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THE PROPERTIES ON DIFFERENTIAL-DIFFERENCE POLYNOMIALS*

ВЛАСТИВОСТІ ДИФЕРЕНЦІАЛЬНО-РІЗНИЦЕВИХ ПОЛІНОМІВ

The main aim of this paper is to improve some classical results on the distribution of zeros for differential polynomials and differential-difference polynomials. We present some results on the distribution of zeros of $[f(z)^n f(z+c)]^{(k)} - \alpha(z)$ and $[f(z)^n(f(z+c) - f(z))]^{(k)} - \alpha(z)$ and give some examples to show that the results are best possible in a certain sense.

Основною метою роботи є поліпшення деяких класичних результатів про розподіли нулів для диференціальних та диференціально-різницевих поліномів. Наведено деякі результати щодо розподілів нулів для $[f(z)^n f(z+c)]^{(k)} -$ $- \alpha(z)$ та $[f(z)^n(f(z+c) - f(z))]^{(k)} - \alpha(z)$, а також деякі приклади, які демонструють, що отримані результати є, в певному розумінні, найкращими.

1. Introduction. We assume that the reader is familiar with standard symbols and fundamental results of Nevanlinna theory [7, 11, 22]. A meromorphic function f means meromorphic in the complex plane. If no poles occur, then f reduces to an entire function. Recall that a meromorphic function $\alpha(z) \not\equiv 0, \infty$ is a small function with respect to $f(z)$, if $T(r, \alpha) = S(r, f)$, where $S(r, f)$ is used to denote any quantity satisfying $S(r, f) = o(T(r, f))$, and $r \to \infty$ outside of a possible exceptional set of finite logarithmic measure. The order of $f(z)$ is defined by

$$\rho(f) := \limsup_{r\to\infty} \frac{\log^+ T(r, f)}{\log r}.$$

A polynomial $p(z)$ is called a Borel exceptional polynomial of $f(z)$ when

$$\lambda(f(z) - p(z)) = \limsup_{r\to\infty} \frac{\log^+ N\left(r, \dfrac{1}{f(z) - p(z)}\right)}{\log r} < \rho(f),$$

where $\lambda(f(z) - p(z))$ is the exponent of convergence of zeros of $f(z) - p(z)$. In this paper, we assume that c is a nonzero complex constant, n is a positive integer, k is a nonnegative integer, unless otherwise specified.

Many mathematicians have made some works on the value distribution of differential polynomials of different types. For example, some results on the zeros of $f(z)$ and its derivatives can be found in [10, 11, 13, 18, 23]. For the aim of the paper, we just recall partial results on the zeros distribution of differential polynomial $f(z)^n f'(z)$ as follows. Hayman [10] (Theorem 10) proved that if $f(z)$ is a transcendental entire function, then $f^n(z)f'(z) - 1$, $n \geq 2$, has infinitely many zeros. Mues [20] proved the above result is also valid for $f(z)$ is a transcendental meromorphic function and $n = 2$. Bergweiler and Eremenko [1], Chen and Fang [3] considered the case $n = 1$ and $f(z)$ is a transcendental meromorphic function, they obtained the following result.

Theorem A. *Let f be a transcendental meromorphic function. If $n \geq 1$, then $f^n f' - 1$ has infinitely many zeros.*

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Remark that $[f^{n+1}]' = (n+1)f^n f'$, Wang and Fang [21] (Corollary 1) improved Theorem A by proving the following result.

Theorem B. *Let f be a transcendental meromorphic function, n, k be two positive integers with $n \geq k+1$. Then $(f^n)^{(k)} - 1$ has infinitely many zeros.*

Recently, there has been an interest to consider the zeros distribution of difference polynomials (see, for example, [2, 4, 14 – 17, 24]). Laine and Yang [15] (Theorem 2) considered the zeros of $f^n(z)f(z+c)-a$, Liu and Yang [16] (Theorems 1.2 and 1.4) also considered the zeros of $f^n(z)f(z+c)-p(z)$ and $f^n(z)\Delta_c f - p(z)$, where $\Delta_c f := f(z+c)-f(z)$ and $p(z)$ is a nonzero polynomial. These results can be viewed as the difference analogues of Theorem A, which can be summarized as follows.

Theorem C. *Let f be a transcendental entire function of finite order and $p(z)$ be a nonzero polynomial. If $n \geq 2$, then $f(z)^n f(z+c) - p(z)$ has infinitely many zeros. If f is not a periodic function with period c, $n \geq 2$, then $f(z)^n \Delta_c f - p(z)$ also has infinitely many zeros.*

Difference analogues of Theorem B also deserve to be considered. Liu, Liu and Cao [17] (Theorems 1.1 and 1.3) obtained the following result.

Theorem D. *Let f be a transcendental entire function of finite order and $\alpha(z)$ be a nonzero small function with respect to $f(z)$. If $n \geq k+2$, then $[f(z)^n f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros. If f is not a periodic function with period c and $n \geq k+3$, then $[f(z)^n \Delta_c f]^{(k)} - \alpha(z)$ has infinitely many zeros.*

Obviously, Theorem D is an extention of Theorem C of the case $k = 0$. Remark that n is related to k in Theorem D. In the following, we will continue to consider what conditions on n and k guarantee that $[f(z)^n f(z+c)]^{(k)} - \alpha(z)$ or $[f(z)^n \Delta_c f]^{(k)} - \alpha(z)$ can have infinitely many zeros? Firstly, we consider the zeros distribution of $[f(z)^n f(z+c)]^{(k)} - \alpha(z)$.

Theorem 1.1. *Let f be a transcendental entire function of finite order. If $n \geq 1$, $k \geq 0$ and $N\left(r, \dfrac{1}{f}\right) = S(r,f)$, then $[f(z)^n f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.*

***Remark* 1.1.** (1) Theorem 1.1 is not true for entire function with infinite order. It can be seen by $f(z) = ze^{e^z}$, $e^c = -n$, $\alpha(z) = p(z)$ is a nonconstant polynomial. Thus $[f(z)^n f(z+c)]^{(k)} - p(z) = [z^n(z+c)]^{(k)} - p(z)$ has finitely many zeros.

(2) The condition $\alpha(z) \not\equiv 0$ can not be deleted, which can be seen by $f(z) = e^z$, $e^c = 2$, thus $[f(z)^n f(z+c)]^{(k)} = 2(n+1)^k e^{(n+1)z}$ has no zeros.

(3) The condition $N\left(r, \dfrac{1}{F}\right) = S(r,f)$ can not be removed, which can be seen by $f(z) = z(e^z - 1)$ and $e^c = -1$. It is easy to get $N\left(r, \dfrac{1}{F}\right) \neq S(r,f)$. Thus $[f(z)f(z+c)]' - 2z - c = -[2z^2 + (2c+2)z + c]e^{2z}$ has only two zeros. $[f(z)f(z+c)]'' - 2 = -[4z^2 + (4c+6)z + 3c + 2]e^{2z}$ has only two zeros.

Theorem 1.2. *Let f be a transcendental entire function of finite order. If $n \geq \dfrac{k}{2} + 1$, $k \geq 0$ and f has infinitely many multiorder zeros, then $[f(z)^n f(z+c)]^{(k)} - p(z)$ has infinitely many zeros, where $p(z)$ is a nonzero polynomial.*

Chen, Huang and Zheng [4] considered the value distribution of $f(z)f(z+c)-a$ if f has a Borel exceptional value and obtained the following result.

Theorem E ([4], Corollary 1.3). *Let $f(z)$ be a transcendental entire function of finite order. If $f(z)$ has a Borel exceptional value 0, then $f(z)f(z+c)$ takes every nonzero value a infinitely often.*

If f has a Borel exceptional polynomial, we can obtain the following results.

Theorem 1.3. *Let f be a transcendental entire function of finite order and $p(z)$ be a nonzero polynomial and $n \geq 1,\ k \geq 0$. If f has a Borel exceptional polynomial $q(z)$, then $[f(z)^n f(z+c)]^{(k)} - p(z)$ has infinitely many zeros, except that $f(z) = q(z) + Aq(z)e^{\alpha z}$, $n = 1$ and $p(z) = [q(z)q(z+c)]^{(k)}$, where $e^{\alpha c} = -1$ and A is nonzero constant.*

***Remark* 1.2.** From the example of Remark 1.1 (3), we know that the exceptional case can occur.

***Corollary* 1.1.** *Under the assumptions of Theorem 1.3. If $f(z) - q(z)$ has no zeros, then $[f(z)^n f(z+c)]^{(k)} - p(z)$ has infinitely many zeros, except that $f(z) = q + he^{\alpha z}$, $n = 1$, $q(z) = q$, $p(z) = q^2$, where $e^{\alpha c} = -1$ and h is nonzero constant.*

***Remark* 1.3.** From Theorems 1.1 – 1.3, if $n \geq 2,\ k \geq 1$ and $[f(z)^n f(z+c)]^{(k)} - p(z)$ has finitely many zeros, then $f(z)$ should satisfy: (i) f has infinitely many zeros, (ii) all zeros of f are simple, (iii) f has no Borel exceptional polynomial.

The following theorems are related to the zeros of $[f(z)^n \Delta_c f]^{(k)} - \alpha(z)$.

Theorem 1.4. *Let f be a transcendental entire function of finite order, not a periodic function with period c. If $n \geq 1,\ k \geq 0$ and $N\left(r, \dfrac{1}{F}\right) = S(r,f)$, then $[f(z)^n \Delta_c f]^{(k)} - \alpha(z)$ has infinitely many zeros.*

***Remark* 1.4.** The condition $N\left(r, \dfrac{1}{f}\right) = S(r,f)$ can not be deleted in Theorem 1.4, which can be seen by function $f(z) = e^z + z$, $e^c = 1$, it is easy to see $N\left(r, \dfrac{1}{f}\right) \neq S(r,f)$, thus $[f(z)(f(z+c) - f(z))]' - c = ce^z$ has no zeros. The condition $\alpha(z) \not\equiv 0$ can not be removed, which can be seen by function $f(z) = e^z$, $e^c = 2$, thus $[f(z)^n f(z+c)]^{(k)} = (n+1)^k e^{(n+1)z}$ has no zeros.

Theorem 1.5. *Let f be a transcendental entire function of finite order, not a periodic function with period c. If $n \geq \dfrac{k}{2} + 1,\ k \geq 0$ and f has infinitely many multiorder zeros, then $[f(z)^n (f(z+c) - f(z))]^{(k)} - p(z)$ has infinitely many zeros.*

Theorem 1.6. *Let f be a transcendental entire function of finite order, not a periodic function with period c, let $n \geq 1,\ k \geq 0$. If f has a Borel exceptional polynomial $q(z)$, then $[f(z)^n (f(z+c) - f(z))]^{(k)} - p(z)$ has infinitely many zeros, except that $f(z) = q(z) + he^{\alpha z}$, $n = 1$ and $p(z) = [q(z)(q(z+c) - q(z))]^{(k)}$, where $e^{\alpha c} = 1$.*

***Remark* 1.5.** The exceptional case also can occur in Theorem 1.6, which can be seen by $f(z) = e^z + z^2$ and $e^{\alpha c} = 1$. Then

$$\big[f(z)(f(z+c) - f(z))\big]' - [z^2(2zc + c^2)]' = (2zc + c^2 + 2c)e^z$$

has only one zero.

2. Some lemmas. The difference logarithmic derivative lemma, given by Halburd and Korhonen [8] (Theorem 2.1), [9] (Theorem 5.6), Chiang and Feng [6] (Corollary 2.5) plays an important part in considering the difference analogues of Nevanlinna theory. Here, we state the version of [9] (Theorem 5.6).

Lemma 2.1. *Let f be a transcendental meromorphic function of finite order. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = S(r, f).$$

Lemma 2.2 ([6], Theorem 2.1)**.** *Let $f(z)$ be a transcendental meromorphic function of finite order. Then*

$$T(r, f(z+c)) = T(r, f) + S(r, f).$$

Lemma 2.3 ([22], Theorem 1.22)**.** *Let $f(z)$ be a transcendental meromorphic function, n be a positive integer. Then*

$$T(r, f^{(n)}) \leq T(r, f) + n\overline{N}(r, f) + S(r, f).$$

Using Lemmas 2.1 and 2.2 and the Valiron – Mohon'ko theorem [19], we obtain the following lemma.

Lemma 2.4. *Let $f(z)$ be a transcendental meromorphic function of finite order with $N(r, f) = S(r, f)$, and let $F = f(z)^n f(z+c)$. Then*

$$T(r, F) = (n+1)T(r, f) + S(r, f).$$

***Remark* 2.1.** If $f(z)$ and $F(z)$ satisfy the conditions of Lemma 2.4, then $\rho(F) = \rho(f) = \rho(F^{(k)})$. However, the above equation is not valid for arbitrary meromorphic functions, see an example in Remark 1.1 (1).

Lemma 2.5. *Let $f(z)$ be a transcendental meromorphic function of finite order with $N(r, f) + N\left(r, \frac{1}{f}\right) = S(r, f)$. Then*

$$T(r, f(z)^n[f(z+c) - f(z)]) = (n+1)T(r, f) + S(r, f). \tag{2.1}$$

Proof. Let $G(z) = f(z)^n[f(z+c) - f(z)]$. Then

$$\frac{1}{f(z)^{n+1}} = \frac{1}{G}\left[\frac{f(z+c) - f(z)}{f(z)}\right].$$

Using the first main theorem of Nevanlinna theory, Valiron – Mohon'ko theorem [19] and Lemma 2.1, we get

$$(n+1)T(r, f) \leq T(r, G(z)) + T\left(r, \frac{f(z+c) - f(z)}{f(z)}\right) + O(1) \leq$$

$$\leq T(r, G(z)) + m\left(r, \frac{f(z+c) - f(z)}{f(z)}\right) + N\left(r, \frac{f(z+c) - f(z)}{f(z)}\right) + O(1) \leq$$

$$\leq T(r, G(z)) + N\left(r, \frac{1}{f(z)}\right) + N(r, f(z)) + N(r, f(z+c)) + S(r, f) \leq$$

$$\leq T(r, G(z)) + S(r, f),$$

thus,
$$T(r,G) \geq (n+1)T(r,f) + S(r,f). \tag{2.2}$$
On the other hand, from Lemma 2.1, we obtain
$$T(r, f(z)^n[f(z+c) - f(z)]) = m(r, f(z)^n[f(z+c) - f(z)]) + S(r,f) \leq$$
$$\leq m\left(r, f(z)^{n+1}\left[\frac{f(z+c)}{f(z)} - 1\right]\right) + S(r,f) \leq$$
$$\leq (n+1)T(r,f) + S(r,f). \tag{2.3}$$
Combining (2.2) with (2.3), (2.1) follows.

***Remark* 2.2.** Remove the condition $N\left(r, \dfrac{1}{f}\right) = S(r,f)$, for entire function $f(z)$, we only get
$$nT(r,f) + S(r,f) \leq T(r, f(z)^n[f(z+c) - f(z)] \leq (n+1)T(r,f) + S(r,f).$$

Following Hayman [12, p. 75, 76], we define an ε-set E to be a countable union of discs not containing the origin, and subtending angles at the origin whose sum is finite. If E is an ε-set, then the set of $r \geq 1$ for which the circle $S(0,r)$ meets E has finite logarithmic measure.

Lemma 2.6 [2]. *Let $g(z)$ be a transcendental meromorphic function of order $\sigma(f) < 1$, $h > 0$. Then there exists an ε-set E such that*
$$\frac{g'(z+c)}{g(z+c)} \to 0 \qquad \textit{and} \qquad \frac{g(z+c)}{g(z)} \to 1 \qquad \textit{as} \quad z \to \infty \quad \textit{in} \quad \mathbb{C} \setminus E,$$
uniformly in c for $|c| \leq h$. Further, E may be chosen so that for large $z \notin E$, the function g has no zeros or poles in $|\zeta - z| \leq h$.

The following result is needed to prove Theorems 1.1 and 1.4.

Lemma 2.7 ([23], Lemma 1). *Let f be a nonconstant meromorphic function and $\alpha(z)$ be a small function of f such that $\alpha(z) \not\equiv 0, \infty$. Then*
$$T(r,f) \leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - \alpha}\right) - N\left(r, \frac{1}{\left(f^{(k)}/\alpha\right)'}\right) + S(r,f).$$

Lemma 2.8 ([22], Theorem 1.62). *Let $f_j(z)$ be a meromorphic functions, $f_k(z)$, $k = 1, 2, \ldots, n-$ -1, are not constants, satisfying $\sum_{j=1}^{n} f_j = 1$ and $n \geq 3$. If $f_n(z) \not\equiv 0$, and*
$$\sum_{j=1}^{n} N\left(r, \frac{1}{f_j}\right) + (n-1)\sum_{j=1}^{n} \overline{N}(r, f_j) < (\lambda + o(1))T(r, f_k),$$
where $\lambda < 1$, $k = 1, 2, \ldots, n-1$, then $f_n(z) \equiv 1$.

Lemma 2.9 ([22], Theorem 1.51). *Let $f_j(z)$, $j = 1, 2, \ldots, n$, $n \geq 2$, be a meromorphic functions, $g_j(z)$, $j = 1, 2, \ldots, n$, be entire functions satisfying*:

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j(z)} \equiv 0$,

(ii) *when $1 \leq j < k \leq n, g_j(z) - g_k(z)$ is not a constant,*

(iii) *when* $1 \le j \le n,\ 1 \le h < k \le n,\ T(r, f_j) = o(T(r, e^{g_h - g_k}))\ (r \to \infty,\ r \notin E)$, *where* $E \subset (1, \infty)$ *is of finite linear measure or finite logarithmic measure.*
Then $f_j(z) \equiv 0,\ j = 1, 2, \ldots, n$.

The following results are related to the growth of solutions of linear difference equations which are needed for the proofs of Theorems 1.3 and 1.6. Here, we give the version with small changes of the type of equations, the proofs are similar.

Lemma 2.10 ([6], Theorem 9.2)**.** *Let* $A_0(z), \ldots, A_n(z)$ *be a entire functions such that there exists an integer* $l,\ 0 \le l \le n$, *such that*

$$\rho(A_l(z)) > \max_{0 \le l \le n, j \ne l} \rho(A_j(z)).$$

If $f(z)$ *is a meromorphic solution of*

$$A_n(z)y(z + c_n) + \ldots + A_1(z)y(z + c_1) + A_0(z)y(z) = 0,$$

then $\rho(f) \ge \rho(A_l(z)) + 1$.

Lemma 2.11 ([5], Theorem 1.2)**.** *Let* $P_0(z), \ldots, P_n(z)$ *be a polynomials such that* $P_n(z)P_0(z) \not\equiv$ $\not\equiv 0$ *and satisfy*

$$\deg(P_n(z) + \ldots + P_0(z)) = \max\{\deg P_j(z) \colon j = 0, \ldots, n\} \ge 1.$$

Then every finite order meromorphic solution $f(z) (\not\equiv 0)$ *of*

$$P_n(z)f(z + c_n) + \ldots + P_1(z)f(z + c_1) + P_0(z)f(z) = 0$$

satisfies $\rho(f) \ge 1$.

3. Proofs of Theorems 1.1 and 1.4. Let $F(z) = f(z)^n f(z + c)$. From the conditions, it is easy to get $N(r, f) + N\left(r, \dfrac{1}{f}\right) = S(r, f)$, thus

$$N(r, F) + N\left(r, \frac{1}{F}\right) = S(r, f). \tag{3.1}$$

From Lemmas 2.4, 2.7 and (3.1), we get

$$(n + 1)T(r, f) + S(r, f) = T(r, F) \le$$
$$\le \overline{N}(r, F) + N\left(r, \frac{1}{F}\right) + N\left(r, \frac{1}{F^{(k)} - \alpha}\right) + S(r, F) \le$$
$$\le N\left(r, \frac{1}{F^{(k)} - \alpha}\right) + S(r, f).$$

Hence, $[f(z)^n f(z + c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Let $G(z) = f(z)^n[f(z + c) - f(z)]$. Since $N(r, f) + N\left(r, \dfrac{1}{F}\right) = S(r, f)$, then

$$N(r, G) + N\left(r, \frac{1}{G}\right) \le N\left(r, \frac{1}{f(z + c) - f(z)}\right) \le T(r, f) + S(r, f).$$

From Lemmas 2.5, 2.7 and above inequality, we get

$$(n+1)T(r,f)+S(r,f)=T(r,G)\le \overline{N}(r,G)+N\left(r,\frac{1}{G}\right)+N\left(r,\frac{1}{G^{(k)}-\alpha}\right)+S(r,G)\le$$

$$\le T(r,f)+N\left(r,\frac{1}{G^{(k)}-\alpha}\right)+S(r,f).$$

Then $nT(r,f)\le N\left(r,\dfrac{1}{G^{(k)}-\alpha}\right)+S(r,f)$. Since $n\ge 1$, then $[f(z)^n(f(z+c)-f(z))]^{(k)}-\alpha(z)$ has infinitely many zeros.

4. Proofs of Theorems 1.2 and 1.5. We first to give the proof of Theorem 1.2. Let $F(z) = f(z)^n f(z+c)$. Assume that $F^{(k)}(z)-p(z)$ has finitely many zeros, from Hadamard factorization theorem, then

$$F^{(k)}(z)-p(z)=h(z)e^{q(z)}, \tag{4.1}$$

where $h(z)$ is a nonzero polynomial, $q(z)$ is a nonconstant polynomial, otherwise, if $q(z)=A$, where A is a constant, then $F^{(k)}(z)-p(z)=h(z)e^{A}$. It implies that $F(z)=f(z)^n f(z+c)$ is also a polynomial, which is a contradiction with Lemma 2.4. Differentiating (4.1), we get

$$F^{(k+1)}(z)-p'(z)=[h'(z)+h(z)q'(z)]e^{q(z)}. \tag{4.2}$$

Combining (4.1) with (4.2), eliminating $e^{q(z)}$, we have

$$\frac{F^{(k+1)}(z)}{F^{(k)}(z)}=\frac{h'(z)+h(z)q(z)}{h(z)}+\left[p'(z)-\frac{h'(z)+h(z)q(z)}{h(z)}p(z)\right]\frac{1}{F^{(k)}(z)}. \tag{4.3}$$

From the left-hand side of (4.3), we remark that the poles of $\dfrac{F^{(k+1)}(z)}{F^{(k)}(z)}$ must be simple. If f has infinitely many multiorder zeros and $n\ge\dfrac{k}{2}+1$, we can find z_0 which is a zero of f and not the zero of $h(z)$ and $p'(z)-\dfrac{h'(z)+h(z)q(z)}{h(z)}p(z)$, then the poles of right-hand side of (4.3) must be multiorder, a contradiction. Thus, we have the proof of Theorem 1.2.

Using the similar method as above, we can get the proof of Theorem 1.5.

5. Proof of Theorem 1.3. Since $q(z)$ is a Borel exceptional polynomial of $f(z)$, thus the value 1 is a Borel exceptional value of $\dfrac{f(z)}{q(z)}$, then $f(z)$ must have positive integer order [22, p. 106] (Corollary). Assume that $\rho(f)=s$, s is a positive integer, then transcendental entire function $f(z)$ can be written as $f(z)=q(z)+h(z)e^{\alpha z^s}$, where α is a nonzero constant and $h(z)$ is a nonzero entire function with $\lambda(h)\le\rho(h)<\rho(f)=s$. Hence,

$$f(z+c)=q(z+c)+h(z+c)e^{\alpha(z+c)^s}=q(z+c)+h_1(z)e^{\alpha z^s}, \tag{5.1}$$

where

$$h_1(z)=h(z+c)e^{\alpha(C_s^1z^{s-1}c+C_s^2z^{s-2}c^2+\ldots+C_s^{s-1}zc^{s-1}+c^s)}. \tag{5.2}$$

Suppose that $\left[f(z)^nf(z+c)\right]^{(k)}-p(z)$ has finitely many zeros, from Hadamard factorization theorem and $\rho\big(f(z)^nf(z+c)\big)=\rho(f)=\rho\big((f(z)^nf(z+c))^{(k)}\big)$, then we assume that

$$[f(z)^nf(z+c)]^{(k)}-p(z)=A(z)e^{\beta z^s}, \tag{5.3}$$

where $A(z)$ is an entire function with $\rho(A)<s$ and has finitely many zeros, β is a nonzero constant.

Case 1. If $k=0$, then $n=1$ from Theorem C, hence,

$$f(z)f(z+c)-p(z)=A(z)e^{\beta z^s}. \tag{5.4}$$

Substituting $f(z)=q(z)+h(z)e^{\alpha z^s}$ into (5.4), we get

$$h(z)h_1(z)e^{2\alpha z^s}+(h(z)q(z+c)+q(z)h_1(z))e^{\alpha z^s}=$$

$$=p(z)-q(z)q(z+c)-A(z)e^{\beta z^s}. \tag{5.5}$$

Combining (5.5) with the definition of type of entire function $f(z)$, that is

$$\tau_f=\limsup_{r\to\infty}\frac{\log M(r,f)}{r^{\rho(f)}},$$

we get $\beta=2\alpha$, where $M(r,f)=\max_{|z|=r}|f(z)|$.

Case 1.1. If $p(z)-q(z)q(z+c)\equiv 0$, from Lemma 2.9 and (5.5), then

$$(h(z)q(z+c)+q(z)h_1(z))e^{\alpha z^s}\equiv 0.$$

It implies that

$$q(z+c)h(z)+q(z)h(z+c)e^{\alpha(C_s^1z^{s-1}c+C_s^2z^{s-2}c^2+\ldots+C_s^{s-1}zc^{s-1}+c^s)}\equiv 0.$$

We affirm that $s=1$, otherwise, if $s\geq 2$, from Lemma 2.10, then $\rho(h)\geq s$ follows, which is a contradiction with $\rho(h)<s$. Thus,

$$q(z+c)h(z)+q(z)h(z+c)e^{\alpha c}\equiv 0. \tag{5.6}$$

Combining (5.6), $\rho(h)<1$ with Lemma 2.11, we get the degree of $q(z+c)+q(z)e^{\alpha c}$ must be less than the degree of $q(z)$. If $q(z)$ is not a constant, then $e^{\alpha c}=-1$. Hence,

$$q(z+c)h(z)-q(z)h(z+c)\equiv 0,$$

which implies that $\dfrac{h(z+c)}{q(z+c)}=\dfrac{h(z)}{q(z)}$, thus $H(z)=\dfrac{h(z)}{q(z)}$ is a periodic function with period c. Since $\rho(H)=\rho(h)<1$, thus $H(z)$ must be a constant A, which implies that $h(z)$ must be a polynomial with the form $h(z)=Aq(z)$.

If $q(z)$ is a constant, we have $h(z)+h(z+c)e^{\alpha c}\equiv 0$. From Lemma 2.6, we get $e^{\alpha c}=-1$. Since that $\rho(h)<s=1$, then $h(z)$ must be a constant. Thus, we get $h(z)$ and $q(z)$ are two constants, which can be written as $h(z)=Aq(z)$.

Thus, we get $f(z)=q(z)+Aq(z)e^{\alpha z}$, where $e^{\alpha c}=-1$.

Case 1.2. If $p(z)-q(z)q(z+c)\not\equiv 0$, then

$$[h(z)h_1(z)-A(z)]e^{2\alpha z^s}+(h(z)q(z+c)+q(z)h_1(z))e^{\alpha z^s}=$$

$$=p(z)-q(z)q(z+c).$$

Let $f_1(z)=\dfrac{[h(z)h_1(z)-A(z)]}{p(z)-q(z)q(z+c)}e^{2\alpha z^s}$ and $f_2(z)=\dfrac{(h(z)q(z+c)+q(z)h_1(z))}{p(z)-q(z)q(z+c)}e^{\alpha z^s}$. Thus, $f_1(z)+$ $+f_2(z)=1$. From the second main theorem, we get

$$T(r,f_1)\leq N(r,f_1)+N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{f_1-1}\right)+S(r,f_1)\leq$$
$$\leq N\left(r,\frac{1}{f_1}\right)+N\left(r,\frac{1}{f_2}\right)+S(r,f_1)\leq$$
$$\leq O(r^{s-1+\varepsilon})+S(r,f_1),$$

which is a contradiction with $\rho(f_1)=s$. Thus, $[f(z)^nf(z+c)]^{(k)}-p(z)$ has infinitely many zeros.

Case 2. If $k\geq 1$, from (5.1) and (5.3), we have

$$\left[(q(z)+h(z)e^{\alpha z^s})^n(q(z+c)+h_1(z)e^{\alpha z^s})\right]^{(k)}-A(z)e^{\beta z^s}=p(z),$$

which implies that

$$\big[q(z)^nq(z+c)+B_1(z)e^{\alpha z^s}+\ldots+B_j(z)e^{j\alpha z^s}+\ldots$$
$$\ldots+B_n(z)e^{n\alpha z^s}+h(z)h_1(z)e^{(n+1)\alpha z^s}\big]^{(k)}-A(z)e^{\beta z^s}=p(z), \tag{5.7}$$

where

$$B_j(z)=C_n^jq(z)^{n-j}q(z+c)h(z)^j+C_n^{j-1}q(z)^{n-j+1}h_1(z)h(z)^{j-1},$$

$\rho(B_j(z))<s,\ j=1,\ldots,n$. For any integer k, from (5.7), we obtain

$$D_1(z)e^{\alpha z^s}+D_2(z)e^{2\alpha z^s}+\ldots+D_j(z)e^{j\alpha z^s}+\ldots$$
$$\ldots+D_n(z)e^{n\alpha z^s}+D_{n+1}(z)e^{(n+1)\alpha z^s}-A(z)e^{\beta z^s}=p(z)-[q(z)^nq(z+c)]^{(k)}, \tag{5.8}$$

where $D_j(z)$ are differential polynomials of $h(z)$, $h_1(z)$, $q(z)$, $q(z+c)$ and their powers derivatives and $\rho(D_j(z))<s,\ j=1,\ldots,n+1$. In the following, we will consider two cases.

Case 2.1. If $p(z)-[q(z)^nq(z+c)]^{(k)}\equiv 0$, from Lemma 2.9, then all $D_j(z)\equiv 0,\ j=1,2,\ldots,n$, and $D_{n+1}(z)-A(z)\equiv 0$. We affirm that $n=1$, otherwise, let $n\geq 2$. If $k=1$, from $D_1(z)\equiv 0$, then we get

$$B_1'(z)+\alpha sz^{s-1}B_1(z)=0,$$

which implies the nontrivial solution $B(z)$ of the above first order differential equation satisfying $\rho(B(z))=s$, which is a contradiction with $\rho(B(z))<s$, thus, $B_1(z)\equiv 0$. If $k=2$, let $g(z)=B_1'(z)+\alpha sz^{s-1}B_1(z)$, then we have $g'(z)+\alpha sz^{s-1}g(z)=0$, which also implies $\rho(g(z))=s$, a contradiction with $\rho(g(z))=\rho(B(z))<s$. Using this method for any positive integer k, we can get $B_1(z)\equiv 0$, that is

$$C_n^1q(z)^{n-1}q(z+c)h(z)+q(z)^nh(z+c)e^{\alpha(C_s^1z^{s-1}c+\ldots+C_s^{s-1}zc^{s-1}+c^s)}\equiv 0.$$

From Lemma 2.10, we get $s=1$, which implies that

$$C_n^1q(z)^{n-1}q(z+c)h(z)+q(z)^nh(z+c)e^{\alpha c}\equiv 0. \tag{5.9}$$

From Lemma 2.11, since $\rho(h(z))<1$, then the degree of $C_n^1q(z)^{n-1}q(z+c)+q(z)^ne^{\alpha c}$ must be less than the degree of $q(z)^n$. Thus, we get

$$e^{\alpha c}=-C_n^1 \tag{5.10}$$

provided that $q(z)$ is not a constant. Using the same discussions as above, we also have $B_2(z) \equiv 0$, that is,

$$C_n^2 q(z)^{n-2} q(z+c) h(z)^2 + C_n^1 q(z)^{n-1} h(z+c) e^{\alpha c} h(z) \equiv 0.$$

Similar as the above, if $q(z)$ is not a constant, we obtain

$$C_n^1 e^{\alpha c} = -C_n^2. \tag{5.11}$$

From (5.10) and (5.11), we get $n = 0$, a contradiction. Thus, we have $n = 1$. From (5.7), Lemmas 2.9 and 2.8, we obtain $B_1(z) \equiv 0$. Then (5.9) is the same to (5.6). Thus, we get $f(z) = q(z) + Aq(z)e^{\alpha z}$, where $e^{\alpha c} = -1$.

Case 2.2. If $p(z) - [q(z)^n q(z+c)]^{(k)} \not\equiv 0$, combining $\rho(D_j(z)) < s$, $\rho(A(z)) < s$ with (5.8) and Lemma 2.8, we get

$$[D_{n+1}(z) - A(z)] e^{(n+1)\alpha z^s} = p(z) - [q(z)^n q(z+c)]^{(k)}$$

or

$$D_j(z) e^{j\alpha z^s} = p(z) - [q(z)^n q(z+c)]^{(k)},$$

which are impossible. Thus $[f(z)^n f(z+c)]^{(k)} - p(z)$ has infinitely many zeros.

Theorem 1.3 is proved.

6. Proof of Theorem 1.6. Similar as the beginning of proof of Theorem 1.3, $f(z)$ also can be written as $f(z) = q(z) + h(z)e^{\alpha z^s}$, where α is a nonzero constant and $h(z)$ is a nonzero entire function with $\lambda(h) \le \rho(h) < \rho(f) = s$. From (5.1), we have

$$f(z+c) - f(z) = q(z+c) - q(z) + (h_1(z) - h(z))e^{\alpha z^s} := q_1(z) + h_2(z)e^{\alpha z^s}.$$

Suppose that $[f(z)^n (f(z+c) - f(z))]^{(k)} - p(z)$ has finitely many zeros, then from Hadamard factorization theorem, we obtain

$$\big[f(z)^n (f(z+c) - f(z))\big]^{(k)} - p(z) = B(z)e^{\gamma z^s},$$

where $B(z)$ is an entire function of finitely many zeros with order $\rho(B) < s$ and γ is a nonzero constant.

Case 1. If $k = 0$. From Theorem C, we get $n = 1$, hence,

$$f(z)\big[f(z+c) - f(z)\big] - p(z) = B(z)e^{\gamma z^s}. \tag{6.1}$$

Substituting $f(z) = q(z) + h(z)e^{\alpha z^s}$ into (6.1), we have

$$\begin{aligned} h(z)(h_1(z) - h(z))e^{2\alpha z^s} + (h(z)(q(z+c) - 2q(z)) + q(z)h_1(z))e^{\alpha z^s} = \\ = p(z) - \big[q(z)(q(z+c) - q(z))\big] + B(z)e^{\gamma z^s}. \end{aligned} \tag{6.2}$$

Case 1.1: $p(z) - q(z)(q(z+c) - q(z)) \equiv 0$. If $h_1(z) \equiv h(z)$, then $\alpha = \gamma$ follows from the above equation. From (5.2) and Lemma 2.10, we get $s = 1$, thus $h(z) = e^{\alpha c} h(z+c)$. From Lemma 2.6, we get $h(z)$ is a constant and $e^{\alpha c} = 1$. Thus, $f(z) = q(z) + he^{\alpha z}$, where $e^{\alpha c} = 1$.

If $h_1(z) \not\equiv h(z)$, then $\gamma = 2\alpha$. From Lemma 2.9, we obtain

$$h(z)\big[q(z+c)-2q(z)\big]+q(z)h(z+c)e^{\alpha(C_s^1 z^{s-1}c+C_s^2 z^{s-2}c^2+\ldots+C_s^{s-1}zc^{s-1}+c^s)}=0.$$

From Lemma 2.10, we get $s=1$, thus,

$$h(z)\big[q(z+c)-2q(z)\big]+q(z)h(z+c)e^{\alpha c}=0.$$

Since $\rho(h)<1$, from Lemma 2.11, then $q(z)$ must be a constant or $e^{\alpha c}=1$ when $q(z)$ is not a constant. If $q(z)$ is a constant, from Lemma 2.6, then $h(z)$ reduces a constant, thus $f(z)$ is a periodic function with period c, a contradiction. If $e^{\alpha c}=1$ and $q(z)$ is not a constant, from (6.2), we have $h(z)(h(z+c)-h(z))=B(z)$, combining with $\rho(h)<1$ and $h(z)$ is an entire function, we obtain $\rho(h(z+c)-h(z))<1$, thus $B(z)$ must have infinitely many zeros, a contradiction.

Case 1.2: $p(z)-q(z)(q(z+c)-q(z))\not\equiv 0$, this case is similar as the Case 1.2 in the proof of Theorem 1.3.

Case 2. If $k\geq 1$, we also can get a similar equation as (5.8),

$$E_1(z)e^{\alpha z^s}+\ldots+E_j(z)e^{j\alpha z^s}+\ldots+E_n(z)e^{n\alpha z^s}E_{n+1}(z)e^{(n+1)\alpha z^s}-B(z)e^{\beta z^s}=$$
$$=p(z)-\big[q(z)^n(q(z+c)-q(z))\big]^{(k)},$$

where E_j are differential polynomials of $h(z)$, $h_2(z)$, $q(z)$ and $q(z+c)$ and their powers derivatives and $\rho(E_j)<s$. In the following, we also divided into two cases $p(z)-[q(z)^n(q(z+c)-q(z))]^{(k)}\equiv 0$ and $p(z)-[q(z)^n(q(z+c)-q(z))]^{(k)}\not\equiv 0$. Using the similar method as the Case 2 of the proof of Theorem 1.3, then we can have the completed proof of Theorem 1.6.

7. Discussions. In the paper, we gave the example to show that $[f(z)f(z+c)]'-\alpha(z)$ and $[f(z)(f(z+c)-f(z))]'-\alpha(z)$ can admit finitely many zeros in Remarks 1.1, 1.4, and 1.5. By theorems of the paper, we see that if f be a transcendental entire function of finite order and $n\geq 2$, $k\geq 1$, then $[f(z)^n f(z+c)]^{(k)}-\alpha(z)$ or $[f(z)^n(f(z+c)-f(z))]^{(k)}-\alpha(z)$ have infinitely many zeros under some additional conditions, for example $N\left(r,\dfrac{1}{F}\right)=S(r,f)$ or $f(z)$ has infinitely many multiorder zeros or $f(z)$ has a Borel exceptional polynomial. Unfortunately, we have no methods to remove these additional conditions. For the further studying, we raise the following two questions, where $\alpha(z)$ is a nonzero small function with respect to $f(z)$.

Question 1: Let f be a transcendental entire function of finite order and $n\geq 2,\ k\geq 1$. Can we get $[f(z)^n f(z+c)]^{(k)}-\alpha(z)$ have infinitely many zeros?

Question 2: Let f be a transcendental entire function of finite order, not a periodic function with period c and $n\geq 2,\ k\geq 1$. Can we get $[f(z)^n(f(z+c)-f(z))]^{(k)}-\alpha(z)$ have infinitely many zeros?

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