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THE NEHARI MANIFOLD APPROACH FOR A $p(x)$-LAPLACIAN PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS

ПІДХІД НА ОСНОВІ МНОГОВИДУ НЕХАРІ ДО ПРОБЛЕМИ $p(x)$-ЛАПЛАСІАНА З НЕЛІНІЙНИМИ ГРАНИЧНИМИ УМОВАМИ

We consider a class of $p(x)$-Laplacian equations that involve nonnegative weight functions with nonlinear boundary conditions. Our technical approach is based on the Nehari manifold, which is similar to the fibering method of Drabek and Pohozaev, together with the recent idea from Brown and Wu.

Розглянуто один клас $p(x)$-рівнянь Лапласа, що включає невід'ємні вагові функції з нелінійними граничними умовами. Наш підхід базується на многовиді Нехарі, що є подібним до методу волокон Драбека та Похожаєва з використанням нових ідей Брауна та Ву.

1. Introduction. The purpose of this paper is to study the existence and multiplicity of positive solutions for the following nonlinear boundary-value problem involving the $p(x)$-Laplacian:

$$\begin{aligned} &-\Delta_{p(x)}u + m(x)|u|^{p(x)-2}u = \lambda f(x)|u|^{q(x)-2}u, \quad x \in \Omega, \\ &|\nabla u|^{p(x)-2}\frac{\partial u}{\partial n} = g(x)|u|^{r(x)-2}u, \quad x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $p(x), q(x), r(x) \in C(\overline{\Omega})$ such that $1 < q(x) < < p(x) < r(x) < p^*(x)$ $\left(p^*(x) = \dfrac{Np(x)}{N-p(x)} \text{ if } N > p(x),\ p^*(x) = \infty \text{ if } N \leq p(x)\right)$, $1 < p^- := := \operatorname{ess\,inf}_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x) < \infty$, $1 < q^- \leq q^+ < p^- \leq p^+ < r^- \leq r^+$, $\lambda > 0 \in \mathbb{R}$ the weight $m(x)$ is a positive bounded function and $f \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$ are nonnegative weight functions with compact support in Ω.

In this paper, we have generalized the articles of Afrouzi and Rasouli [1] and Wu [19–21], to the $p(x)$-Laplacian by using the Nehari manifold under the certain conditions.

2. The space $W^{1,p(x)}(\Omega)$. To discuss problem (1.1) we need some results on the space $W^{1,p(x)}(\Omega)$ which we call variable exponent Sobolev space.

Let Ω be a bounded domain of $\mathbb{R}^n$, we have

$$L^\infty_+(\Omega) = \{p \in L^\infty(\Omega) : p^- > 1\}.$$

Let's define by $U(\Omega)$ the set of all measurable real functions defined on Ω. For any $p \in L^\infty_+(\Omega)$,

$$L^{p(x)}(\Omega) = \left\{ u \in U(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < \infty \right\},$$

we introduce a norm on $L^{p(x)}(\Omega)$, that so-called Luxemburg norm [11, 14],

$$|u|_{p(x)} = \inf\left\{ \delta > 0 : \int_\Omega \left|\frac{u(x)}{\delta}\right|^{p(x)} dx \leq 1 \right\},$$

then $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, we call it as variable exponent Lebesgue space, and

$$L^{p(x)}_{c(x)}(\Omega) = \left\{ u \in U(\Omega) \colon \int\limits_{\Omega} c(x)|u(x)|^{p(x)}\,dx < \infty \right\},$$

where c is a measurable real-valued function and $c(x) > 0$ for $x \in \Omega$.

We define

$$|u|_{(p(x)\cdot c(x))} = \inf \left\{ \delta > 0 \colon \int\limits_{\Omega} c(x) \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\}.$$

Let us define the space

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \colon |\nabla u| \in L^{p(x)}(\Omega)\}.$$

In the Banach space $W^{1,p(x)}(\Omega)$ we introduce the norm which is equivalent to the standard one:

$$\|u\|^p = \left(\int\limits_{\Omega} (|\nabla u|^{p(x)} + m(x)|u(x)|^{p(x)})\,dx \right)^{p(x)} \qquad \forall u \in W^{1,p(x)}(\Omega).$$

Let $p^*_\partial(x) = \dfrac{(N-1)\,p(x)}{N - p(x)}$ if $N > p(x)$.

Theorem 2.1 [11]. *The space* $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ *is a separable, uniformly convex Banach space, and has conjugate* $L^{p'(x)}(\Omega)$, *where* $\dfrac{1}{p'(x)} + \dfrac{1}{p(x)} = 1$. *For any* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$, *we have*

$$\left| \int\limits_{\Omega} uv\,dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}.$$

Theorem 2.2 [11]. *Let* $\rho(u) = \int_{\Omega} |u(x)|^{p(x)}\,dx \;\forall u \in L^{p(x)}(\Omega)$, *then*

(i) $|u|_{p(x)} < 1\ (= 1;\ > 1)$ *if and only if* $\rho(u) < 1\ (= 1; > 1)$,

(ii) $|u|_{p(x)} > 1$ *implies* $|u|^{p^-}_{p(x)} \leq \rho(u) \leq |u|^{p^+}_{p(x)}$,

(iii) $|u|_{p(x)} < 1$ *implies* $|u|^{p^+}_{p(x)} \leq \rho(u) \leq |u|^{p^-}_{p(x)}$.

Theorem 2.3 [11]. *Let* $p(x)$ *and* $q(x)$ *be measurable functions such that* $p(x) \in L^\infty(\Omega)$ *and* $1 \leq p(x)q(x) \leq \infty$ *for a.e.* $x \in \Omega$. *Let* $u \in L^{q(x)}(\Omega)$. *Then*

$$|u|_{p(x)q(x)} \leq 1 \quad \textit{implies} \quad |u|^{p^+}_{p(x)q(x)} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$

$$|u|_{p(x)q(x)} \geq 1 \quad \textit{implies} \quad |u|^{p^-}_{p(x)q(x)} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ *is constant, then*

$$\left| |u|^p \right|_{q(x)} = |u|^p_{pq(x)}.$$

Theorem 2.4 [11]. *If* $u, u_n \in L^{p(x)}(\Omega),\ n = 1, 2, \ldots,$ *then*
(i) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0,$
(ii) $\lim_{n\to\infty} \rho(u_n - u) = 0,$
(iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho(u_n) = \rho(u)$
are equivalent.

Theorem 2.5 [11]. *If* $p^- > 1$ *and* $p^+ < \infty$, *then the spaces* $L^{p(x)}(\Omega)$, $L^{p(x)}_{c(x)}(\Omega)$, *and* $W^{1,p(x)}(\Omega)$ *are separable and reflexive Banach spaces.*

Theorem 2.6 [11]. (i) *Let* $p \in C(\overline{\Omega})$ *and* $\partial\Omega$ *possesses the cone property. If* $q \in C(\overline{\Omega})$ *and* $1 \leq q(x) < p^*(x)$ *for any* $x \in \overline{\Omega}$, *then* $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$.

(ii) *Let* $p \in C(\overline{\Omega})$ *and* $\partial\Omega$ *possesses the cone property. If* $q \in C(\overline{\Omega})$ *and* $1 \leq q(x) < p^*_\partial(x)$ *for any* $x \in \overline{\Omega}$, *then* $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}(\partial\Omega)$.

(iii) *If* $p, q \in C(\overline{\Omega})$ *and* $p(x) \leq q(x) \leq p^*(x)$ *for any* $x \in (\overline{\Omega})$, *then* $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ *and*

$$|u|_{q(x)} \leq c\|u\| \quad \forall u \in W_0^{1,p(x)}(\Omega),$$

where $c > 0$ *is constant.*

Theorem 2.7. *Let* $p \in C(\overline{\Omega})$ *and* $\partial\Omega$ *possesses the cone property. Suppose that* $g \in L^{\beta(x)}(\partial\Omega)$, $g(x) > 0$ *for* $x \in \Omega, \beta \in C(\overline{\Omega})$ *and* $\beta^- > 1,\ \beta_0^- \leq \beta_0(x) \leq \beta_0^+$, *where* $\beta_0(x) = \dfrac{\beta(x)}{\beta(x) - 1}$. *If* $r \in C(\overline{\Omega})$ *and*

$$1 < r(x) < \frac{\beta(x) - 1}{\beta(x)} p^*_\partial(x) \quad \forall x \in \overline{\Omega}, \tag{2.1}$$

or

$$1 < \beta(x) < \frac{N\beta(x)}{N\beta(x) - r(x)(N - p(x))},$$

then $W^{1,p(x)}(\Omega) \hookrightarrow L^{r(x)}_{g(x)}(\partial\Omega)$ *is compact. Moreover, there is a constant* $c_5 > 0$ *such that the inequality*

$$\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS \leq c_5\big(\|u\|^{r^-} + \|u\|^{r^+}\big) \tag{2.2}$$

holds.

Proof. We must remark that our proof of the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}_{g(x)}(\partial\Omega)$ is similar to Fan [12]. Let $u \in W^{1,p(x)}(\Omega)$ and set $h(x) = \dfrac{\beta(x)}{\beta(x) - 1} r(x) = \beta_0(x) r(x)$. Then (2.1) implies $h(x) < p^*_\partial(x)$. Hence, by Theorem 2.6 we have the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{h(x)}(\partial\Omega)$. So, for $u \in W^{1,p(x)}(\Omega)$, we have $|u|^{r(x)} \in L^{\beta_0(x)}(\partial\Omega)$. By Theorem 2.1,

$$\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS \leq c_1 |g|_{\beta(x)} \big||u|^{r(x)}\big|_{\beta_0(x)} < \infty.$$

This implies that $W^{1,p(x)}(\Omega) \subset L^{r(x)}_{g(x)}(\partial\Omega)$. Now let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ and

$$u_n \rightharpoonup 0 \quad \text{(weakly) in} \quad W^{1,p(x)}(\Omega).$$

Then we have

$$u_n \to 0 \quad \text{(strongly) in} \quad L^{h(x)}(\partial\Omega).$$

So, it follows that $\big||u_n|^{r(x)}\big|_{\beta_0(x)} \to 0$. Thus,

$$\int\limits_{\partial\Omega} g(x)|u_n|^{r(x)}\,dS \le c_1|g|_{\beta(x)}\big||u_n|^{r(x)}\big|_{\beta_0(x)} \to 0,$$

which implies $|u_n|_{(r(x),g(x))} \to 0$. Hence, $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}_{g(x)}(\partial\Omega)$.

Now let's show the inequality (2.2) holds. Since $r^- \le r(x) \le r^+$ and $|u|^{r(x)} \le |u|^{r^-} + |u|^{r^+}$, thus

$$\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\,dS \le \int\limits_{\partial\Omega} g(x)|u|^{r^-}\,dS + \int\limits_{\partial\Omega} g(x)|u|^{r^+}\,dS,$$

and $r^-\beta_0(x) \le r^+\beta_0(x) < p^*_\partial(x)$, we get

$$\int\limits_{\partial\Omega} g(x)|u|^{r^-}\,dS \le c_2|g|_{\beta(x)}\Big||u|^{r^-}\Big|_{\beta_0(x)} = c_2|g|_{\beta(x)}|u|^{r^-}_{r^-\beta_0(x)} \le c_3\|u\|^{r^-}. \tag{2.3}$$

Moreover,

$$\int\limits_{\partial\Omega} g(x)|u|^{r^+}\,dS \le c_4\|u\|^{r^+}. \tag{2.4}$$

As a result, from (2.3) and (2.4) it follows that

$$\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\,dS \le c_5(\|u\|^{r^-} + \|u\|^{r^+}).$$

Theorem 2.7 is proved.

Theorem 2.8. *Let* $p \in C(\overline{\Omega})$ *and* $\partial\Omega$ *possesses the cone property. Suppose that* $f \in L^{\alpha(x)}(\Omega)$, $f(x) > 0$ *for* $x \in \Omega$, $\alpha \in C(\overline{\Omega})$ *and* $\alpha^- > 1$, $\alpha_0^- \le \alpha_0(x) \le \alpha_0^+$, *where* $\alpha_0(x) = \dfrac{\alpha(x)}{\alpha(x)-1}$. *If* $q \in C(\overline{\Omega})$, $p(x) < \dfrac{\alpha(x)}{\alpha(x)-1}q(x)$ *and*

$$1 < q(x) < \frac{\alpha(x)-1}{\alpha(x)}p^*(x) \quad \forall x \in \overline{\Omega} \tag{2.5}$$

or

$$\frac{Np(x)}{Np(x) - q(x)(N-p(x))} < \alpha(x) < \frac{p(x)}{p(x)-q(x)},$$

then $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}_{f(x)}(\Omega)$ *is compact. Moreover, there is a constant* $c_7 > 0$ *such that the following inequality is holds*:

$$\int\limits_{\Omega} f(x)|u|^{q(x)}\,dx \le c_7(\|u\|^{q^-} + \|u\|^{q^+}). \tag{2.6}$$

Proof. Let $u \in W^{1,p(x)}(\Omega)$. Set $a(x) = \dfrac{\alpha(x)}{\alpha(x)-1}q(x) = \alpha_0(x)q(x)$. Then (2.5) implies $a(x) < p^*(x)$. Hence, by Theorem 2.6 there is the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{m(x)}(\Omega)$. For $u \in W^{1,p(x)}(\Omega)$ we have $|u|^{q(x)} \in L^{\alpha_0(x)}(\Omega)$. By Theorem 2.1,

$$\int\limits_{\Omega} f(x)|u|^{q(x)}\,dx \le c_6|f|_{\alpha(x)}\big||u|^{q(x)}\big|_{\alpha_0(x)}(\Omega) \to 0.$$

This implies that $W^{1,p(x)}(\Omega) \subset L^{q(x)}_{f(x)}(\Omega)$. Now let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ and

$$u_n \rightharpoonup 0 \quad \text{in} \quad W^{1,p(x)}(\Omega).$$

Then we obtain

$$u_n \to 0 \quad \text{in} \quad L^{m(x)}(\Omega).$$

So $\big||u_n|^{q(x)}\big|_{\alpha_0(x)} \to 0$. Thus,

$$\int\limits_{\Omega} f(x)|u_n|^{q(x)}\,dx \le c_6|f|_{\alpha(x)}\big||u_n|^{q(x)}\big|_{\alpha_0(x)} \to 0,$$

which implies $|u_n|_{(q(x),f(x))} \to 0$. Hence, we have the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{f(x)}(\Omega)$. Now, by the above inequality we show that the inequality (2.6) holds. By $q^-\alpha_0(x) \le q^+\alpha_0(x) < p^*(x)$ and applying the similar steps as we did in proof of Theorem 2.7, we have

$$\int\limits_{\Omega} f(x)|u|^{q(x)}\,dx \le c_7(\|u\|^{q^-} + \|u\|^{q^+}).$$

Theorem 2.8 is proved.

By Theorems 2.7 and 2.8, we conclude that for $u \in W^{1,p(x)}(\Omega)$, there exist positive constants $c_8, c_9, c_{10}, c_{11} > 0$ such that

(i) $\displaystyle\int_{\partial\Omega} g(x)|u|^{r(x)}\,dS \le \begin{cases} c_8\|u\|^{r^+} & \text{if} \quad \|u\| > 1, \\ c_9\|u\|^{r^-} & \text{if} \quad \|u\| < 1, \end{cases}$

(ii) $\displaystyle\int_{\Omega} f(x)|u|^{q(x)}\,dx \le \begin{cases} c_{10}\|u\|^{q^+} & \text{if} \quad \|u\| > 1, \\ c_{11}\|u\|^{q^-} & \text{if} \quad \|u\| < 1, \end{cases}$

hold.

3. Assumptions and statement of main result. The Euler functional associated with (1.1) is defined by

$$\mathcal{J}_\lambda(u) = \int\limits_{\Omega} \frac{1}{p(x)}(|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\,dx - \lambda\int\limits_{\Omega} \frac{1}{q(x)}f(x)|u|^{q(x)}\,dx - \int\limits_{\partial\Omega} \frac{1}{r(x)}g(x)|u|^{r(x)}\,dS.$$

Then

$$\mathcal{J}_\lambda(u) \ge \frac{1}{p^+}\int\limits_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right)dx - \frac{\lambda}{q^-}\int\limits_{\Omega} f(x)|u|^{q(x)}\,dx - \frac{1}{r^-}\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\,dS \ge$$

$$\geq \frac{1}{p^+}\|u\|^{p^-} - \frac{\lambda}{q^-}c_7(\|u\|^{q^-} + \|u\|^{q^+}) - \frac{1}{r^-}c_5(\|u\|^{r^-} + \|u\|^{r^+}).$$

Since $q^+ < p^- \leq p^+ < r^- \leq h^+$, this shows $\mathcal{J}_\lambda$ is not bounded below on whole $W^{1,p(x)}(\Omega)$. However, it is useful to consider the functional on the Nehari manifold $\mathcal{N}_\lambda$ which is given by

$$\mathcal{N}_\lambda = \big\{u \in W^{1,p(x)}(\Omega)\backslash\{0\} : \langle \mathcal{J}'_\lambda(u), u\rangle = 0\big\},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $W^{1,p(x)}(\Omega)$ and $(W^{1,p(x)}(\Omega))^{-1}$. Clearly, the critical points of $\mathcal{J}_\lambda$ correspond to points on the Nehari manifold. In particular, $u \in \mathcal{N}_\lambda$ if and only if

$$K_\lambda(u) = \langle \mathcal{J}'_\lambda(u), u\rangle = \int\limits_\Omega \big(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\big)\,dx - \lambda\int\limits_\Omega f(x)|u|^{q(x)}\,dx - \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\,dS = 0. \tag{3.1}$$

Then for $u \in \mathcal{N}_\lambda$ we have

$$\langle K'_\lambda(u), u\rangle = \int\limits_\Omega p(x)\big(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\big)\,dx -$$

$$-\lambda\int\limits_\Omega q(x)f(x)|u|^{q(x)}\,dx - \int\limits_{\partial\Omega} r(x)g(x)|u|^{r(x)}\,dS \leq$$

$$\leq (p^+ - q^-)\lambda\int\limits_\Omega f(x)|u|^{q(x)}\,dx + (p^+ - r^-)\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\,dS.$$

We can write

$$\mathcal{N}^+_\lambda = \{u \in \mathcal{N}(\Omega) : \langle K'_\lambda(u), u\rangle > 0\},$$
$$\mathcal{N}^-_\lambda = \{u \in \mathcal{N}(\Omega) : \langle K'_\lambda(u), u\rangle < 0\},$$
$$\mathcal{N}^0_\lambda = \{u \in \mathcal{N}(\Omega) : \langle K'_\lambda(u), u\rangle = 0\}.$$

Lemma 3.1. *There exists $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$ we have $\mathcal{N}^0_\lambda(\Omega) = \varnothing$.*

Proof. Let $\mathcal{N}^0_\lambda(\Omega) \neq \varnothing$ for all $\lambda \in \mathbb{R}\backslash\{0\}$ and $u \in \mathcal{N}^0_\lambda(\Omega)$ such that $\|u\| > 1$. Then using (2.4), (3.1) and definition of $\mathcal{N}^0_\lambda(\Omega)$, we obtain

$$0 = \langle K'_\lambda(u), u\rangle = \int\limits_\Omega p(x)\big(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\big)\,dx -$$

$$-\lambda\int\limits_\Omega q(x)f(x)|u|^{q(x)}\,dx - \int\limits_{\partial\Omega} r(x)g(x)|u|^{r(x)}\,dS \geq$$

$$\geq p^-\int\limits_\Omega \big(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\big)\,dx -$$

$$-q^+\left(\int\limits_\Omega |\nabla u|^{p(x)}\,dx - \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\,dS\right) - r^+\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\,dS \geq$$

$$\geq (p^- - q^+) \int\limits_\Omega \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx + (q^+ - r^+) \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS.$$

Hence,

$$0 \geq (p^- - q^+)\|u\|^{p^-} + c_8(q^+ - r^+)\|u\|^{r^+},$$

and then

$$\|u\| \geq c_{12} \left(\frac{p^- - q^+}{r^+ - q^+}\right)^{\frac{1}{r^+ - p^-}}. \tag{3.2}$$

Similarly,

$$0 = \langle K'_\lambda(u), u\rangle \leq$$

$$\leq p^+ \int\limits_\Omega (|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\, dx - \lambda q^- \int\limits_\Omega f(x)|u|^{q(x)}\, dx - r^- \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS \leq$$

$$\leq p^+ \int\limits_\Omega (|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\, dx - \lambda q^- \int\limits_\Omega f(x)|u|^{q(x)}\, dx -$$

$$-r^- \left(\int\limits_\Omega (|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\, dx - \lambda \int\limits_\Omega f(x)|u|^{q(x)}\, dx \right).$$

Therefore,

$$0 \leq (p^+ - r^-)\|u\|^{p^-} + \lambda c_{10}(r^- - q^-)\|u\|^{q^+}$$

and

$$\|u\| \leq c_{13} \left(\lambda \frac{r^- - q^-}{r^- - p^+}\right)^{\frac{1}{p^- - q^+}}. \tag{3.3}$$

If λ is sufficiently small $\left(\text{e.g., } \lambda = \left(\frac{h^- - p^+}{r^- - q^-}\right)\left(\frac{p^- - q^+}{r^+ - q^+}\right)^{\frac{p^- - q^+}{r^+ - p^-}}\right)$, then from (3.2) and (3.3) we get $\|u\| < 1$ which contradicts with our assumption. Hence, we conclude $\mathcal{N}^0_\lambda(\Omega) = \varnothing$.

Lemma 3.1 is proved.

For $0 < \lambda < \lambda_1$, we can write $\mathcal{N}_\lambda(\Omega) = \mathcal{N}^+_\lambda(\Omega) \cup \mathcal{N}^-_\lambda(\Omega)$ and

$$\alpha^+_\lambda = \inf_{u \in \mathcal{N}^+_\lambda(\Omega)} \mathcal{J}_\lambda(u), \qquad \alpha^-_\lambda = \inf_{u \in \mathcal{N}^-_\lambda(\Omega)} \mathcal{J}_\lambda(u).$$

Theorem 3.1. *Suppose that u_0 is a local maximum or minimum for $\mathcal{J}$ on $\mathcal{N}_\lambda(\Omega)$. If $u_0 \notin \mathcal{N}^0_\lambda(\Omega)$, then u_0 is a critical point of $\mathcal{J}$.*

Proof. The proof of Theorem 3.1 can be obtained directly from the following lemmas.

Lemma 3.2. *The energy functional $\mathcal{J}$ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$.*

Proof. Let $u \in \mathcal{N}_\lambda(\Omega)$ and $\|u\| > 1$. Then using (3.1) and Theorem 2.2 we have

$$\mathcal{J}_\lambda(u) = \int\limits_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\, dx -$$

$$-\lambda \int\limits_{\Omega} \frac{1}{q(x)} f(x)|u|^{q(x)}\, dx - \int\limits_{\partial\Omega} \frac{1}{r(x)} g(x)|u|^{r(x)}\, dS \geq$$

$$\geq \frac{1}{p^+} \int\limits_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx - \frac{\lambda}{q^-} \int\limits_{\Omega} f(x)|u|^{q(x)}\, dx -$$

$$-\frac{1}{r^-} \left(\int\limits_{\Omega} (|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\, dx - \lambda \int\limits_{\Omega} f(x)|u|^{q(x)}\, dx \right) \geq$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) \int\limits_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx + \lambda \left(\frac{1}{r^-} - \frac{1}{q^-}\right) \int\limits_{\Omega} f(x)|u|^{q(x)}\, dx \geq$$

$$\geq \left(\frac{r^- - p^+}{r^- p^+}\right) \|u\|^{p^-} - c_{10}\lambda \left(\frac{r^- - q^-}{r^- q^-}\right) \|u\|^{q^+}.$$

Since $p^- > q^+$ so, $\mathcal{J}(u) \to \infty$ as $\|u\| \to \infty$. This implies $\mathcal{J}_\lambda$ is coercive and bounded below on $\mathcal{N}_\lambda(\Omega)$.

Lemma 3.3. *If* $0 < \lambda < \lambda_1$, *then*

(i) $\mathcal{J}_\lambda(u) < 0$ *for all* $u \in \mathcal{N}_\lambda^+(\Omega)$,

(ii) $\mathcal{J}_\lambda(u) > 0$ *for all* $u \in \mathcal{N}_\lambda^-(\Omega)$.

Proof. (i) Let $u \in \mathcal{N}_\lambda^+(\Omega)$. By definition of $\mathcal{J}_\lambda(u)$, we can write

$$\mathcal{J}_\lambda(u) \leq \frac{1}{p^-} \int\limits_{\Omega} (|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\, dx - \frac{\lambda}{q^+} \int\limits_{\Omega} f(x)|u|^{q(x)}\, dx - \frac{1}{r^+} \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS.$$

Since $u \in \mathcal{N}_\lambda^+(\Omega)$, we have

$$p^+ \int\limits_{\Omega} (|\nabla u|^{p(x)} + m(x)|u|^{p(x)})\, dx - \lambda q^- \int\limits_{\Omega} f(x)|u|^{q(x)}\, dx - r^- \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS > 0.$$

We get

$$\int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS < \frac{p^+ - q^-}{r^- - q^-} \int\limits_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx.$$

Moreover,

$$\mathcal{J}_\lambda(u) \leq \left(\frac{1}{p^-} - \frac{1}{q^+}\right) \int\limits_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx + \left(\frac{1}{q^+} - \frac{1}{r^+}\right) \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS.$$

Therefore,

$$\mathcal{J}_\lambda(u) < -\frac{(p^- - q^+)(r^+ - p^-)}{r^+ p^- q^+} \|u\|^{p^-} < 0.$$

Hence, we have $\alpha_\lambda^+ = \inf_{u \in \mathcal{N}_\lambda^+}(\Omega) \mathcal{J}_\lambda(u) < 0$.

(ii) Let $u \in \mathcal{N}_\lambda^-(\Omega)$. By definition of $\mathcal{J}_\lambda(\Omega)$ and (3.1), we obtain

$$\mathcal{J}_\lambda(u) \geq \frac{1}{p^+} \int\limits_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx - \frac{\lambda}{q^-} \int\limits_{\Omega} f(x)|u|^{q(x)}\, dx - \frac{1}{r^-} \int\limits_{\partial\Omega} g(x)|u|^{r(x)}\, dS$$

and

$$\int_{\partial\Omega} g(x)|u|^{r(x)}\,dS = \int_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx - \lambda \int_{\Omega} f(x)|u|^{q(x)}\,dx.$$

Therefore

$$\mathcal{J}_\lambda(u) \geq \frac{1}{p^+} \int_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx - \frac{\lambda}{q^-} \int_{\Omega} f(x)|u|^{q(x)}\,dx -$$

$$-\frac{1}{r^-}\left(\int_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx - \lambda \int_{\Omega} f(x)|u|^{q(x)}\,dx\right) \geq$$

$$\geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) \int_{\Omega} \left(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\right) dx + \lambda \left(\frac{1}{r^-} - \frac{1}{q^-}\right) \int_{\Omega} f(x)|u|^{q(x)}\,dx.$$

By Theorem 2.2 and the condition $p^- > q^+$, we obtain

$$\mathcal{J}_\lambda(u) \geq \left(\frac{1}{p^+} - \frac{1}{r^-}\right) \|u\|^{p^-} + c_{10}\lambda \left(\frac{1}{r^-} - \frac{1}{q^-}\right) \|u\|^{q^+} \geq$$

$$\geq \left(\frac{r^- - p^+}{p^+ r^-} + c_{10}\lambda \left(\frac{q^- - r^-}{q^- r^-}\right)\right) \|u\|^{p^-}.$$

So, if we choose $\lambda < \dfrac{q^-(r^- - p^+)}{c_{10}p^+(r^- - q^-)}$, we get $\mathcal{J}_\lambda(u) > 0$. If we consider the facts $\mathcal{N}_\lambda(\Omega) = \mathcal{N}_\lambda^+(\Omega) \cup \mathcal{N}_\lambda^-(\Omega)$ (see Lemma 3.1), $\mathcal{N}_\lambda^+(\Omega) \cap \mathcal{N}_\lambda^-(\Omega) = \varnothing$, and by the Lemma 3.3, we must have $u \in \mathcal{N}_\lambda^-(\Omega)$.

Theorem 3.2. *If $0 < \lambda < \lambda_1$, then the functional $\mathcal{J}_\lambda$ has a minimizer u_0^+ in $\mathcal{N}_\lambda^+(\Omega)$ and $\mathcal{J}_\lambda(u_0^+) = \alpha_\lambda^+$.*

Proof. Since $\mathcal{J}_\lambda$ is bounded below on $\mathcal{N}_\lambda(\Omega)$. Then there exist a minimizing sequence $\{u_n^+\} \subseteq \mathcal{N}_\lambda^+(\Omega)$ such that

$$\lim_{n\to\infty} \mathcal{J}_\lambda(u_n^+) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda(u) = \alpha_\lambda^+ < 0.$$

Since $\mathcal{J}_\lambda$ is coercive, u_n^+ is bounded in $W^{1,p(x)}(\Omega)$. Let $u_n^+ \rightharpoonup u_0^+$ in $W^{1,p(x)}(\Omega)$,

$$u_n^+ \to u_0^+ \quad \text{in} \quad L^{q(x)}_{f(x)}(\Omega),$$

and

$$u_n^+ \to u_0^+ \quad \text{in} \quad L^{r(x)}_{g(x)}(\Omega).$$

Now, we prove that $u_n^+ \to u_0^+$ in $W^{1,p(x)}(\Omega)$. Otherwise, suppose $u_n^+ \nrightarrow u_0^+$ in $W^{1,p(x)}(\Omega)$. Then

$$\int_{\Omega} \left(|\nabla u_0^+|^{p(x)} + m(x)|u_0^+|^{p(x)}\right) dx < \lim_{n\to\infty}\inf \int_{\Omega} \left(|\nabla u_n^+|^{p(x)} + m(x)|u_n^+|^{p(x)}\right) dx.$$

Moreover,

$$\int_{\Omega} f(x)|u_0^+|^{q(x)}\,dx = \lim_{n\to\infty}\inf \int_{\Omega} f(x)|u_n^+|^{q(x)}\,dx,$$

$$\int\limits_{\partial\Omega} g(x)|u_0^+|^{r(x)}\,dS = \lim_{n\to\infty}\inf \int\limits_{\partial\Omega} g(x)|u_n^+|^{r(x)}\,dS.$$

By $\langle \mathcal{J}'_\lambda(u_n^+),(u_n^+)\rangle = 0$ and Theorem 2.8, we have

$$\mathcal{J}_\lambda(u_n^+) \geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int\limits_\Omega \left(|\nabla u_n^+|^{p(x)} + m(x)|u_n^+|^{p(x)}\right)dx + \lambda\left(\frac{1}{r^-}-\frac{1}{q^-}\right)\int\limits_\Omega f(x)|u_n^+|^{q(x)}\,dx,$$

and then

$$\alpha_\lambda^+ = \lim_{n\to\infty}\mathcal{J}_\lambda(u_n^+) \geq$$

$$\geq \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\lim_{n\to\infty}\int\limits_\Omega \left(|\nabla u_n^+|^{p(x)} + m(x)|u_n^+|^{p(x)}\right)dx +$$

$$+\lambda\left(\frac{1}{r^-}-\frac{1}{q^-}\right)\lim_{n\to\infty}\int\limits_\Omega f(x)|u_n^+|^{q(x)}\,dx >$$

$$> \left(\frac{1}{p^+}-\frac{1}{r^-}\right)\|u_0^+\|^{p^-} + c_7\lambda\left(\frac{1}{r^-}-\frac{1}{q^-}\right)\left(\|u_0^+\|^{q^-} + \|u_0^+\|^{q^+}\right).$$

Since $p^- > q^+$, for $\|u_0^+\| > 1$, we obtain

$$\alpha_\lambda^+ = \inf_{u\in\mathcal{N}_\lambda^+}\mathcal{J}_\lambda(u) > 0.$$

By Lemma 3.3, for any $u \in \mathcal{N}_\lambda^+(\Omega), \mathcal{J}_\lambda(u) < 0$.

So, this is a contradiction. Hence, $u_n^+ \to u_0^+$ in $W_0^{1,p(x)}(\Omega)$ and

$$\mathcal{J}_\lambda(u_0^+) = \lim_{n\to\infty}\mathcal{J}_\lambda(u_n^+) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)}\mathcal{J}_\lambda(u).$$

Thus, u_0^+ is a minimizer for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^+(\Omega)$.

Theorem 3.2 is proved.

Theorem 3.3. *If* $0 < \lambda < \lambda_1$, *then the functional* $\mathcal{J}_\lambda$ *has a minimizer* u_0^- *in* $\mathcal{N}_\lambda^-(\Omega)$ *and* $\mathcal{J}_\lambda(u_0^-) = \alpha_\lambda^-$.

Proof. Since $\mathcal{J}_\lambda$ is bounded below on $\mathcal{N}_\lambda(\Omega)$ and so on $\mathcal{N}_\lambda^-(\Omega)$, then there exists a minimizing sequence $\{u_n^-\} \subseteq \mathcal{N}_\lambda^-(\Omega)$ such that

$$\lim_{n\to\infty}\mathcal{J}_\lambda(u_n^-) = \inf_{u\in\mathcal{N}_\lambda^-(\Omega)}\mathcal{J}_\lambda(u) = \alpha_\lambda^- > 0.$$

Since $\mathcal{J}_\lambda$ is coercive, u_n^- is bounded in $W^{1,p(x)}(\Omega)$. Thus, $u_n^- \rightharpoonup u_0^-$ in $W^{1,p(x)}(\Omega)$ and

$$u_n^- \to u_0^- \quad \text{in} \quad L^{q(x)}_{f(x)}(\Omega),$$

$$u_n^- \to u_0^- \quad \text{in} \quad L^{r(x)}_{g(x)}(\Omega).$$

Moreover, if $u_0^- \in \mathcal{N}_\lambda^-(\Omega)$, then there is a constant $t > 0$ such that $tu_0^- \in \mathcal{N}_\lambda^-(\Omega)$ and $\mathcal{J}_\lambda(u_0^-) \geq$ $\geq \mathcal{J}_\lambda(tu_0^-)$. Since

$$K_\lambda'(u) = \int\limits_\Omega p(x)\big(|\nabla u|^{p(x)} + m(x)|u|^{p(x)}\big)\,dx - \lambda\int\limits_\Omega q(x)f(x)|u|^{q(x)}\,dx - \int\limits_{\partial\Omega} r(x)g(x)|u|^{r(x)}\,dS,$$

then

$$K_\lambda'(tu_0^-) = \int\limits_\Omega p(x)\big(|\nabla tu_0^-|^{p(x)} + m(x)|u_0^-|^{p(x)}\big)\,dx -$$
$$-\lambda\int\limits_\Omega q(x)f(x)|tu_0^-|^{q(x)}\,dx - \int\limits_{\partial\Omega} r(x)g(x)|tu_0^-|^{r(x)}\,dS \leq$$
$$\leq t^{p^+}p^+\int\limits_\Omega (|\nabla u_0^-|^{p(x)} + m(x)|u_0^-|^{p(x)})\,dx -$$
$$-\lambda t^{q^-}q^-\int\limits_\Omega f(x)|u_0^-|^{q(x)}\,dx - t^{r^-}r^-\int\limits_{\partial\Omega} g(x)|u_0^-|^{r(x)}\,dS.$$

Since $q^- < p^+ < r^-$, and by the assumptions on f and g, it follows $K_\lambda'(tu_0^-) < 0$. Hence, by the definition of $\mathcal{N}_\lambda^-(\Omega)$, $tu_0^- \in \mathcal{N}_\lambda^-(\Omega)$.

Now, we prove $u_n^- \to u_0^-$ in $W_0^{1,p(x)}(\Omega)$. Let $u_n^- \nrightarrow u_0^-$ in $W_0^{1,p(x)}(\Omega)$. By

$$\int\limits_\Omega (|\nabla u_0^-|^{p(x)} + m(x)|u_0^-|^{p(x)})\,dx < \lim_{n\to\infty}\inf\int\limits_\Omega (|\nabla u_n^-|^{p(x)} + m(x)|u_n^-|^{p(x)})\,dx,$$

we have

$$\mathcal{J}_\lambda'(tu_0^-) \leq \frac{t^{p^+}}{p^-}\int\limits_\Omega (|\nabla u_0^-|^{p(x)} + m(x)|u_0^-|^{p(x)})\,dx -$$
$$-\lambda\frac{t^{q^-}}{q^+}\int\limits_\Omega f(x)|u_0^-|^{q(x)}\,dx - \frac{t^{r^-}}{r^+}\int\limits_{\partial\Omega} g(x)|u_0^-|^{r(x)}\,dS <$$
$$< \lim_{n\to\infty}\left[\frac{t^{p^+}}{p^-}\int\limits_\Omega (|\nabla u_n^-|^{p(x)} + m(x)|u_n^-|^{p(x)})\,dx -\right.$$
$$\left.-\lambda\frac{t^{q^-}}{q^+}\int\limits_\Omega f(x)|u_n^-|^{q(x)}\,dx - \frac{t^{r^-}}{r^+}\int\limits_{\partial\Omega} g(x)|u_n^-|^{r(x)}\,dS\right] \leq$$
$$\leq \lim_{n\to\infty}\mathcal{J}_\lambda(tu_n^-) \leq \lim_{n\to\infty}\mathcal{J}_\lambda(u_n^-) = \inf_{u\in\mathcal{N}_\lambda^-(\Omega)}\mathcal{J}_\lambda(u).$$

Thus, u_0^- is a minimizer for $\mathcal{J}_\lambda$ on $\mathcal{N}_\lambda^-(\Omega)$.

Theorem 3.3 is proved.

Conclusions. By Theorems 3.2 and 3.3 we conclude that there exist $u_0^+ \in \mathcal{N}_\lambda^+(\Omega)$ and $u_0^- \in \in \mathcal{N}_\lambda^-(\Omega)$ such that $\mathcal{J}_\lambda(u_0^+) = \inf_{u\in\mathcal{N}_\lambda^+(\Omega)} \mathcal{J}_\lambda(u)$ and $\mathcal{J}_\lambda(u_0^-) = \inf_{u\in\mathcal{N}_\lambda^-(\Omega)} \mathcal{J}_\lambda(u)$. Moreover, since $\mathcal{J}_\lambda(u_0^\pm) = \mathcal{J}_\lambda(|u_0^\pm|)$ and $|u_0^\pm| \in \mathcal{N}_\lambda^\pm(\Omega)$, we may assume $u_0^\pm \geq 0$. By Theorem 3.1, $u_0^\pm$ are critical points $\mathcal{J}_\lambda$ on $W_0^{1,p(x)}(\Omega)$ and hence are weak solutions of (1.1). Finally, by the Harnack inequality due to [22], we obtain that $u_0^\pm$ are positive solutions of (1.1).

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