon \mathbb{R}^+ \to \mathbb{R}$ is given by $F(t) = \ln t$ and denoting $e^{-\tau} = k$, then we have the following corollary:

***Corollary* 6.** *Let (X,d) be a metric space, $f\colon X \to X$ and $T\colon X \to CB(X)$. Suppose that there exist $k \in (0,1)$, $\varphi \in \Phi$ such that*

$$\mathcal{H}^p(Tx,Ty) \leq k\varphi\begin{pmatrix}\alpha d^p(fx,fy)+\dfrac{\beta[1+d^p(fx,Tx)]d^p(fy,Ty)}{1+d^p(fx,fy)}+\gamma[d^p(fx,Tx)+d^p(fy,Ty)]+\\ +\delta[d^p(fx,Ty)+d^p(fy,Tx)]\end{pmatrix}$$

for all $x,y \in X$ with $Tx \neq Ty$, where $p \geq 1$, $\alpha,\beta,\gamma,\delta \geq 0$, $\alpha+\beta+2\gamma+2\delta \leq 1$, and the hybrid pair (f,T) enjoys the CLR_f. Then the mappings f and T have a coincidence point.

Moreover, if the pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

In view of Remark 1 and increasingness of the members of $\mathcal{F}$ and Φ, we have the following natural corollary:

***Corollary* 7.** *Let (X,d) be a metric space, $f\colon X \to X$ and $T\colon X \to CB(X)$. Suppose there exist $F \in \mathcal{F}$, $\varphi \in \Phi$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + F(\mathcal{H}^p(Tx,Ty)) \leq$$

$$\leq F\left(\varphi\left(\alpha d^p(fx,fy)+\beta\left[d^p(fx,Tx)+d^p(fy,Ty)\right]+\gamma\left[d^p(fx,Ty)+d^p(fy,Tx)\right]\right)\right)$$

for all $x,y \in X$ with $Tx \neq Ty$, where $p \geq 1$, $\alpha,\beta,\gamma \geq 0$, $\alpha+2\beta+2\gamma \leq 1$, and enjoys the property (E.A.) along with the closedness of $f(X)$. Then the mappings f and T have a coincidence point.

Moreover, if the hybrid pair (f,T) is occasionally coincidentally idempotent, then the pair (f,T) has a common fixed point.

3. Illustrative example. In this section, we provide an example to establish the genuineness of our extension.

***Example* 5.** Let $X = [0,3]$ be a metric space equipped with the metric $d(x,y) = |x-y|$. Define $f : X \to X$ and $T : X \to CB(X)$ as follows:

$$fx = \begin{cases} 3-x, & \text{if } x \in [0,2],\\ 3, & \text{if } x \in (2,3],\end{cases} \qquad Tx = \begin{cases} [1,2], & \text{if } x \in [0,2],\\ \left[0,\dfrac{1}{2}\right], & \text{if } x \in (2,3].\end{cases}$$

Let $F\colon \mathbb{R}^+ \to \mathbb{R}$ such that $F(t) = t + ln(t)$, $\varphi\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(t) = \dfrac{9}{10}t$ and $\tau = \dfrac{1}{5} > 0$ and $p \geq 1$, then the condition (3) takes the form

$$\mathcal{H}^p(Tx,Ty) \le \frac{9}{10}\Theta(x,y)e^{\frac{9}{10}\Theta(x,y)-\mathcal{H}^p(Tx,Ty)-\frac{1}{5}}, \tag{5}$$

where

$$\Theta(x,y)=\max\left\{\begin{array}{c} d^p(fx,Tx), d^p(fy,Ty), d^p(fy,fx), \dfrac{1}{2}\left[d^p(fx,Ty)+d^p(fy,Tx)\right], \\ \dfrac{d^p(fx,Tx)d^p(fy,Ty)}{1+d^p(fy,fx)}, \dfrac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(fy,fx)}, \dfrac{d^p(fx,Ty)d^p(fy,Tx)}{1+d^p(Tx,Ty)} \end{array}\right\}.$$

Then it is easy to verify that

$F \in \mathcal{F}$; $\varphi \in \Phi$; $f(X) = [1,3] \cup \{3\}$, a closed set in X; $\mathcal{C}(f,F) = [1,2]$;

the hybrid pair (f,T) satisfies CLR_f property, as for the sequence $\left\{1+\dfrac{1}{n}\right\}_{n\in\mathbb{N}}$,

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty}\left(2-\frac{1}{n}\right) = 2 = f1 \in [1,2] = \lim_{n\to\infty} T\left(1+\frac{1}{n}\right);$$

(f,T) is not coincidentally idempotent because $ff1 = f2 = 1 \ne 2 = f1$;

(f,T) is occasionally coincidentally idempotent, because $ff\dfrac{3}{2} = f\dfrac{3}{2} = \dfrac{3}{2}$.

Now, in order to verify condition (5), we distinguish two cases:

Case I. If $x \in [0,2]$ and $y \in (2,3]$, then

$$\mathcal{H}(Tx,Ty) = \mathcal{H}\left([1,2], \left[0,\frac{1}{2}\right]\right) =$$

$$= \max\left\{d\left([1,2], \left[0,\frac{1}{2}\right]\right),\ d\left(\left[0,\frac{1}{2}\right],[1,2]\right)\right\} = \max\left\{\frac{3}{2},1\right\} = \frac{3}{2},$$

and $d(fy,Ty) = d\left(3, \left[0,\dfrac{1}{2}\right]\right) = \dfrac{5}{2}$. Therefore, (5) reduces to

$$\left(\frac{3}{2}\right)^p < \frac{9}{10}\left(\frac{5}{2}\right)^p e^{\frac{9}{10}\left(\frac{5}{2}\right)^p-\left(\frac{3}{2}\right)^p-\frac{1}{5}}$$

which is true for all $p \ge 1$.

Case II. If $x \in (2,3]$ and $y \in [1,2]$, then

$$\mathcal{H}(Tx,Ty) = \mathcal{H}\left(\left[0,\frac{1}{2}\right],[1,2]\right) = \frac{3}{2} \quad \text{and} \quad d(fx,Tx) = d\left(3,\left[0,\frac{1}{2}\right]\right) = \frac{5}{2}.$$

Therefore, (5) reduces to

$$\left(\frac{3}{2}\right)^p < \frac{9}{10}\left(\frac{5}{2}\right)^p e^{\frac{9}{10}\left(\frac{5}{2}\right)^p-\left(\frac{3}{2}\right)^p-\frac{1}{5}}$$

which is true for all $p \ge 1$.

Notice that for $x,y \in [1,2]$ (or $x,y \in (2,3]$) $\mathcal{H}(Tx,Ty) = 0$ and so (5) is true.

Thus, all the hypotheses of Theorem 2 are satisfied and the hybrid pair (f,T) has the common fixed point $\left(\text{namely } \dfrac{3}{2}\right)$.

With a view to establish genuineness of our extension, notice that for $x = 1$, $y = 3$, we have

$$\mathcal{H}(Tx,Ty)=\frac{3}{2};\quad d(fx,fy)=d(2,3)=1;$$

$$\frac{1}{2}\left[d(fx,Tx)+d(fy,Ty)\right]=\frac{1}{2}\left[d(2,[1,2])+d\left(3,\left[0,\frac{1}{2}\right]\right)\right]=\frac{1}{2}\left(0+\frac{5}{2}\right)=\frac{5}{4}$$

and

$$\frac{1}{2}\left[d(fx,Ty)+d(fy,Tx)\right]=\frac{1}{2}\left[d\left(2,\left[0,\frac{1}{2}\right]\right)+d(3,[1,2])\right]=\frac{1}{2}\left(\frac{3}{2}+1\right)=\frac{5}{4},$$

which shows that the contractive condition of Theorem 11 (due to Kadelburg et al. [18]) is not satisfied. Thus, in all our results (Corollary 4 as well as Theorem 2) are applicable to the present example while Theorem 11 of Kadelburg et al. [18] is not which substantiates the utility of Theorem 2.

4. Applications. As applications of our main results, we prove an existence theorem on bounded solutions of a system of functional equations. Also, an existence theorem on the solution of integral inclusion is proved.

4.1. Application to dynamic programming. In 1978, Bellman and Lee [5] first studied the existence of solutions for functional equations wherein authors notice that the basic form of functional equations in dynamic programming can be described as follows:

$$q(x)=\sup_{y\in D}\{G(x,y,q(\tau(x,y)))\},\quad x\in W,$$

where $\tau\colon W\times D\to W$, $G\colon W\times D\times\mathbb{R}\to\mathbb{R}$ are mappings, while $W\subseteq U$ is a state space, $D\subseteq V$ is a decision space, and U, V are Banach spaces.

In 1984, Bhakta and Mitra [6] obtained some existence theorems for the following functional equation which arises in multistage decision process related to dynamic programming:

$$q(x)=\sup_{y\in D}\big\{g(x,y)+G(x,y,q(\tau(x,y)))\big\},\quad x\in W,$$

where $\tau\colon W\times D\to W$, $g\colon W\times D\to\mathbb{R}$, $G\colon W\times D\times\mathbb{R}\to\mathbb{R}$ are mappings, while $W\subseteq U$ is a state space, $D\subseteq V$ is a decision space, and U, V are Banach spaces.

In recent years, a lot of work have been done in this direction wherein a multitude of existence and uniqueness results have been obtained for solutions and common solutions of some functional equations, including systems of functional equations in dynamic programming using suitable fixed point results. For more details one can consults [26, 27, 31–33, 37] and the references therein.

Consider now a multistage process, reduced to the system of functional equations

$$q_i(x)=\sup_{y\in D}\big\{g(x,y)+G_i(x,y,q_i(\tau(x,y)))\big\},\quad x\in W,\quad i\in\{1,2\},\tag{6}$$

where $\tau\colon W\times D\to W$, $g\colon W\times D\to\mathbb{R}$, $G_i\colon W\times D\times\mathbb{R}\to\mathbb{R}$ are given mappings, while $W\subseteq U$ is a state space, $D\subseteq V$ is a decision space, and U, V are Banach spaces. The purpose of this section is to prove the existence of solutions for a system of functional equations (6) using Theorem 2.

Let $B(W)$ be the set of all bounded real-valued functions on W. For an arbitrary $h\in B(W)$ define $||h||=\sup_{x\in W}|h(x)|$, with respective metric d. Also, $(B(W),||\cdot||)$ is a Banach space wherein convergence is uniform. Therefore, if we consider a Cauchy sequence $\{h_n\}$ in $B(W)$, then the sequence $\{h_n\}$ converges uniformly to a function, say h^*, so that $h^*\in B(W)$.

We consider the operators $T_i\colon B(W)\to B(W)$ given by

$$T_ih_i(x) = \sup_{y\in D}\Big\{g(x,y) + G_i(x,y,h_i(\tau(x,y)))\Big\}, \tag{7}$$

for $h_i \in B(W),\ x \in W$, for $i = 1,2$; these mappings are well-defined if the functions g and G_i are bounded. Also, denote

$$\Theta(h,k) = \max\left\{\begin{matrix} d(T_2h,T_2k), d(T_2h,T_1h), d(T_2k,T_1k), \dfrac{d(T_1h,T_2k)+d(T_1k,T_2h)}{2}, \\ \dfrac{d(T_1h,T_2h)d(T_1k,T_2k)}{1+d(T_2k,T_2h)}, \dfrac{d(T_1h,T_2k)d(T_1k,T_2h)}{1+d(T_2k,T_2h)}, \dfrac{d(T_1h,T_2k)d(T_1k,T_2h)}{1+d(T_1h,T_1k)} \end{matrix}\right\} \tag{8}$$

for $h,k \in B(W)$.

Theorem 4. *Let $T_i\colon B(W)\to B(W)$ be given by* (7)*, for $i=1,2$. Suppose that the following hypotheses hold*:

(i) *there exist $\tau\in\mathbb{R}^+$ and $\varphi\in\Phi$ such that*

$$|G_1(x,y,h(x)) - G_2(x,y,k(x))| \le \frac{\varphi(\Theta(h,k)(x))}{(1+\tau\sqrt{\sup_{x\in W}\varphi(\Theta(h,k)(x))})^2}$$

for all $x\in W,\ y\in D$;

(ii) *$g\colon W\times D\to\mathbb{R}$ and $G_i\colon W\times D\times\mathbb{R}\to\mathbb{R}$ are bounded functions, for $i=1,2$*;

(iii) *there exists a sequence $\{h_n\}$ in $B(W)$ and a function $h^*\in B(W)$ such that*

$$\lim_{n\to\infty} T_1h_n = \lim_{n\to\infty} T_2h_n = T_1h^*;$$

(iv) *$T_1T_1h = T_1h$, whenever $T_1h = T_2h$, for some $h\in B(W)$.*

Then the system of functional equations (6) *has a bounded solution.*

Proof. By hypothesis (iii), the pair (T_1,T_2) shares the common limit range property with respect to T_1. Now, let λ be an arbitrary positive number, $x\in W$ and $h_1,h_2\in B(W)$. Then there exist $y_1,y_2\in D$ such that

$$T_1h_1(x) < g(x,y_1) + G_1(x,y_1,h_1(\tau(x,y_1))) + \lambda, \tag{9}$$

$$T_2h_2(x) < g(x,y_2) + G_2(x,y_2,h_2(\tau(x,y_2))) + \lambda, \tag{10}$$

$$T_1h_1(x) \ge g(x,y_2) + G_1(x,y_2,h_1(\tau(x,y_2))), \tag{11}$$

$$T_2h_2(x) \ge g(x,y_1) + G_2(x,y_1,h_2(\tau(x,y_1))). \tag{12}$$

Next, by using (9) and (12), we obtain

$$\begin{aligned} T_1h_1(x) - T_2h_2(x) &< G_1(x,y_1,h_1(\tau(x,y_1))) - G_2(x,y_1,h_2(\tau(x,y_1))) + \lambda \le \\ &\le |G_1(x,y_1,h_1(\tau(x,y_1))) - G_2(x,y_1,h_2(\tau(x,y_1)))| + \lambda \le \\ &\le \frac{\varphi(\Theta(h,k)(x))}{(1+\tau\sqrt{\sup_{x\in W}\varphi(\Theta(h,k)(x))})^2} + \lambda \end{aligned}$$

and so we have

$$T_1h_1(x) - T_2h_2(x) < \frac{\varphi(\Theta(h,k)(x))}{(1+\tau\sqrt{\sup_{x\in W}\varphi(\Theta(h,k)(x))})^2} + \lambda. \tag{13}$$

Analogously, by using (10) and (11), we get

$$T_2h_2(x) - T_1h_1(x) < \frac{\varphi(\Theta(h,k)(x))}{(1+\tau\sqrt{\sup_{x\in W}\varphi(\Theta(h,k)(x))})^2} + \lambda. \tag{14}$$

Combining (13) and (14), we obtain

$$|T_1h_1(x) - T_2h_2(x)| < \frac{\varphi(\Theta(h,k)(x))}{(1+\tau\sqrt{\sup_{x\in W}\varphi(\Theta(h,k)(x))})^2} + \lambda,$$

implying thereby

$$d(T_1h_1, T_2h_2) \leq \frac{\varphi(\Theta(h,k))}{(1+\tau\sqrt{\varphi(\Theta(h,k))})^2} + \lambda.$$

Notice that, the last inequality does not depend on $x \in W$ and $\lambda > 0$ is taken arbitrarily, therefore we have

$$d(T_1h_1, T_2h_2) \leq \frac{\varphi(\Theta(h,k))}{(1+\tau\sqrt{\varphi(\Theta(h,k))})^2}.$$

If we consider $F \in \mathcal{F}$ defined by $F(t) = \dfrac{-1}{\sqrt{t}}$, for each $t \in (0,+\infty)$, and put $f = T_1$, $T = T_2$, then we get condition

$$\tau + F(d(fh_1, Th_2)) \leq F(\varphi(\Theta(h,k)))$$

where $\Theta(h,k)$ is given in (8). Thus all the hypotheses of Theorem 2 are satisfied for the pair (f,T) and $p = 1$. Moreover, in view of the hypotheses (iv), the pair (T_1, T_2) is occasionally coincidentally idempotent, so by using Theorem 2, the mapping T_1 and T_2 have a common fixed point, that is, the system of functional equations (6) has a bounded solution.

4.2. Application to Volterra integral inclusions. Here, we present yet another application of Theorem 3. This application is essentially inspired by [46].

We establish new results on the existence of solutions of integral inclusion of the type

$$x(t) \in q(t) + \int_0^{\sigma(t)} k(t,s)F(s,x(s))\,ds \tag{15}$$

for $t \in J = [0,1] \subset \mathbb{R}$, where $\sigma\colon J \to J$, $q\colon J \to E$, $k\colon J \times J \to \mathbb{R}$ are continuous and $F\colon J \times E \to C(E)$, where E is a Banach space with norm $\|\cdot\|_E$ and $C(E)$ denotes the class of all nonempty closed subsets of E.

Let $C(J,E)$ be the space of all continuous E-valued functions on J. Define a norm $\|\cdot\|$ on $C(J,E)$ by

$$\|x\| = \sup_{t\in J}\|x(t)\|_E.$$

Definition 7. *A continuous function $a \in C(J,E)$ is called a lower solution of the integral inclusion* (15), *if it satisfies*

$$a(t) \leq q(t) + \int_0^{\sigma(t)} k(t,s)v_1(s)ds \text{ for all } v_1 \in B(J,E)$$

such that $v_1(t) \in F(t, a(t))$ almost everywhere $(a.e.)$ for $t \in J$, *where* $B(J,E)$ *is the space of all* E*-valued Bochner-integrable functions on* J. *Similarly, a continuous function* $b \in C(J,E)$ *is called an upper solution of the integral inclusion* (15), *if it satisfies*

$$b(t) \geq q(t) + \int\limits_0^{\sigma(t)} k(t,s)v_2(s)ds, \quad for\ all \quad v_2 \in B(J,E)$$

such that $v_2(t) \in F(t, b(t))$ *a.e. for* $t \in J$.

Notice that, all the solution lies between lower solution 'a' as well upper solution 'b'. We can denote the solution set as an interval $[a, b]$.

Definition 8. *A continuous function* $x\colon J \to E$ *is said to be a solution of the integral inclusion* (15), *if*

$$x(t) = q(t) + \int\limits_0^{\sigma(t)} k(t,s)v(s)\,ds$$

for some $v \in B(J,E)$ *satisfying* $v(t) \in F(t, x(t))$ *for all* $t \in J$.

In what follows, we also need the following definitions:

Definition 9. *A multivalued mapping* $F\colon J \to 2^E$ *is said to be measurable if for any* $y \in E$, *the function* $t \mapsto d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ *is measurable.*

Definition 10. *A multivalued mapping* $\beta\colon J \times E \to 2^E$ *is called Carathéodory if*

(i) $t \mapsto (t, x)$ *is measurable for each* $x \in E$, *and*

(ii) $x \mapsto (t, x)$ *is upper semicontinuous almost everywhere for* $t \in J$.

Denote

$$\|F(t,x)\| = \sup\{\|u\|_E : u \in F(t,x)\}.$$

Definition 11. *A Carathéodory multimapping* $F(t,x)$ *is called* L^1*-Carathéodory if for every real number* $r > 0$, *there exists a function* $h_r \in L^1(J, \mathbb{R})$ *such that*

$$\|F(t,x)\| \leq h_r(t) \quad \textit{for almost every} \quad t \in J$$

and for all $x \in E$ *with* $\|x\|_E \leq r$.

Denote

$$S_F^1(x) = \big\{v \in B(J,E) : v(t) \in F(t, x(t)) \ \ a.e. \ \ t \in J\big\}.$$

Lemma 1 [25]. *If* $\operatorname{diam}(E) < \infty$ *and* $F\colon J \times E \to 2^E$ *is* L^1*-Carathéodory, then* $S_F^1(x) \neq \varnothing$ *for each* $x \in C(J,E)$.

Lemma 2 [46]. *Let* E *be a Banach space,* F *a Carathéodory multimappping with* $S_F^1 \neq \varnothing$ *and* $\mathcal{L}\colon L^1(J,E) \to C(J,E)$ *a continuous linear mapping. Then the operator*

$$\mathcal{L} \circ S_F^1 \colon C(J,E) \to 2^{C(J,E)}$$

is a closed graph operator on $C(J,E) \times C(J,E)$.

Let us list the following set of conditions:

(H_0) the function $k(t,s)$ is continuous and nonnegative on $J \times J$ with

$$\sup_{t,s \in J} k(t,s) \leq 1;$$

(H_1) the multivalued mapping $F(t,x)$ is Carathéodory;

(H_2) the multivalued mapping $F(t,x)$ is increasing in x almost everywhere for $t \in J$;

(H_3) There exist $\tau \in \mathbb{R}^+$ and $\varphi \in \Phi$ such that

$$|F(s,x(s)) - F(s,y(s))| \leq \sqrt{[e^{-\tau}\left[(\varphi(\Delta(x,y)))^2 + \varphi(\Delta(x,y))] + \frac{1}{4}\right]} - \frac{1}{2}$$

for all $s \in J,\ x \in E$, where

$$\Delta(x,y) = \alpha|fx - fy| + \frac{\beta\,[1 + |fx - Tx|]\,|fy - Ty|}{1 + |fx - fy|} + \gamma\,[|fx - Tx| + |fy - Ty|] +$$
$$+\delta\,[|fx - Ty| + |fy - Tx|]$$

with $\alpha, \beta, \gamma, \delta \geq 0, \alpha + \beta + 2\gamma + 2\delta \leq 1$;

(H_4) $S^1_F(x) \neq \varnothing$ for each $x \in C(J,E)$.

Theorem 5. *Suppose that the conditions* (H_0) – (H_4) *hold. Then the integral inclusion* (15) *has a solution in* $[a,b]$ *defined on* J.

Proof. Let $X = C(J,E)$. Define a multivalued mapping $T\colon [a,b] \subset X \to 2^X$ given by

$$Tx = \left\{ u \in [a,b] : u(t) = q(t) + \int\limits_0^{\sigma(t)} k(t,s)v(s)\,ds; \quad v \in S^1_F(x), \quad \text{for every} \quad t \in [0,1] \right\}.$$

Observe that T is well-defined, as owing to (H_4), $S^1_F(x) \neq \varnothing$. To show that T satisfies all hypotheses of Theorem 3 defined on $[a,b]$.

For all $\vartheta, \mu \in 2^X$ on $t \in J$ and making use of (H_0) and (H_3), we have $\big($for $v_1, v_2 \in S^1_F(x)\big)$

$$\|\vartheta(t) - \mu(t)\|_E = \left\| \int\limits_0^{\sigma(t)} k(t,s)v_1(s)\,ds - \int\limits_0^{\sigma(t)} k(t,s)v_2(s)\,ds \right\|_E \leq$$
$$\leq \int\limits_0^{\sigma(t)} k(t,s)\,ds \|v_1(s) - v_2(s)\|_E \leq$$
$$\leq \sup_{t,s \in J} k(t,s) \sqrt{[e^{-\tau}\left[(\varphi(\Delta(v_1,v_2)))^2 + \varphi(\Delta(v_1,v_2))] + \frac{1}{4}\right]} - \frac{1}{2}.$$

This implies that

$$\|\vartheta(t) - \mu(t)\|_E \leq \sqrt{[e^{-\tau}\left[(\varphi(\Delta(v_1,v_2)))^2 + \varphi(\Delta(v_1,v_2))] + \frac{1}{4}\right]} - \frac{1}{2},$$

for each $t \in J$.

On considering $F \in \mathcal{F}$ defined by $F(t) = ln(t^2 + t)$, for each $t \in (0, +\infty)$, then we have condition

$$\tau + F(\|\vartheta(t) - \mu(t)\|_E) \leq F(\varphi(\Delta(v_1, v_2))).$$

Thus we deduce that the operator T satisfy condition (4) where f is an identity mapping and $p = 1$. Also T is a closed mapping, using Theorem 3, we conclude that the given integral inclusion has a solution in $[a, b]$.

Theorem 5 is proved.

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References

1. *Aamri M., El Moutawakil D.* Some new common fixed point theorems under strict contractive conditions // J. Math. Anal. and Appl. – 2002. – **270**, № 1. – P. 181 – 188.
2. *Ali J., Imdad M.* Common fixed points of nonlinear hybrid mappings under strict contractions in semi-metric spaces // Nonlinear Anal. Hybrid Syst. – 2010. – **4**, № 4. – P. 830 – 837.
3. *Alsulami H. H., Karapinar E., Piri H.* Fixed points of modified F-contractive mappings in complete metric-like spaces // J. Funct. Spaces. – 2014. – Article ID 270971.
4. *Aubin J. P., Cellina A.* Differential Inclusions. – Berlin: Springer, 1984.
5. *Bellman R., Lee E. S.* Functional equations in dynamic programming // Aequat. Math. – 1978. – **17**. – P. 1 – 18.
6. *Bhakta T. C., Mitra S.* Some existence theorems for functional equations arising in dynamic programming // J. Math. Anal. and Appl. – 1984. – **98**. – P. 348 – 362.
7. *Chauhan S., Khan M. A., Kadelburg Z., Imdad M.* Unified common fixed point theorem for a hybrid pair of mappings via an implicit relation involving altering distance function // Abstr. and Appl. Anal. – Vol. 2014. – Article ID 718040. – 8 p.
8. *Deimling K.* Multivalued differential equations. – Berlin: Walter de Gruyter, 1992.
9. *Dhage B. C.* A functional integral inclusion involving Carathéodories // Electron. J. Qual. Theory Different. Equat. – 2003. – **14**. – P. 1 – 18.
10. *Dhompongsa S., Yingtaweesittikul H.* Fixed points for multivalued mappings and the metric completeness // Fixed Point Theory and Appl. – 2009. – Article ID 972395.
11. *Fisher B.* Common fixed point theorems for mappings and set-valued mappings // Rostock. Math. Kolloq. – 1981. – **18**. – P. 69 – 77.
12. *Hu T.* Fixed point theorems for multivalued mappings // Can. Math. Bull. – 1980. – **23**. – P. 193 – 197.
13. *Imdad M., Ahmad A., Kumar S.* On nonlinear nonself hybrid contractions // Rad. Mat. – 2001. – **10**, № 2. – P. 233 – 244.
14. *Imdad M., Chauhan S., Soliman A. H., Ahmed M. A.* Hybrid fixed point theorems in symmetric spaces via common limit range property // Demonstr. Math. – 2014. – **47**, № 4. – P. 949 – 962.
15. *Imdad M., Khan M. S., Sessa S.* On some weak conditions of commutativity in common fixed point theorems // Int. J. Math. and Math. Sci. – 1988. – **11**, № 2. – P. 289 – 296.
16. *Imdad M., Soliman A. H.* Some common fixed point theorems for a pair of tangential mappings in symmetric spaces // Appl. Math. Lett. – 2010. – **23**, № 4. – P. 351 – 355.
17. *Jungck G., Rhoades B. E.* Fixed points for set valued functions without continuity // Indian J. Pure Appl. and Math. – 1998. – **29**, № 3. – P. 227 – 248.
18. *Kadelburg Z., Chauhan S., Imdad M.* A hybrid common fixed point theorem under certain recent properties // Sci. World J. – 2014. – Article ID 860436. – 6 p.
19. *Kamran T.* Coincidence and fixed points for hybrid strict contractions // J. Math. Anal. and Appl. – 2004. – **299**, № 1. – P. 235 – 241.
20. *Kaneko H.* Single-valued and multi-valued f-contractions // Boll. Unione Mat. Ital. – 1985. – **4**. – P. 29 – 33.
21. *Kaneko H.* A common fixed point of weakly commuting multi-valued mappings // Math. Jap. – 1988. – **33**, № 5. – P. 741 – 744.

22. *Kaneko H., Sessa S.* Fixed point theorems for compatible multi-valued and single-valued mappings // Int. J. Math. and Math. Sci. – 1989. – **12**, № 2. – P. 257 – 262.
23. *Karapinar E., Piri H., Al-Sulami H.* Fixed points of generalized F-Suzuki type contraction in complete b-metric spaces // Discrete Dyn. Nat. Sci. – 2015. – Article ID 969726. – 8 p.
24. *Kubiak T.* Fixed point theorems for contractive type multivalued mappings // Math. Jap. – 1985. – **30**, № 1. – P. 89 – 101.
25. *Lasota A., Opial Z.* An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations // Bull. acad. pol. sci. Ser. sci. math., astron. et phys. – 1965. – **13**. – P. 781 – 786.
26. *Liu Z., Ume J. S.* On properties of solutions for a class of functional equations arising in dynamic programming // J. Optim. Theory and Appl. – 2003. – **117**. – 533.
27. *Liu Z., Wong L., Kug Kim H., Kang S. M.* Common fixed point theorems for contactive mappings and their applications in dynamic programming // Bull. Korean Math. Soc. – 2008. – **45**, № 3. – P. 573 – 585.
28. *Nadler Jr. S. B.* Multivalued contraction mappings // Pacif. J. Math. – 1969. – **20**, № 2. – P. 457 – 488.
29. *Naimpally S. A., Singh S. L., Whitfield J. H. M.* Coincidence theorems for hybrid contractions // Math. Nachr. – 1986. – **127**. – P. 177 – 180.
30. *Pathak H. K.* Fixed point theorems for weak compatible multi-valued and single-valued mappings // Acta Math. Hung. – 1995. – **67**, № 1-2. – P. 69 – 78.
31. *Pathak H. K., Cho Y. J., Kang S. M., Lee B. S.* Fixed point theorems for compatible mappings of type (P) and applications to dynamic programming // Matematiche. – 1995. – **50**. – P. 15 – 33.
32. *Pathak H. K., Deepmala* Some existing theorems for solvability of certain functional equations arising in dynamic programming // Bull. Calcutta Math. Soc. – 2012. – **104**, № 3. – P. 237 – 244.
33. *Pathak H. K., Fisher B.* Common fixed point theorems with applications in dynamic programming // Glas. Mat. – 1996. – **31**. – P. 321 – 328.
34. *Pathak H. K., Kang S. M., Cho Y. J.* Coincidence and fixed point theorems for nonlinear hybrid generalized contractions // Math. J. – 1998. – **48(123)**. – P. 341 – 357.
35. *Pathak H. K., Khan M. S.* Fixed and coincidence points of hybrid mappings // Arch. Math. – 2002. – **38**. – P. 201 – 208.
36. *Pathak H. K., Rodríguez – López R.* Noncommutativity of mappings in hybrid fixed point results // Bound. Value Probl. – 2013. – **145**. – 2013. – 21 p.
37. *Pathak H. K., Tiwari R.* Common fixed points for weakly compatible mappings and applications in dynamic programming // Ital. J. Pure and Appl. Math. – 2013. – **30**. – P. 253 – 268.
38. *O'Regan D.* Integral inclusions of upper semi-continuous or lower semi-continuous type // Proc. Amer. Math. Soc. – 1996. – **124**. – P. 2391 – 2399.
39. *Sgroi M., Vetro C.* Multi-valued F-contractions and the solution of certain functional and integral equations // Filomat. – 2013. – **27**, № 7. – P. 1259 – 1268.
40. *Singh S. L., Ha K. S., Cho Y. J.* Coincidence and fixed points of nonlinear hybrid contractions // Int. J. Math. and Math. Sci. – 1989. – **12**, № 2. – P. 247 – 256.
41. *Singh S. L., Hashim A. M.* New coincidence and fixed point theorems for strictly contractive hybrid maps // Aust. J. Math. Anal. and Appl. – 2005. – **2**, № 1. – Article 12. – 7 p.
42. *Singh S. L., Mishra S. N.* Coincidence and fixed points of non-self hybrid contractions // J. Math. Anal. and Appl. – 2001. – **256**. – P. 486 – 497.
43. *Singh S. L., Mishra S. N.* Coincidence theorems for certain classes of hybrid contractions // Fixed Point Theory and Appl. – 2010. – Article ID 898109.
44. *Sintunavarat W., Kumam P.* Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces // J. Appl. Math. – 2011. – Article ID 637958. – 14 p.
45. *Sessa S., Khan M. S., Imdad M.* Common fixed point theorem with a weak commutativity condition // Glas. Mat. – 1986. – **21(41)**. – P. 225 – 235.
46. *Türkoğlu D., Altun I.* A fixed point theorem for multi-valued mappings and its applications to integral inclusions // Appl. Math. Lett. – 2007. – **20**. – P. 563 – 570.
47. *Wardowski D.* Fixed points of a new type of contractive mappings in complete metric spaces // Fixed Point Theory and Appl. – 2012. – **2012**, № 94.

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