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ON MATRIX OPERATORS ON THE SERIES SPACE $|\overline{N}_p^{\theta}|_k$

ПРО МАТРИЧНІ ОПЕРАТОРИ В ПРОСТОРАХ РЯДІВ $\left|\overline{N}_p^{\theta}\right|_k$

Recently, the space $\left|\overline{N}_p^{\theta}\right|_k$ has been generated from the set of k-absolutely convergent series ℓ_k as the set of series summable by the absolute weighted method. In the paper, we investigate some properties of this space, such as β-duality and the relationship with ℓ_k and then show that each element in the classes $\left(\left|\overline{N}_p\right|, \left|\overline{N}_q^{\theta}\right|_k\right)$ and $\left(\left|\overline{N}_p^{\theta}\right|_k, \left|\overline{N}_q\right|\right)$ of infinite matrices corresponds to a continuous linear operator and also characterizes these classes. Hence, in the special case, we deduce some well-known results of Sarıgöl, Bosanquet, Orhan, and Sunouchi.

Нещодавно простір $\left|\overline{N}_p^{\theta}\right|_k$ було згенеровано з множини ℓ_k всіх k-абсолютно збіжних рядів, як множину рядів сумовних за абсолютним ваговим методом. В роботі досліджено деякі властивості цього простору такі, як β-дуальність і зв'язок з ℓ_k і, крім того, доведено, що кожний елемент класів $\left(\left|\overline{N}_p\right|, \left|\overline{N}_q^{\theta}\right|_k\right)$ та $\left(\left|\overline{N}_p^{\theta}\right|_k, \left|\overline{N}_q\right|\right)$ нескінченних матриць відповідає неперервному лінійному оператору і також характеризує ці класи. Таким чином, у частинному випадку, нами виведено загальновідомі результати Саріголя, Босанке, Орхана та Сунучі.

1. Introduction. Let ω be the set of all complex sequences and ℓ_k be the set of k-absolutely convergent series. Let $A=(a_{nv})$ be an arbitrary infinite matrix of complex numbers and (θ_n) be a positive sequence. By $A(x)=(A_n(x))$, we denote the A-transform of the sequence $x=(x_v)$, i.e.,

$$A_n(x)=\sum_{v=0}^{\infty} a_{nv}x_v,$$

provided that the series are convergent for $v,n\geq 0$. For (U,V), we write the set of all infinite matrices A which map a sequence space U into a sequence V, and also the sets $U_A=\{x\in\omega\colon A(x)\in U\}$ and

$$U^{\beta}=\left\{\psi=(\psi_n)\colon \sum_{n=0}^{\infty}\psi_n x_n \ \text{ is convergent for all } \ x\in U\right\}$$

are said to be the domain of a matrix A in U and the β dual of U, respectively.

Now let Σa_v be a given infinite series with nth partial sum s_n. Then the series Σa_v is said to be summable $|A,\theta|_k,\ k\geq 1$, if [16]

$$\sum_{n=1}^{\infty}\theta_n^{k-1}\mid A_n(s)-A_{n-1}(s)\mid^k<\infty.$$

In the case $A=\left(\overline{N},p_n\right)$ and $\theta_n=P_n/p_n$, the summability $|A,\theta|_k$ is reduced to the summability methods $|\overline{N},p_n,\theta_n|_k$ and $|\overline{N},p_n|_k$, [3, 19], respectively. Also $|A,\theta|_k=|C,\alpha|_k$ for $A=(C,\alpha)$ and $\theta_n=n$, in Flett's notation [5]. By a weighted mean matrix $A=(a_{nv})$, we mean that

$$a_{nv}=\begin{cases} p_v/P_n, & 0\leq v\leq n,\\ 0, & v>n,\end{cases} \tag{1.1}$$

where (p_n) is a positive sequence with $P_n = p_0 + p_1 + \ldots + p_n \to \infty$ as $n \to \infty,\ P_{-1} = p_{-1} = 0$. Throughout the paper, (q_v) denotes a positive sequence with $Q_n = q_0+q_1+\ldots+q_n \to \infty$ as $n \to \infty$, and k^* also denotes the conjugate of $k > 1$, i.e., $1/k + 1/k^* = 1,\ 1/k^* = 0$ for $k = 1$. The series space $|\overline{N}_p^\theta|_k$ has been defined in [14] as the set of all series summable by the summability method $|\overline{N}, p_n, \theta_n|_k$. Say $\mu_k = |\overline{N}_p^\theta|_k$ and $\lambda_k = |\overline{N}_q^\theta|_k$, for brevity. Then the series $\Sigma\, a_v$ is $|\overline{N}, p_n, \theta_n|_k$ summable if and only if the sequence $a = (a_v) \in \mu_k$. On the other hand, if we take the matrix A as in (1.1), then, we can write

$$A_n(s) = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v = \frac{1}{P_n}\sum_{v=0}^{n}\left(P_n - P_{v-1}\right)a_v, \quad P_{-1} = 0,$$

which implies

$$A_0(s) = a_0,\ A_n(s) - A_{n-1}(s) = \frac{p_n}{P_n P_{n-1}}\sum_{v=0}^{n} P_{v-1}a_v \quad \text{for} \quad n \geq 1.$$

We define the sequence space μ_k by

$$\mu_k = \left\{ a = (a_n) \in \omega : \sum_{n=1}^{\infty} \theta_n^{k-1}\left|\frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n} P_{v-1}a_v\right|^k < \infty \right\}, \quad 1 \leq k < \infty.$$

One can restate the space μ_k as the domain of the matrix $T = (t_{nv})$ in the space ℓ_k of k-absolutely convergent series, i.e., $\mu_k = (l_k)_T$, where

$$t_{nv} = \begin{cases} 1 & n = 0, v = 0, \\ \dfrac{\theta_n^{1/k^*} p_n P_{v-1}}{P_n P_{n-1}}, & 1 \leq v \leq n, \\ 0, & v > n \quad \text{or} \quad n \geq 1, \quad v = 0, \end{cases} \tag{1.2}$$

for all $n, v \in \mathbb{N} = \{0, 1, 2, \ldots\}$. Besides, it is well known that l_k is the BK-space (i.e., Banach space with continuos coordinates) with respect to its natural norm $||x||_{l_k} = \left(\sum_{v=0}^{\infty}|x_v|^k\right)^{1/k}$ for $k \geq 1$. Hence, since the matrix T is triangle and $\mu_k = (l_k)_T$, it is immediate by Theorem 4.3.2 of Wilansky [22, p. 63] that μ_k is also BK-space with respect to the norm $||x||_{\mu_k} = ||T(x)||_{l_k},\ k \geq 1$. We refer the reader to [13] for the case $\theta_n = P_n/p_n$ and $p_n = 1$, and also to [1] for full knowledge on the normed sequence spaces and domain of triangle matrices in normed or paranormed sequence spaces and the matrix transformations and summability theory.

The problems of absolute summability factors and comparision of these methods goes to old rather and uptill now were widely examined by many authors, (see, for example, [3–6, 8–20]). Now, by different standpoint we note that most of these results correspond to the special matrices $I, W \in (\mu_1, \lambda_k)$ and (μ_k, λ_1), where I is an infinite identity matrix and the matrix $W = (w_{nv})$ defined by $w_{nv} = \varepsilon_n$ for $v = n$, zero otherwise. More recently, the above mentioned space μ_k has been derived by the matrix T from the space ℓ_k and triangle matrix operators defined on that space have been investigated in [14]. Note that although in the most cases the new space generated by the summability matrix from ℓ_k is the expansion or the contraction of the original space, it can be seen in some cases that these spaces overlap.

2. Main results. In this paper we compute β-dual of μ_k and exhibit some inclusion relations between the spaces μ_k and ℓ_k. We also show that each element in the classes (μ_1, λ_k) and (μ_k, λ_1) of infinite matrices corresponds to a continous linear operator and characterize these operators, which includes some known results of Sarıgöl [14], Bosanquet [4], Orhan and Sarıgöl [12] and Sunouchi [20] as a special case. More precisely, we prove the following theorems.

Theorem 2.1. *Let* $1 < k < \infty$ *and* (θ_n) *be a positive sequence. Then*

$$\mu_1^\beta = \left\{ \psi\colon \sup_v \left\{ \left| \frac{1}{p_v} \left(P_v \psi_v - P_{v-1} \psi_{v+1} \right) \right| + \frac{P_v}{p_v} |\psi_v| \right\} < \infty \right\}$$

and

$$\mu_k^\beta = \left\{ \psi\colon \sum_{v=1}^\infty \frac{1}{\theta_v} \left| \frac{1}{p_v} \left(P_v \psi_v - P_{v-1} \psi_{v+1} \right) \right|^{k^*} < \infty, \ \sup_v |L_v| < \infty \right\}, \tag{2.1}$$

where

$$L_v = \theta_v^{-1/k^*} \frac{P_v}{p_v} \psi_v.$$

Theorem 2.2. *Let* (θ_n) *be a positive sequence. Then*

(i) $\ell_k \subset \mu_k$ *holds for* $1 < k < \infty$ *if and only if*

$$\sup_m \left\{ \sum_{v=1}^m P_{v-1}^{k^*} \right\}^{1/k^*} \left\{ \sum_{n=m}^\infty \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k \right\}^{1/k} < \infty; \tag{2.2}$$

(ii) $\mu_k \subset \ell_k$ *holds for* $1 < k < \infty$ *if and only if*

$$\sup_m \theta_m^{-1/k^*} \frac{P_m}{p_m} < \infty; \tag{2.3}$$

(iii) $\mu_k \subset \ell_1$ *holds for* $1 \leq k < \infty$ *if and only if*

$$\sum_{v=1}^\infty \frac{1}{\theta_v} \left(\frac{P_v}{p_v} \right)^{k^*} < \infty;$$

(iv) $\ell_1 \subset \mu_k$ *holds for* $1 \leq k < \infty$ *if and only if*

$$\sup_v P_{v-1} \sum_{n=v}^\infty \theta_n^{k-1} \left(\frac{p_n}{P_n P_{n-1}} \right)^k < \infty.$$

Note that, $\mu_k = \ell_k$ for $\theta_n p_n = O(P_n)$ and $P_n = O(p_n)$, and also if $n p_n \theta_n = O(P_n)$, then (2.2) holds but not (2.3) and so the inclusion $\mu_k \subset \ell_k$ holds strictly. Besides, $\ell_1 \subset \mu_1$ for all sequence (p_n), since

$$\sum_{n=v}^\infty \frac{p_n}{P_n P_{n-1}} = \frac{1}{P_{v-1}}.$$

Theorem 2.3. *Let $1 \le k < \infty$. Assume that $A = (a_{nv})$ is an arbitrary infinite matrix and (θ_n) is a positive sequence. Then $(\mu_1, \lambda_k) \subset B(\mu_1, \lambda_k)$, i.e., there exists a continuous linear operator L_A such that $L_A(x) = A(x)$, and $A \in (\mu_1, \lambda_k)$ if and only if, for $n = 0, 1, \ldots,$*

$$\sup_v \left\{ \left| \frac{1}{p_v} \left(P_v a_{nv} - P_{v-1} a_{n,v+1}\right) \right| + \frac{P_v}{p_v} |a_{nv}| \right\} < \infty, \tag{2.4}$$

$$\sup_j \sum_{n=1}^{\infty} \left| \frac{\theta_n^{1/k^*} q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} \frac{Q_{v-1}}{p_j} \left(P_j a_{vj} - P_{j-1} a_{v,,j+1}\right) \right|^k < \infty. \tag{2.5}$$

Theorem 2.4. *Let $1 < k < \infty$. Assume that $A = (a_{nv})$ is an arbitrary infinite matrix and (θ_n) is a positive sequence. Then, $(\mu_k, \lambda_1) \subset B(\mu_k, \lambda_1)$, i.e., there exists a bouned linear operator L_A such that $L_A(x) = A(x)$, and $A \in (\mu_k, \lambda_1)$ if and only if*

$$\sup_v \theta_v^{-1/k^*} \frac{P_v}{p_v} |a_{nv}| < \infty, \quad n = 0, 1, \ldots, \tag{2.6}$$

$$\sum_{v=1}^{\infty} \frac{1}{\theta_v} \left| \frac{1}{p_v} \left(P_v a_{nv} - P_{v-1} a_{n,v+1}\right) \right|^{k^*} < \infty, \quad n = 0, 1, \ldots, \tag{2.7}$$

$$\sum_{j=0}^{\infty} \frac{1}{\theta_j} \left(\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left| \sum_{v=1}^{n} \frac{Q_{v-1}}{p_j} \left(P_j a_{vj} - P_{j-1} a_{v,,j+1}\right) \right| \right)^{k^*} < \infty. \tag{2.8}$$

3. Needed lemmas. We need the following lemmas for the proof of our theorems.

Lemma 3.1 [18]. *Let $1 < k < \infty$. Then, $A \in (\ell_k, \ell_1)$ if and only if*

$$\sum_{v=0}^{\infty} \left(\sum_{n=0}^{\infty} |a_{nv}| \right)^{k^*} < \infty,$$

Lemma 3.2 [7]. *Let $1 \le k < \infty$. Then $A \in (\ell_1, \ell_k)$ if and only if*

$$\sup_v \sum_{n=0}^{\infty} |a_{nv}|^k < \infty.$$

Lemma 3.3 [21]. *Let $1 < k < \infty$. Then*

(i) *$A \in (\ell_1, c)$ if and only if*

$$\lim_n a_{nv} \quad \text{exists for} \quad v \ge 0 \quad \text{and} \quad \sup_{n,v} |a_{nv}| < \infty;$$

(ii) *$A \in (\ell_k, c)$ if and only if*

$$\text{holds for} \quad v \ge 0 \quad \text{and} \quad \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty.$$

Lemma 3.4 [2]. *Let $1 < k \le s < \infty$. Then $A \in (\ell_k, \ell_k)$ if and only if*

$$\sup_m \left(\sum_{v=0}^{m} c_v^{k^*} \right)^{1/k^*} \left(\sum_{n=m}^{\infty} a_v^k \right)^{1/k} < \infty,$$

where $A = (a_{nv})$ is a factorable matrix of nonnegative numbers, i.e., $a_{nv} = a_n c_v$ for $0 \le v \le n$, and zero otherwise.

4. Proofs of theorems.

***Proof of Theorem* 2.1.** Let $\psi\in\mu_k^\beta$. Then, the series $\sum_{v=0}^{\infty}\psi_v x_v$ is convergent for every $x\in\mu_k$, and also $x\in\mu_k$ if and only if $R\in\ell_k$, where $R=(R_n)_{n\in N}$ is defined by

$$R_0=x_0,\ R_n=\frac{\theta_n^{1/k^*}p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}x_v \quad \text{for} \quad n\geq 1. \tag{4.1}$$

Furher, it can be written from Abel's partial summation and (4.1) that

$$\sum_{v=0}^{m}\psi_v x_v=\psi_0R_0+\sum_{v=1}^{m-1}\left(P_v\psi_v-P_{v-1}\psi_{v+1}\right)\frac{R_v}{p_v}+\frac{P_m}{p_m}\psi_mR_m=$$

$$=\psi_0R_0+\sum_{v=1}^{m-1}\left(P_v\psi_v-P_{v-1}\psi_{v+1}\right)\frac{\theta_v^{-1/k^*}}{p_v}R_v+\frac{P_m}{p_m}\psi_m\theta_m^{-1/k^*}R_m=\sum_{v=0}^{\infty}w_{mv}R_v,$$

where

$$w_{mv}=\begin{cases}\psi_0, & v=0,\\[2mm] \left(P_v\psi_v-P_{v-1}\psi_{v+1}\right)\dfrac{\theta_v^{-1/k^*}}{p_v}, & 1\leq v\leq m-1,\\[2mm] \dfrac{P_m}{p_m}\psi_m\theta_m^{-1/k^*}, & v=m,\\[2mm] 0, & v>m.\end{cases}$$

Now, $\psi\in\mu_k^\beta\Leftrightarrow W\in(\ell_k,c)$. Therefore, it follows from Lemma 3.3 that

$$\sup_m\left\{\sum_{v=1}^{m-1}\frac{1}{\theta_v}\left|\frac{1}{p_v}\left(P_v\psi_v-P_{v-1}\psi_{v+1}\right)\right|^{k^*}+\left|\frac{P_m}{p_m}\psi_m\theta_m^{-1/k^*}\right|^{k^*}\right\}<\infty$$

if and only if (2.2) is satisfied.

Theorem 2.1 is proved.

***Proof of Theorem* 2.2.** (i) Let us define the matrix T by (1.2). Then, it is easily seen that $l_k\subset\mu_k$ if and only if $T\in(l_k,l_k)$. Hence, applying Lemma 3.4 with the matrix T, it follows that $T\in(l_k,l_k)$ if and only if (2.2) holds, which completes the proof.

(ii) Take $x\in\mu_k$, then $y=T(x)\in l_k$ and so $\mu_k\subset l_k$ states that if $y\in l_k$ then $T^{-1}(y)\in l_k$, or, equivalently, $T^{-1}\in(l_k,l_k)$. Now, if we say $T^{-1}=S=(s_{nv})$, then we have

$$s_{nv}=\begin{cases}1, & n=0,\quad v=0,\\[2mm] \dfrac{-d_{n-1}}{P_{n-1}}, & v=n-1,\quad n\geq 2,\\[2mm] \dfrac{d_n}{P_{n-1}}, & v=n,\quad n\geq 1,\\[2mm] 0 & v>n \quad or \quad n\geq 1,\quad v=0,\end{cases}$$

where

$$d_n = \frac{P_{n-1}P_n}{\theta_n^{1/k^*} p_n} \quad \text{for} \quad n \geq 1.$$

Now, if $y \in l_k$, then$S_0(y) = y_0$,

$$S_n(y) = \sum_{v=1}^{n} s_{nv} y_v = \frac{1}{P_{n-1}} \left(-d_{n-1} y_{n-1} + d_n y_n\right) \quad \text{for} \quad n \geq 1$$

and so, since $P_{n-1} < P_n$ for all $n \geq 1$,

$$||S(y)||_{l_k} \leq \left\{ |y_0|^k + \sum_{n=1}^{\infty} \left(\left| \frac{d_{n-1}}{P_{n-1}} y_{n-1} \right|^k + \left| \frac{d_n}{P_{n-1}} y_n \right|^k \right) \right\}^{1/k} =$$

$$= O(1) \sup_n \frac{d_n}{P_{n-1}} ||y||_{l_k} < \infty,$$

which shows that (2.3) is sufficient for $T^{-1} \in (l_k, l_k)$. Conversely, if $T^{-1} \in (l_k, l_k)$, then $S\colon l_k \to l_k$ is a bounded linear operator since l_k is a BK-space. Hence, there exists some constant M such that

$$||S(x)||_{l_k} \leq M ||x||_{l_k} \quad \text{for all} \quad x \in l_k. \tag{4.2}$$

Applying (4.2) with $x_v = 1$ for $v = n$ and otherwise, we get, for all $v \geq 0$,

$$|s_{vv}| \leq \left(\sum_{n=0}^{\infty} |s_{nv}|^k \right)^{1/k} \leq M,$$

which implies (2.3).

Theorem 2.2 is proved.

The other parts can be easily proved by Lemmas 3.2 and 3.3.

Proof of Theorem **2.3.** It is clear from the definition of matrix transformation that L_A is a linear operator, and since μ_k is a BK-space, it is also continuous by Theorem 4.2.8 of Wilanky [22, p. 57]. For the second part, $A \in (\mu_1, \lambda_k)$ if and only if $(a_{nj})_{j=0}^{\infty} \in \mu_1^{\beta}$ and $A(x) = (A_n(x)) \in \lambda_k$. But, by Theorem 2.1, $(a_{nj})_{j=0}^{\infty} \in \mu_1^{\beta}$ if and only if (2.8) holds. Further, since $(x_n) \in \mu_1 \Leftrightarrow (R_n) \in \ell_1$ by (4.1) with $k = 1$, then $\dfrac{P_m}{p_m} a_{nm} R_m \to 0$ as $m \to \infty$, for each n. By the inversion of (4.1), we get

$$\sum_{v=0}^{m} a_{nv} x_v = \sum_{v=0}^{m-1} \left(P_v a_{nv} - P_{v-1} a_{n,v+1}\right) \frac{R_v}{p_v} + \frac{P_m}{p_m} a_{nm} R_m$$

which implies

$$A_n(x) = \sum_{v=0}^{\infty} \left(P_v a_{nv} - P_{v-1} a_{n,v+1}\right) \frac{R_v}{p_v}.$$

On the other hand, $t^* \in \ell_k$ if and only if $A(x) = (A_n(x)) \in \lambda_k$, whenever $t_0^* = A_0(x)$ and, for $n \geq 1$,

$$t_n^* = \frac{\theta_n^{1/k^*} q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n} Q_{v-1} A_v(x) =$$

$$= \sum_{j=0}^{\infty} \sum_{v=1}^{n} \frac{\theta_n^{1/k^*} q_n Q_{v-1}}{Q_n Q_{n-1}} \left(P_j a_{vi} - P_{j-1} a_{v,j+1}\right) \frac{R_j}{p_j} = \sum_{j=0}^{\infty} c_{nj} R_j,$$

where, for $j = 0, 1, 2, \ldots$,

$$c_{nj} = \begin{cases} \left(P_j a_{nj} - P_{j-1} a_{n,j+1}\right) \dfrac{1}{p_j}, & n = 0, \\ \sum_{v=1}^{n} \dfrac{\theta_n^{1/k^*} q_n Q_{v-1}}{Q_n Q_{n-1}} \left(P_j a_{vj} - P_{j-1} a_{v,j+1}\right) \dfrac{1}{p_j}, & n \geq 1. \end{cases}$$

Now, $A \in (\mu_1, \lambda_k) \Leftrightarrow C \in (\ell_1, \ell_k)$, i.e., equivalently, $\sup_j \sum_{n=0}^{\infty} |c_{nj}|^k < \infty$, by Lemma 3.2. Thus, it follows from the definition of the matrix C that

$$\sup_j \sum_{n=0}^{\infty} |c_{nj}|^k = \sup_j \left\{ \left| \left(P_j a_{0j} - P_{j-1} a_{0,j+1}\right) \frac{1}{p_j} \right|^k + \right.$$

$$\left. + \sum_{n=1}^{\infty} \theta^{k-1} \left| \sum_{v=1}^{n} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} \left(P_j a_{vi} - P_{j-1} a_{v,j+1}\right) \frac{1}{p_j} \right|^k \right\} < \infty,$$

if and only if (2.9) is satisfied by (2.4).

Theorem 2.3 is proved.

***Proof of Theorem* 2.4.** The first part is as in the proof of Theorem 2.3. For the second part, $A \in (\mu_k, \lambda_1)$ if and only if $(a_{nj})_{j=0}^{\infty} \in \mu_k^{\beta}$ and $A(x) = (A_n(x)) \in \lambda_1$ for every $x \in \mu_k$. But, it follows from Lemma 3.3 that $(a_{nj})_{j=0}^{\infty} \in \mu_k^{\beta}$ if and only if (2.6) and (2.7) hold. Also, by (4.1), since $x \in \mu_k \Leftrightarrow R \in \ell_k$, then $\dfrac{P_m}{p_m} a_{nm} \theta_m^{-1/k^*} R_m \to 0$ as $m \to \infty$ for each n, by (2.6). By Abel's partial summation and (4.1), we get

$$\sum_{v=0}^{m} a_{nv} x_v = a_{n0} R_0 + \sum_{v=1}^{m-1} \left(P_v a_{nv} - P_{v-1} a_{n,v+1}\right) \frac{\theta_v^{-1/k^*}}{p_v} R_v + \frac{P_m}{p_m} \theta_m^{-1/k^*} a_{nm} R_m,$$

and so

$$A_n(x) = \sum_{v=0}^{\infty} \left(P_v a_{nv} - P_{v-1} a_{n,v+1}\right) \frac{\theta_v^{-1/k^*}}{p_v} R_v.$$

On the other hand, $t^* \in \ell_1$ if and only if $A(x) = (A_n(x)) \in \lambda_1$ whenever

$$t_0^* = \sum_{v=0}^{\infty} \left(P_v a_{0v} - P_{v-1} a_{0,v+1}\right) \frac{\theta_v^{-1/k^*}}{p_v} R_v$$

and, for $n \geq 1,$

$$t_n^* = \sum_{v=1}^{n} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} A_v(x) =$$

$$= \sum_{j=1}^{\infty} \left(\sum_{v=1}^{n} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} \left(P_j a_{vi} - P_{j-1} a_{v,j+1}\right) \frac{\theta_j^{-1/k^*}}{p_j} \right) R_j = \sum_{j=0}^{\infty} d_{nj} R_j,$$

where

$$d_{nj} = \begin{cases} \left(P_j a_{0j} - P_{j-1} a_{0,j+1}\right) \dfrac{\theta_j^{-1/k^*}}{p_j}, & n = 0, \quad j \geq 0, \\ \sum_{v=1}^{n} \dfrac{q_n Q_{v-1}}{Q_n Q_{n-1}} \left(P_j a_{vi} - P_{j-1} a_{v,j+1}\right) \dfrac{\theta_j^{-1/k^*}}{p_j}, & n \geq 1, \quad j \geq 0. \end{cases}$$

Now, $A \in (\mu_k, \lambda_1) \Leftrightarrow D \in (\ell_k, \ell_1)$. Applying Lemma 3.1 to the matrix D gives

$$\sum_{j=0}^{\infty} \left(\sum_{n=0}^{\infty} |d_{nj}| \right)^{k^*} = \sum_{j=0}^{\infty} \left(|d_{0j}| + \sum_{n=1}^{\infty} |d_{nj}| \right)^{k^*} < \infty$$

which holds if and only if condition (2.8) is satisfied by (2.7).

Theorem 2.4 is proved.

5. Applications. Theorem 2.3 and 2.4 have several consequences depending on the choice of an infinite matrix A, the sequences $\theta = (\theta_n)$, $p = (p_n)$ and $q = (q_n)$, For example, if we take $A = W$ (resp. $\varepsilon_v = 1$) then $A \in (\mu_1, \lambda_k)$ gives us summability factors of the form that if $\Sigma\, a_v$ is summable μ_1, then $\Sigma\, \varepsilon_v a_v$ is summable λ_k (resp. $\mu_1 \subset \lambda_k$).

If A is chosen any triangular matrix in Theorem 2.3, then, it is obvious that (2.4) holds, so it can be omitted, and also (2.5) is reduced to

$$\sup_i \frac{1}{p_j^k} \sum_{n=j}^{\infty} \theta_n^{k-1} \left| \sum_{v=j}^{n} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} \left(P_j a_{vj} - P_{j-1} a_{v,,j+1}\right) \right|^k < \infty, \tag{5.1}$$

equivalently,

$$\sup_j \left\{ \left| \frac{\theta_j^{1/k^*} q_{jP_j} a_{jj}}{Q_j p_j} \right|^k + \sum_{n=j+1}^{\infty} \theta_n^{k-1} |\Gamma(n,j)|^k \right\} < \infty, \tag{5.2}$$

where

$$\Gamma(n,j) = \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} \left[\frac{P_j}{p_j} (a_{vj} - a_{v,,j+1}) + a_{v,,j+1} \right].$$

Further, if (5.1) is satisfied, then $A\colon \mu_1 \to \lambda_k$ is a continuous linear mapping and so there exists a number M such that

$$||A(x)||_{\lambda_k} \leq M ||x||_{\mu_1} \quad \text{for all} \quad x \in \mu_1. \tag{5.3}$$

Taking any $v \geq 0$, we apply (5.3) with $x_{v+1} = 1, x_m = 0, m \neq v+1$. Hence, it can be obtained that $v = 0, 1, \ldots$

$$\sum_{n=v+1}^{\infty} \theta_n^{k-1} \left| \sum_{m=v+1}^{n} \frac{q_n Q_{m-1}}{Q_n Q_{n-1}} a_{m,v+1} \right|^k \leq M^k \tag{5.4}$$

(see, also, [14). Thus, it is easily seen from (5.2) that (5.1) implies (5.4),

$$\sup_j \frac{\theta_j^{1/k^*} q_j P_j}{Q_j p_j} |a_{jj}| < \infty \tag{5.5}$$

and

$$\sup_j \left(\frac{P_j}{p_j} \right)^k \sum_{n=j+1}^{\infty} \theta_n^{k-1} \left| \sum_{v=j}^{n} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} (a_{vj} - a_{v,j+1}) \right|^k < \infty. \tag{5.6}$$

Conversely, by considering (5.2), it is deduced from (5.4), (5.5) and (5.6) that (5.1) is satisfied. So (5.1) is equivalent to (5.4), (5.5) and (5.6), which gives the following result of [14].

***Corollary* 5.1.** *Let A be any triangle matrix and (θ_n) be a positive sequence.Then, $A \in (\mu_1, \lambda_k)$, $1 \leq k < \infty$, if and only if* (5.4), (5.5) *and* (5.6) *are satisfied.*

It is well known that the case $A = I$ and $k = 1$ of this result was given by Bosanquet [4] and Sunouchi [20], and also the case $A = I$, $\theta_n = n$ and $k \geq 1$ was established by Orhan and Sarıgöl [12].

Further, if we take A as triangular matrix in Theorem 2.4, then (2.6) and (2.7) hold directly, and (2.8) reduces to

$$\sum_{j=0}^{\infty} \frac{1}{p_j^{k^*} \theta_j} \left(\sum_{n=j}^{\infty} \left| \sum_{v=j}^{n} \frac{q_n Q_{v-1}}{Q_n Q_{n-1}} (P_j a_{vj} - P_{j-1} a_{v,,j+1}) \right| \right)^{k^*} < \infty. \tag{5.7}$$

So we get the following main result of [14].

***Corollary* 5.2.** *Let $1 < k < \infty$ and $1/k + 1/k^* = 1$. Let A be a triangle matrix and (θ_n) be a positive sequence. Then $A \in (\mu_k, \lambda_1)$ if and only if* (5.7) *is satisfied.*

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