UDC 517.5

Ü. Totur, İ. Çanak (Adnan Menderes Univ., Aydin and Ege Univ., Izmir, Turkey)

A TAUBERIAN THEOREM FOR THE POWER-SERIES SUMMABILITY METHOD

ТЕОРЕМА ТАУБЕРА ДЛЯ МЕТОДУ ПІДСУМОВУВАННЯ СТЕПЕНЕВИХ РЯДІВ

We introduce a one-sided Tauberian condition in terms of the weighted general control modulo oscillatory behavior of integer order m with $m \geq 1$ for the power-series summability method.

Введено односторонню умову Таубера в термінах вагового загального управління по модулю коливної поведінки порядку m, де $m \geq 1$, для методу підсумовування степеневих рядів.

1. Introduction and preliminaries. Let $u = (u_n)$ be a sequence of real numbers. Assume that $p = (p_n)$ be a sequence of nonnegative numbers with $p_0 > 0$,

$$P_n := \sum_{k=0}^{n} p_k \to \infty \qquad \text{as} \qquad n \to \infty,$$

and

$$p(x) = \sum_{k=0}^{\infty} p_k x^k < \infty \qquad \text{for} \qquad 0 \leq x < 1.$$

The nth weighted mean of (u_n) is defined by

$$\sigma_{n,p}^{(1)}(u) := \frac{1}{P_n} \sum_{k=0}^{n} p_k u_k.$$

A sequence (u_n) is said to be summable by the weighted mean method determined by the sequence p, in short, $(\overline{N}, p)$ summable to a finite number s if

$$\lim_{n\to\infty} \sigma_{n,p}^{(1)}(u) = s. \tag{1}$$

If $p_n = 1$ for all nonnegative n, then $(\overline{N}, p)$ summability method reduces to Cesàro summability method.

If $\sum_{k=0}^{\infty} p_k u_k x^k$ is convergent for $0 \leq x < 1$ and

$$\lim_{x\to 1^-} \frac{1}{p(x)} \sum_{k=0}^{\infty} p_k u_k x^k = s, \tag{2}$$

we say that (u_n) is summable to s by the power series method (J, p) and we write $u_n \to s \ (J, p)$. If $p_n = 1$ for all nonnegative n, then the (J, p) summability method reduces to Abel summability method.

The sequence $\Delta u = (\Delta u_n)$, which is the backward difference of (u_n), is defined by $\Delta u_n = u_n - u_{n-1}$ for $n \geq 1$ and $\Delta u_0 = u_0$. For any nonnegative integer m, we define $\Delta_m u_n = \Delta(\Delta_{m-1} u_n) = \Delta_{m-1}(\Delta u_n)$ with $\Delta_0 u_n = u_n$ and $(n\Delta)_m u_n = n\Delta((n\Delta)_{m-1} u_n) = (n\Delta)_{m-1}(n\Delta u_n)$ with $(n\Delta)_0 u_n = u_n$ and $(n\Delta)_1 u_n = n\Delta u_n = n(u_n - u_{n-1})$.

A sequence (u_n) of real numbers is called totally monotone if $\Delta_m u_n \geq 0$ for nonnegative integers m and n.

The difference between u_n and $\sigma^{(1)}_{n,p}(u)$ which is called the weighted Kronecker identity is given by

$$u_n - \sigma^{(1)}_{n,p}(u) = V^{(0)}_{n,p}(\Delta u), \tag{3}$$

where $V^{(0)}_{n,p}(\Delta u) := \dfrac{1}{P_n} \sum_{k=1}^{n} P_{k-1} \Delta u_k$ (see [1]).

For each integer $m \geq 0$, we define $\sigma^{(m)}_{n,p}(u)$ and $V^{(m)}_{n,p}(\Delta u)$ by

$$\sigma^{(m)}_{n,p}(u) = \begin{cases} \dfrac{1}{P_n} \displaystyle\sum_{k=0}^{n} p_k \sigma^{(m-1)}_{k,p}(u), & m \geq 1, \\ u_n, & m = 0, \end{cases}$$

and

$$V^{(m)}_{n,p}(\Delta u) = \begin{cases} \dfrac{1}{P_n} \displaystyle\sum_{k=0}^{n} p_k V^{(m-1)}_{k,p}(\Delta u), & m \geq 1, \\ V^{(0)}_{n,p}(\Delta u), & m = 0, \end{cases}$$

respectively.

For a sequence $u = (u_n)$ and any integer $m \geq 1$, the identities

$$\frac{P_{n-1}}{p_n} \Delta \sigma^{(m)}_{n,p}(u) = V^{(m-1)}_{n,p}(\Delta u) \tag{4}$$

and

$$\sigma^{(1)}_{n,p}\left(\frac{P_{n-1}}{p_n} \Delta V^{(m-1)}_{n,p}(\Delta u) \right) = \frac{P_{n-1}}{p_n} \Delta V^{(m)}_{n,p}(\Delta u) \tag{5}$$

are given in [2].

The weighted classical control modulo of (u_n) is defined by

$$\omega^{(0)}_{n,p}(u) = \frac{P_{n-1}}{p_n} \Delta u_n$$

and the weighted general control modulo of the oscillatory behavior of integer order m with $m \geq 1$ of (u_n) is defined by

$$\omega^{(m)}_{n,p}(u) = \omega^{(m-1)}_{n,p}(u) - \sigma^{(1)}_{n,p}(\omega^{m-1}(u)).$$

We remark that the notation ω^{m-1} above stands for the result of application of the operator $\sigma^{(1)}$ to the sequence $(\omega^{m-1}(u))$. For a sequence $u = (u_n)$ and any nonnegative integer m, we define

$$\left(\frac{P_{n-1}}{p_n}\Delta\right)_m u_n = \left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1}\left(\frac{P_{n-1}}{p_n}\Delta u_n\right) =$$
$$= \frac{P_{n-1}}{p_n}\Delta\left(\left(\frac{P_{n-1}}{p_n}\Delta\right)_{m-1} u_n\right)$$

with $\left(\frac{P_{n-1}}{p_n}\Delta\right)_0 u_n = u_n$ and $\left(\frac{P_{n-1}}{p_n}\Delta\right)_1 u_n = \frac{P_{n-1}}{p_n}\Delta u_n = \frac{P_{n-1}}{p_n}(u_n - u_{n-1})$. The different writing of the weighted general control modulo of the oscillatory behavior of integer order m with $m \geq 1$ of (u_n) is obtained in [2] by

$$\omega_{n,p}^{(m)}(u) = \left(\frac{P_{n-1}}{p_n}\Delta\right)_m V_{n,p}^{(m-1)}(\Delta u). \tag{6}$$

The identity (6) can be also written as

$$\omega_{n,p}^{(m)}(u) = V_{n,p}^{(0)}(\Delta\omega^{(m-1)}(u)) = \frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(1)}(\omega^{m-1}(u)).$$

Throughout this work, the symbols $u_n = o(1)$ and $u_n = O(1)$ mean that (u_n) converges to zero and (u_n) is bounded, respectively. The symbol $[\lambda n]$ denotes the integral part of the product λn.

A sequence (u_n) is said to be slowly decreasing if

$$\lim_{\lambda\to 1+}\liminf_{n\to\infty}\min_{n<k\leq[\lambda n]}(u_k - u_n) \geq 0 \tag{7}$$

or, equivalently,

$$\lim_{\lambda\to 1-}\liminf_{n\to\infty}\min_{[\lambda n]<k\leq n}(u_n - u_k) \geq 0.$$

The condition (7) is satisfied if there exists a constant $C > 0$ such that

$$\frac{P_{n-1}}{p_n}\Delta u_n \geq -C$$

with $\frac{np_n}{P_{n-1}} = O(1)$. Indeed, we can estimate as follows. For any $k > n$, we have

$$u_k - u_n = \sum_{j=n+1}^{k}\Delta u_j \geq -C\sum_{j=n+1}^{k}\frac{p_j}{P_{j-1}} \geq -C\sum_{j=n+1}^{k}\frac{1}{j} \geq -C\left(\frac{k-n}{n}\right)$$

whence we conclude that

$$\liminf_{n\to\infty}\min_{n<k\leq[\lambda n]}(u_k - u_n) \geq -C(\lambda - 1), \qquad \lambda > 1.$$

Letting $\lambda \to 1^+$, the inequality (7) follows immediately. Note that we use C to denote a constant, possibly different at each occurrence.

A sequence (u_n) is slowly increasing if and only if the sequence $(-u_n)$ is slowly decreasing (see [3]). An equivalent definition of a slowly increasing sequence is given as follows:

A sequence (u_n) is said to be slowly increasing if

$$\lim_{\lambda\to 1+}\limsup_{n\to\infty}\max_{n<k\le[\lambda n]}(u_k-u_n)\le 0. \tag{8}$$

The condition (8) is reformulated as follows:

$$\lim_{\lambda\to 1-}\limsup_{n\to\infty}\max_{[\lambda n]<k\le n}(u_n-u_k)\le 0.$$

A sequence $u=(u_n)$ is slowly oscillating if

$$\lim_{\lambda\to 1^+}\limsup_{n\to\infty}\max_{n<k\le[\lambda n]}|u_k-u_n|=0 \tag{9}$$

or, equivalently,

$$\lim_{\lambda\to 1^-}\limsup_{n\to\infty}\max_{[\lambda n]<k\le n}|u_n-u_k|=0.$$

A sequence (u_n) is said to be slowly oscillating if and only if (u_n) is both slowly increasing and slowly decreasing (see [3]).

It is clear from the Cauchy criterion that every convergent sequence is slowly oscillating, but the converse of this statement is not always true. For example, the sequence $(\log(n+1))$ is not convergent, but it is slowly oscillating (here, the logarithm is to the natural base e):

To justify (9), let $\lambda>1$ and $n<k\le[\lambda n]$. It follows from

$$u_k-u_n=\log(k+1)-\log(n+1)=\log\frac{k+1}{n+1}$$

that

$$\limsup_{n\to\infty}\max_{n<k\le[\lambda n]}|u_k-u_n|\le\log\lambda.$$

Letting $\lambda\to 1^+$ gives (9).

One can easily show that the sequences $(\sin\log(n+1))$ and $(\cos\log(n+1))$ are slowly oscillating sequences. To justify (9), let $\lambda>1$ and $n<k\le[\lambda n]$. It follows from

$$|\sin\log(k+1)-\sin\log(n+1)|\le\frac{k-n}{n}$$

that

$$\limsup_{n\to\infty}\max_{n<k\le[\lambda n]}|\sin\log(k+1)-\sin\log(n+1)|\le\lambda-1.$$

Letting $\lambda\to 1^+$ gives (9).

Similarly, it can be shown that $(\cos\log(n+1))$ is slowly oscillating.

2. History of classical theory. The $(\overline{N},p)$ and (J,p) summability methods are regular. In other words if the limit

$$\lim_{n\to\infty}u_n=s \tag{10}$$

exists, then both (1) and (2) also exist. However, the converses are not always true. Notice that (1) or (2) may imply (10) under certain conditions, which are called Tauberian conditions. Any theorem which states that convergence of sequences follows from the $(\overline{N}, p)$ summability method or the (J, p) summability method with some Tauberian conditions is said to be a Tauberian theorem.

Hardy [4] proved that if

$$\omega_{n,p}^{(0)}(u) = O(1)$$

and (1) exists, then (u_n) converges.

Çanak and Totur [1] obtained the following one-sided Tauberian theorem for the weighted mean method of summability.

Theorem 1. *Let*

$$1 < \liminf_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} \le \limsup_{n\to\infty} \frac{P_{[\lambda n]}}{P_n} < \infty \qquad \text{for} \quad \lambda > 1, \tag{11}$$

$$1 < \liminf_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} \le \limsup_{n\to\infty} \frac{P_n}{P_{[\lambda n]}} < \infty \qquad \text{for} \quad 0 < \lambda < 1, \tag{12}$$

and

$$\frac{np_n}{P_{n-1}} = O(1). \tag{13}$$

If (u_n) is $(\overline{N}, p)$ summable to s and

$$\omega_{n,p}^{(1)}(u) \ge -C$$

for some $C > 0$, then (u_n) converges to s.

Later, Totur and Çanak [2] proved the next theorem which is an extension of Hardy's Tauberian theorem for the weighted mean method of summability.

Theorem 2. *If (u_n) is $(\overline{N}, p)$ summable to s and*

$$\omega_{n,p}^{(m)}(u) = O(1)$$

for some nonnegative integer m, (u_n) converges to s.

Tietz [5] obtained some classical Tauberian theorems which are analogous to Hardy – Littlewood's theorem [6] and Schmidt's theorem [7] for the power series method of summability.

Theorem 3 [5]. *Let*

$$1 \le \frac{P_m}{P_n} \to 1 \quad \text{when} \quad 1 < \frac{m}{n} \to 1\,(n \to \infty). \tag{14}$$

If (u_n) is (J, p) summable to s and

$$\omega_{n,p}^{(0)}(u) \ge -C$$

for some $C > 0$ (or (u_n) is slowly decreasing), then (u_n) converges to s.

Ishiguro [8] showed that (1) implies the limit (2) and obtained some Tauberian theorems which state that the $(\overline{N}, p)$ summability of (u_n) follows from the (J, p) summability of (u_n).

Tietz and Trautner [9] obtained the limit (1) from the existence of the limit (2) under the conditions $u_n \ge -C$ for some $C > 0$ and (14).

Mikhalin [10] proved that if (u_n) is (J, p) summable to s and

$$\sigma_{n,p}^{(1)}(\omega^{(0)}(u)) \ge -C$$

for some $C > 0$, the condition (14) holds and $\left(\frac{p_n}{P_n}\right)$ is totally monotone, then (u_n) is $(\overline{N}, p)$ summable to s.

A number of authors such as Tietz [11], Kratz and Stadtmüller [12], Tietz and Zeller [13], Çanak and Totur [14, 15] have studied Tauberian theorems for the $(\overline{N}, p)$ and (J, p) summability methods.

3. Main result. In this paper our aim is to obtain convergence of (u_n) from the existence of (2) and one-sided boundedness of the weighted general control modulo of the oscillatory behavior of integer order m with $m \geq 1$ with respect to a sequence satisfying some certain conditions. The main theorem extends Tietz's Tauberian theorem for the (J, p) summability method. Further, we have Hardy–Littlewood's Tauberian theorem for Abel summability method as a corollary.

Theorem 4. *Let* (p_n) *satisfy the conditions* (11), (12) *and* (13) *and let* $\left(\frac{p_n}{P_n}\right)$ *be totally monotone. If* (u_n) *is* (J, p) *summable to* s *and*

$$\omega_{n,p}^{(m)}(u) \geq -M_n \tag{15}$$

for some nonnegative sequence $M = (M_n)$ *and some nonnegative integer* m *such that* $A = (A_n)$ *is slowly oscillating where* $A_n = \sum_{j=1}^{n} \frac{p_j}{P_{j-1}} M_j$ *for* $n \geq 1$ *and* $A_0 = 0$, *then* (u_n) *converges to* s.

The following corollaries of Theorem 4 are obvious.

***Corollary* 1.** *Let* (p_n) *satisfy the conditions* (11), (12) *and* (13) *and let* $\left(\frac{p_n}{P_n}\right)$ *be totally monotone. If* (u_n) *is* (J, p) *summable to* s *and*

$$\omega_{n,p}^{(m)}(u) \geq -C$$

for some $C > 0$ *and some nonnegative integer* m, *then* (u_n) *converges to* s.

To prove that Corollary 1 follows from Theorem 4, we need to show that the sequence (a_n) defined by $a_n = \sum_{k=1}^{n} \frac{p_k}{P_{k-1}}$ for $n \geq 1$ and $a_0 = 0$ is slowly oscillating. To justify that (a_n) is slowly oscillating, let $\lambda > 1$ and $n < k \leq [\lambda n]$. By (13), we have

$$|a_k - a_n| = \sum_{j=n+1}^{k} \frac{p_j}{P_{j-1}} \leq C \sum_{j=n+1}^{k} \frac{1}{j} \leq C\left(\frac{k-n}{n}\right)$$

whence we conclude that

$$\limsup_{n\to\infty} \max_{n<k\leq[\lambda n]} |a_k - a_n| \leq C(\lambda - 1).$$

Letting $\lambda \to 1^+$ shows that (a_n) is slowly oscillating.

***Corollary* 2.** *If* (u_n) *is Abel summable to* s *and*

$$\omega_{n,1}^{(m)}(u) \geq -C$$

for some $C > 0$ *and some nonnegative integer* m, *then* (u_n) *converges to* s.

Corollary 2 is given by Çanak et al. [17].

***Corollary* 3.** *If* (u_n) *is Abel summable to* s *and*

$$\omega_{n,1}^{(0)}(u) \geq -C$$

for some $C > 0$, *then* (u_n) *converges to* s.

4. Lemmas. For the proof of the main theorem in the next section we need the following lemmas.

Lemma 1. *For each integer m with $m \geq 1$,*

$$\omega_{n,p}^{(m)}(u) = \sum_{j=0}^{m-1} (-1)^j \binom{m-1}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u),$$

where $\binom{m-1}{j} = \dfrac{(m-1)(m-2)\ldots(m-j)}{j!}$.

Proof. We do the proof by induction. For $m = 1$, we have

$$\omega_{n,p}^{(1)}(u) = \frac{P_{n-1}}{p_n}\Delta u_n - V_{n,p}^{(0)}(\Delta u) = \frac{P_{n-1}}{p_n}\Delta V_{n,p}^{(0)}(\Delta u) =$$

$$= \sum_{j=0}^{0} (-1)^j \binom{0}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u)$$

by (4) and (5). Assume the observation is true for $m = k$. That is, assume that

$$\omega_{n,p}^{(k)}(u) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u). \tag{16}$$

We must show that the observation is true for $m = k+1$. That is, we must show that

$$\omega_{n,p}^{(k+1)}(u) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u).$$

By definition,

$$\omega_{n,p}^{(k+1)}(u) = \omega_{n,p}^{(k)}(u) - \sigma_{n,p}^{(1)}(\omega^{(k)}(u)).$$

By (16),

$$\omega_{n,p}^{(k+1)}(u) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u) -$$

$$- \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j+1)}(\Delta u).$$

Letting $j+1 = i$ in the second sum. Using this substitution

$$\omega_{n,p}^{(k+1)}(u) = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u) +$$

$$+ \sum_{i=1}^{k} (-1)^i \binom{k-1}{i-1} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(i)}(\Delta u). \tag{17}$$

In the second sum of (17) we rename the index of summation j, split the first term off in the first sum and the last term in the second sum of (17), we get

$$\omega_{n,p}^{(k+1)}(u) = (-1)^0 \binom{k-1}{0} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u) + \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u) +$$

$$+ \sum_{j=1}^{k-1} (-1)^j \binom{k-1}{j-1} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u) + (-1)^k \binom{k-1}{k-1} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(k)}(\Delta u).$$

Thus, we obtain

$$\omega_{n,p}^{(k+1)}(u) = (-1)^0 \binom{k-1}{0} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u) +$$

$$+ \sum_{j=1}^{k-1} (-1)^j \left[\binom{k-1}{j} + \binom{k-1}{j-1} \right] \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u) +$$

$$+ (-1)^k \binom{k-1}{k-1} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(k)}(\Delta u).$$

Since $\binom{k-1}{j} + \binom{k-1}{j} = \binom{k}{j}$, the last identity can be written

$$\omega_{n,p}^{(k+1)}(u) = (-1)^0 \binom{k-1}{0} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(0)}(\Delta u) +$$

$$+ \sum_{j=1}^{k-1} (-1)^j \binom{k}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u) +$$

$$+ (-1)^k \binom{k-1}{k-1} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(k)}(\Delta u) =$$

$$= \sum_{j=1}^{k} (-1)^j \binom{k}{j} \frac{P_{n-1}}{p_n} \Delta V_{n,p}^{(j)}(\Delta u).$$

Thus, we conclude that Lemma 1 is true for every positive integer m.

Lemma 2. *Let* (p_n) *satisfy the condition* (13). *If* (u_n) *is slowly decreasing, then* $V_{n,p}^{(0)}(\Delta u) \geq \geq -C$ *for some* $C > 0$ *and* $(\sigma_{n,p}^{(1)}(u))$ *is slowly decreasing.*

Proof. In [10, p. 568], it is shown that if (u_n) is slowly decreasing and the condition (13) holds true for all nonnegative integer n, then there exist a constant D defined by the inequality

$$1 \leq \left(1 + \frac{p_{n+1}}{P_n}\right)^{\frac{P_n}{p_{n+1}}} \leq D$$

for all nonnegative integer n and numbers $a > 0, \quad b > 0$ for which

$$\sigma_{m,p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u) \geq \frac{P_m - P_n}{P_m} \left[-a \left(\ln \frac{P_m}{P_n} - \ln D \right) - b \right] \tag{18}$$

for all nonnegative integers m and n with $m \geq n \geq 0$. In (18), if m and n are replaced by n and $n-1$, respectively, we get

$$\Delta\sigma_{n,p}^{(1)}(u) \geq \frac{p_n}{P_n}\left[-a\left(\ln\frac{P_n}{P_{n-1}} - \ln D\right) - b\right]. \tag{19}$$

Multiplying both sides of (19) by $\dfrac{P_{n-1}}{p_n}$, we have

$$V_{n,p}^{(0)}(\Delta u) = \frac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(1)}(u) \geq \frac{P_{n-1}}{P_n}\left[-a\left(\ln\frac{P_n}{P_{n-1}} - \ln D\right) - b\right]. \tag{20}$$

Taking the fact that the right-hand side of the inequality (20) tends to $a\ln D - b$ as $n \to \infty$ and (13) into consideration, we obtain

$$V_{n,p}^{(0)}(\Delta u) \geq -C$$

for some $C > 0$.

It easily follows from the identity $V_{n,p}^{(0)}(\Delta u) = \dfrac{P_{n-1}}{p_n}\Delta\sigma_{n,p}^{(1)}(u)$ that $(\sigma_{n,p}^{(1)}(u))$ is slowly decreasing.

Baron and Tietz [16] obtained the following result stating that if a sequence is summable to a finite number by the power series method (J,p), then the sequence of its weighted means is summable to the same finite number by the power series method (J,p) under a certain condition.

Lemma 3 [16, p. 17]. *Let $\left(\dfrac{p_n}{P_n}\right)$ be totally monotone. If (u_n) is (J,p) summable to s, then $(\sigma_{n,p}^{(1)}(u))$ is (J,p) summable to s.*

The difference between a sequence and the sequence of its weighted means is given as follows:

Lemma 4 [1]. *Let $u = (u_n)$ be a sequence of real numbers.*

(i) *For $\lambda > 1$ and sufficiently large n,*

$$u_n - \sigma_{n,p}^{(1)}(u) = \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(\sigma_{[\lambda n],p}^{(1)}(u) - \sigma_{n,p}^{(1)}(u)\right) - \frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k(u_k - u_n).$$

(ii) *For $0 < \lambda < 1$ and sufficiently large n,*

$$u_n - \sigma_{n,p}^{(1)}(u) = \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}}\left(\sigma_{n,p}^{(1)}(u) - \sigma_{[\lambda n],p}^{(1)}(u)\right) + \frac{1}{P_n - P_{[\lambda n]}}\sum_{k=[\lambda n]+1}^{n} p_k(u_n - u_k).$$

5. Proof of Theorem 4. Using the fact that the slow oscillation of (A_n) implies the slow increase of (A_n), we obtain that $(-A_n)$ is slowly decreasing. Then, we have

$$V_{n,p}^{(0)}(\Delta(-A)) = -\sigma_{n,p}^{(1)}(M) \geq -C_1$$

for some $C_1 > 0$ by Lemma 2. Taking the weighted mean of both sides of (15), we obtain

$$\sigma_{n,p}^{(1)}(\omega^{(m)}(u)) \geq -\sigma_{n,p}^{(1)}(M) \geq -C_1$$

for some $C_1 > 0$. By (6), we have the following identity:

$$\sigma^{(1)}_{n,p}(\omega^{(m)}(u)) = \sigma^{(1)}_{n,p}\left(\frac{P_{n-1}}{p_n}\Delta\sigma^{(1)}_{n,p}(\omega^{(m-1)}(u))\right) = \frac{P_{n-1}}{p_n}\Delta\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u)).$$

Then, we obtain $\dfrac{P_{n-1}}{p_n}\Delta\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u)) \geq -C_1$ for some $C_1 > 0$. Since (u_n) is (J,p) summable to s, we get, by using the weighted Kronecker identity (3),

$$(\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u))) \text{ is } (J,p) \text{ summable to } 0$$

for each nonnegative integer m by Lemma 3. If we apply Theorem 3 to the sequence $(\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u)))$, we obtain $\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u)) = o(1)$. It follows from the weighted Kronecker identity (3) applied to the sequence $(\sigma^{(1)}_{n,p}(\omega^{(m-1)}(u)))$ that we have $\sigma^{(1)}_{n,p}(\omega^{(m)}(u)) = \sigma^{(1)}_{n,p}(\omega^{(m-1)}(u)) - \sigma^{(2)}_{n,p}(\omega^{(m-1)}(u))$. It follows that $\sigma^{(1)}_{n,p}(\omega^{(m-1)}(u)) \geq -C_2$ for some $C_2 > 0$. Similarly, from the identity $\sigma^{(1)}_{n,p}(\omega^{(m)}(u)) = \dfrac{P_{n-1}}{p_n}\Delta\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u))$, we have $\dfrac{P_{n-1}}{p_n}\Delta\sigma^{(2)}_{n,p}(\omega^{(m-2)}(u)) \geq -C_3$ for some $C_3 > 0$. Since (u_n) is (J,p) summable to s, then $(\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u)))$ is (J,p) summable to 0. By Theorem 3, we get $\sigma^{(2)}_{n,p}(\omega^{(m-1)}(u)) = o(1)$. Continuing in this way, we obtain

$$\sigma^{(2)}_{n,p}(\omega^{(0)}(u)) = V^{(1)}_{n,p}(\Delta u) = o(1). \tag{21}$$

By Lemma 1, we have from (15)

$$\frac{P_{n-1}}{p_n}\Delta V^{(0)}_{n,p}(\Delta u) \geq -(M_n + C_0)$$

for some $C_0 \geq 0$. Applying Lemma 4 (i) to $(V^{(0)}_{n,p}(\Delta u))$, we get

$$V^{(0)}_{n,p}(\Delta u) - V^{(1)}_{n,p}(\Delta u) = \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(V^{(1)}_{[\lambda n],p}(\Delta u) - V^{(1)}_{n,p}(\Delta u)\right) -$$

$$-\frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k (V^{(0)}_{k,p}(\Delta u) - V^{(0)}_{n,p}(\Delta u)) =$$

$$= \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(V^{(1)}_{[\lambda n],p}(\Delta u) - V^{(1)}_{n,p}(\Delta u)\right) -$$

$$-\frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k \sum_{j=n+1}^{k} \Delta V^{(0)}_{j,p}(\Delta u) \leq$$

$$\leq \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(V^{(1)}_{[\lambda n],p}(\Delta u) - V^{(1)}_{n,p}(\Delta u)\right) +$$

$$+\frac{1}{P_{[\lambda n]} - P_n}\sum_{k=n+1}^{[\lambda n]} p_k \sum_{j=n+1}^{k} \frac{p_j}{P_{j-1}}(M_j + C_0)$$

for some $C > 0$ and from this we obtain

$$V^{(0)}_{n,p}(\Delta u) - V^{(1)}_{n,p}(\Delta u) \leq \frac{P_{[\lambda n]}}{P_{[\lambda n]} - P_n}\left(V^{(1)}_{[\lambda n],p}(\Delta u) - V^{(1)}_{n,p}(\Delta u)\right) +$$

$$+\max_{n<k\leq[\lambda n]}\left(\sum_{j=1}^{k}\frac{p_j}{P_{j-1}}M_j-\sum_{j=1}^{n}\frac{p_j}{P_{j-1}}M_j\right)+$$
$$+C\left(\frac{[\lambda n]-n}{n}\right)$$

for some constant C. Taking $\limsup$ of both sides of the inequality above as $n\to\infty$, we have

$$\limsup_{n\to\infty}\left(V_{n,p}^{(0)}(\Delta u)-V_{n,p}^{(1)}(\Delta u)\right)\leq$$
$$\leq\limsup_{n\to\infty}\frac{P_{[\lambda n]}}{P_{[\lambda n]}-P_n}\limsup_{n\to\infty}\left(V_{[\lambda n],p}^{(1)}(\Delta u)-V_{n,p}^{(1)}(\Delta u)\right)+$$
$$+\limsup_{n\to\infty}\max_{n<k\leq[\lambda n]}\left(\sum_{j=1}^{k}\frac{p_j}{P_{j-1}}M_j-\sum_{j=1}^{n}\frac{p_j}{P_{j-1}}M_j\right)+C(\lambda-1)\qquad(22)$$

for some $C>0$.

Since the first term on the right-hand side of the inequality (22) vanishes by (21), we get

$$\limsup_{n\to\infty}\left(V_{n,p}^{(0)}(\Delta u)-V_{n,p}^{(1)}(\Delta u)\right)\leq$$
$$\leq\limsup_{n\to\infty}\max_{n<k\leq[\lambda n]}\left(\sum_{j=1}^{k}\frac{p_j}{P_{j-1}}M_j-\sum_{j=1}^{n}\frac{p_j}{P_{j-1}}M_j\right)+C(\lambda-1)$$

for some $C>0$.

Taking the limit of both sides as $\lambda\to1^+$ and using the slow oscillation of (A_n), we obtain

$$\limsup_{n\to\infty}\left(V_{n,p}^{(0)}(\Delta u)-V_{n,p}^{(1)}(\Delta u)\right)\leq 0.$$

From Lemma 4 (ii), we have

$$V_{n,p}^{(0)}(\Delta u)-V_{n,p}^{(1)}(\Delta u)=\frac{P_{[\lambda n]}}{P_n-P_{[\lambda n]}}\left(V_{n,p}^{(1)}(\Delta u)-V_{[\lambda n],p}^{(1)}(\Delta u)\right)+$$
$$+\frac{1}{P_n-P_{[\lambda n]}}\sum_{k=[\lambda n]+1}^{n}p_k(V_{n,p}^{(0)}(\Delta u)-V_{k,p}^{(0)}(\Delta u))\geq$$
$$\geq\frac{P_{[\lambda n]}}{P_n-P_{[\lambda n]}}\left(V_{n,p}^{(1)}(\Delta u)-V_{[\lambda n],p}^{(1)}(\Delta u)\right)+$$
$$+\frac{1}{P_n-P_{[\lambda n]}}\sum_{k=[\lambda n]+1}^{n}p_k\sum_{j=k+1}^{n}\Delta V_{j,p}^{(0)}(\Delta u)\geq$$
$$\geq\frac{P_{[\lambda n]}}{P_n-P_{[\lambda n]}}\left(V_{n,p}^{(1)}(\Delta u)-V_{[\lambda n],p}^{(1)}(\Delta u)\right)-$$

$$-\frac{1}{P_n - P_{[\lambda n]}} \sum_{k=[\lambda n]+1}^{n} p_k \sum_{j=k+1}^{n} \frac{p_j}{P_{j-1}} (M_j + C_0)$$

for some $C_0 > 0$ and from this we obtain

$$V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u) \geq \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}} \left(V_{n,p}^{(1)}(\Delta u) - V_{[\lambda n],p}^{(1)}(\Delta u) \right) -$$

$$- \max_{[\lambda n] < k \leq n} \left(\sum_{j=1}^{n} \frac{p_j}{P_{j-1}} M_j - \sum_{j=1}^{k} \frac{p_j}{P_{j-1}} M_j \right) -$$

$$-C \left(\frac{n - [\lambda n]}{n} \right)$$

for some constant C. Taking $\liminf$ of both sides of the inequality above as $n \to \infty$, we get

$$\liminf_{n \to \infty} \left(V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u) \right) \geq$$

$$\geq \liminf_{n \to \infty} \frac{P_{[\lambda n]}}{P_n - P_{[\lambda n]}} \liminf_{n \to \infty} \left(V_{n,p}^{(1)}(\Delta u) - V_{[\lambda n],p}^{(1)}(\Delta u) \right) -$$

$$- \limsup_{n \to \infty} \max_{[\lambda n] < k \leq n} \left(\sum_{j=1}^{n} \frac{p_j}{P_{j-1}} M_j - \sum_{j=1}^{k} \frac{p_j}{P_{j-1}} M_j \right) - C(1 - \lambda) \tag{23}$$

for some $C > 0$. Since the first term on the right-hand side of the inequality (23) vanishes by (21), we have

$$\liminf_{n \to \infty} \left(V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u) \right) \geq$$

$$\geq - \limsup_{n \to \infty} \max_{[\lambda n] < k \leq n} \left(\sum_{j=1}^{n} \frac{p_j}{P_{j-1}} M_j - \sum_{j=1}^{k} \frac{p_j}{P_{j-1}} M_j \right) - C(1 - \lambda)$$

for some $C > 0$.

Taking the limit of both sides as $\lambda \to 1^-$ and using the slow oscillation of (A_n), we obtain

$$\liminf_{n \to \infty} \left(V_{n,p}^{(0)}(\Delta u) - V_{n,p}^{(1)}(\Delta u) \right) \geq 0.$$

Finally, by (18) and (21), it follows that

$$\lim_{n \to \infty} V_{n,p}^{(0)}(\Delta u) = 0. \tag{24}$$

Since (u_n) is (J, p) summable to s, then $(\sigma_{n,p}^{(1)}(u))$ is (J, p) summable to s by Lemma 3. By (4), we have

$$\omega_{n,p}^{(0)}(\sigma^{(1)}(u)) = \frac{P_{n-1}}{p_n} \Delta \sigma_{n,p}^{(1)}(u) = V_{n,p}^{(0)}(\Delta u). \tag{25}$$

Taking (24) and (25) into consideration, we obtain

$$\omega_{n,p}^{(0)}(\sigma^{(1)}(u)) \geq -C$$

for some constant C. By Theorem 3, $(\sigma_{n,p}^{(1)}(u))$ converges to s. It follows from the weighted Kronecker identity (3) that (u_n) converges to s.

References

1. *Çanak İ., Totur Ü.* Some Tauberian theorems for the weighted mean methods of summability // Comput. and Math. Appl. – 2011. – **62**, № 6. – P. 2609 – 2615.
2. *Totur Ü., Çanak İ.* Some general Tauberian conditions for the weighted mean summability method // Comput. and Math. Appl. – 2012. – **63**, № 5. – P. 999 – 1006.
3. *Móricz F.* Ordinary convergence follows from statistical summability $(C, 1)$ in the case of slowly decreasing or oscillating sequences // Colloq. Math. – 2004. – **99**, № 2. – P. 207 – 219.
4. *Hardy G. H.* Divergent series. – Oxford: Clarendon Press, 1949.
5. *Tietz H.* Schmidtsche Umkehrbedingungen für Potenzreihenverfahren // Acta Sci. Math. (Szeged). – 1990. – **54**, № 3-4. – P. 355 – 365.
6. *Hardy G. H., Littlewood J. E.* Tauberian theorems concerning power series and Dirichlet's series whose coefficients are positive // Proc. London Math. Soc. – 1914. – **13**, № 2. – P. 174 – 191.
7. *Schmidt R.* Uber divergente Folgen und lineare Mittelbildungen // Math. Z. – 1925. – **22**, № 1. – S. 89 – 152.
8. *Ishiguro K.* A Tauberian theorem for (J, p_n) summability // Proc. Jap. Acad. – 1964. – **40**. – P. 807 – 812.
9. *Tietz H., Trautner R.* Tauber-Sätze für Potenzreihenverfahren // Arch. Math. (Basel). – 1988. – **50**, № 2. – P. 164 – 174.
10. *Mikhalin G. A.* Theorems of Tauberian type for (J, p_n) summation methods // Ukr. Mat. Zh. – 1977. – **29**, № 6. – P. 763 – 770.
11. *Tietz H.* Tauberian theorems of $J_p \to M_p$-type // Math. J. Okayama Univ. – 1989. – **31**. – P. 221 – 225.
12. *Kratz W., Stadtmüller U.* Tauberian theorems for J_p-summability // J. Math. Anal. and Appl. – 1989. – **139**, № 2. – P. 362 – 371.
13. *Tietz H., Zeller K.* Tauber-Bedingungen für Verfahren mit Abschnittskonvergenz // Acta Math. Hungar. – 1998. – **81**, № 3. – P. 241 – 247.
14. *Çanak İ., Totur Ü.* Extended Tauberian theorem for the weighted mean method of summability // Ukrainian Math. J. – 2013. – **65**, № 7. – P. 1032 – 1041.
15. *İ. Çanak İ., Totur Ü.* Tauberian theorems for the (J, p) summability method // Appl. Math. Lett. – 2012. – **25**, №. 10. – P. 1430 – 1434.
16. *Baron S., Tietz H.* Produktsätze für Potenzreihenverfahren und verallgemeinerte Nörlund – Mittel // Tartu Ül. Toimetised. – 1993. – **960**. – P. 13 – 22.
17. *Çanak İ., Totur Ü., Dik M.* One-sided Tauberian conditions for (A, k) summability method // Math. and Comput. Model. – 2010. – **51**, № 5-6. – P. 425 – 430.

Received 03.04.14,
after revision — 05.07.17