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# REDUCING SEQUENCES

# ЗВІДНІ ПОСЛІДОВНОСТІ

We introduce and examine two new classes of distinguished sequences in the unit disk for the space of bounded analytic functions. One of these sequences is intermediate between interpolating and zero sequences.

Введено та вивчено два нових класи виділених послідовностей в одиничному колі для простору обмежених аналітичних функцій. Одна з цих послідовностей є проміжною між інтерполяційною та нульовою послідовностями.

**1. Introduction.** Let $\mathbb{D}$ be the unit disk of the complex plane. We consider the space $H^\infty$ of all analytic functions $f$ in $\mathbb{D}$ such that $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty$ and the Bergman space $A^1$ of all analytic functions in $\mathbb{D}$ such that

$$\|f\|_1 = \int\limits_{\mathbb{D}} |f(z)|\, dm(z) < \infty.$$

Let $l^1$ and $l^\infty$ be the Banach spaces of all complex sequences $(a_n)$ such that $\|(a_n)\|_1 = \sum_{n=1}^{\infty} |a_n| < \infty$ and $\|(a_n)\|_\infty = \sup_{n\in\mathbb{N}} |a_n| < \infty$, respectively. We denote by $\sigma$ any sequence $(z_n)$ of points in $\mathbb{D}$. Recall that $\sigma$ is said to be a *Blaschke sequence* if it satisfies the condition

$$\sum_{n=1}^{\infty}(1-|z_n|) < \infty.$$

For a Blaschke sequence $\sigma$, the analytic function $B = B_\sigma$ defined in $\mathbb{D}$ by

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z}$$

is called the *Blaschke product with zeros at* $\sigma$ and it is bounded by one. For a fixed $n \in \mathbb{N}$, we denote by $B_n$ the Blaschke product $B_{\sigma\setminus\{z_n\}}$. We put $\rho(z,w)$ for the pseudohyperbolic distance between $z$, $w \in \mathbb{D}$, that is

$$\rho(z,w) = \left|\frac{z-w}{1-z\overline{w}}\right|.$$

Given a point $\eta$ in the boundary of $\mathbb{D}$ and a number $t \in [1,\infty)$, the domain

$$\big\{z \in \mathbb{D} \,/\, |1 - \overline{\eta}\, z| \le t\,(1-|z|)\big\}$$

is called *Stolz angle with vertex at* $\eta$. A sequence $\sigma$ is called *uniformly separated* if there exists $\delta > 0$ such that

$$\sup_{n\in\mathbb{N}} |B_n(z_n)| \ge \delta.$$

As usual, we will write $c$ for positive constants when they appear. First, we recall two types of distinguished sequences in $\mathbb{D}$ for $H^\infty$:

**Definition 1.** *$\sigma = (z_n)$ is called a zero sequence if there exists a function $f \in H^\infty$, not identically zero, such that $f(z_n) = 0$ $\forall n \in \mathbb{N}$.*

**Definition 2.** *$\sigma = (z_n)$ is called an interpolating sequence if given any sequence $(a_n) \in l^\infty$, there exists a function $f \in H^\infty$ such that $f(z_n) = a_n$ $\forall n \in \mathbb{N}$.*

Clearly, interpolating sequences are zero sequences. It is well known that $\sigma$ is a zero sequence if and only if it is a Blaschke sequence [3]. On the other hand, it is proved in [2] that $\sigma$ is an interpolating sequence if and only if it is uniformly separated. Let $\mathcal{R}$ be the space of real sequences defined by

$$\mathcal{R} = \big\{(r_n) \,/\, r_n \in (0,1)\ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} r_n = 0\big\}.$$

Next, we introduce another class of distinguished sequences in $\mathbb{D}$ for $H^\infty$:

**Definition 3.** *For a fixed sequence $(r_n)$ in $\mathcal{R}$, we say that $\sigma = (z_n)$ is a $[r_n]$-reducing sequence if given any sequence $(a_n) \in l^\infty$, there exists a function $f \in H^\infty$ such that $f(z_n) = r_n\, a_n$ $\forall n \in \mathbb{N}$.*

We will say that the function $f$ in Definition 3 is a $[r_n]$-*reducing function*. Note that if $\sigma$ is a $[r_n]$-reducing sequence, then $|f(z_n) - a_n| = (1 - r_n)\,|a_n|$ $\forall n \in \mathbb{N}$, that is, each value $f(z_n)$ is on a circle centred in $a_n$ and radius strictly between 0 and $|a_n|$, approaching $|a_n|$. Since

$$\rho(f(z_n), a_n) = \frac{1 - r_n}{1 - r_n\,|a_n|^2}\,|a_n| \quad \forall n \in \mathbb{N},$$

it follows that if $(a_n) \subset \mathbb{D}$, then values $f(z_n)$ are also on pseudohyperbolic circles with the same properties as before. If $l^\infty[r_n]$ denotes the subspace of $l^\infty$ defined by

$$l^\infty[r_n] = \big\{(b_n) \in l^\infty / \, b_n = r_n\, a_n \ \text{ for a certain } \ (a_n) \in l^\infty\big\},$$

then $[r_n]$-reducing sequences can be viewed as interpolating sequences for the target space $l^\infty[r_n]$. Thus, it follows that every interpolating sequence is $[r_n]$-reducing for any $(r_n)$ in $\mathcal{R}$.

We also state one variation of Definition 3:

**Definition 4.** *For a fixed sequence $(r_n)$ in $\mathcal{R}$, we say that $\sigma = (z_n)$ is a $[r_n]$-reducing sequence in the $l^1$-sense if given any sequence $(a_n) \in l^1$, there exists a function $f \in H^\infty$ such that $f(z_n) = r_n\, a_n$ $\forall n \in \mathbb{N}$.*

Clearly, every $[r_n]$-reducing sequence is a $[r_n]$-reducing sequence in the $l^1$-sense. Taking $(a_n) \equiv (0)$ in Definitions 3 and 4, it turns out that all $[r_n]$-reducing sequences and $[r_n]$-reducing sequences in the $l^1$-sense are zero sequences. Then $[r_n]$-reducing sequences are intermediate between interpolating and zero sequences. The following section is devoted to examine both types of introduced sequences.

**2. Statement and proof of results.** First, we characterize $[r_n]$-reducing sequences in the $l^1$-sense:

**Theorem 1.** *A sequence $\sigma = (z_n)$ is $[r_n]$-reducing in the $l^1$-sense if and only if it is a Blaschke sequence and the sequence $(r_n)$ in $\mathcal{R}$ verifies*

$$r_n \le c\,|B_n(z_n)| \quad \forall n \in \mathbb{N}.$$

***Proof.*** *Necessity.* Suppose that the restriction operator on $\sigma$ maps $H^\infty$ onto the subspace of $l^1$ defined by

$$l^1[r_n] = \{(b_n) \in l^1 / \, b_n = r_n \, a_n \;\; \text{for a certain} \;\; (a_n) \in l^1\},$$

with the norm $\|(b_n)\|_{l^1[r_n]} = \|(a_n)\|_1$. Fixed $m \in \mathbb{N}$, we apply the open mapping theorem to the sequence $(b_n) \in l^1[r_n]$ defined by means of $(a_n)$ such that $a_m = 1$ and $a_n = 0$, if $n \neq m$. Then we obtain that there exists $f_m \in H^\infty$ such that

$$\|f_m\|_\infty \leq c \, \|(b_n)\|_{l^1[r_n]} = c \, \|(a_n)\|_1 = c.$$

Since $f_m = g_m \, B_m$, for a certain $g_m$ not vanishing on $\sigma$ and verifying $\|g_m\|_\infty = \|f_m\|_\infty$, it follows that

$$r_m = |f_m(z_m)| = |g_m(z_m)| \, |B_m(z_m)| \leq \|g_m\|_\infty \, |B_m(z_m)| \leq c \, |B_m(z_m)|.$$

*Sufficiency.* For a sequence $(a_n)$ in $l^1$, we define the function $f$ in $\mathbb{D}$ by

$$f(z) = \sum_{n=1}^{\infty} r_n \, a_n \, \frac{B_n(z)}{B_n(z_n)}. \tag{1}$$

Then $f(z_k) = r_k \, a_k \; \forall k \in \mathbb{N}$. On the other hand, $f \in H^\infty$, because

$$|f(z)| \leq c \sum_{n=1}^{\infty} |a_n| < \infty \quad \forall z \in \mathbb{D}.$$

Next, we put a condition on $\sigma$ and another on $(r_n)$ in $\mathcal{R}$ so that $\sigma$ is $[r_n]$-reducing:

**Theorem 2.** *If $\sigma = (z_n)$ is a Blaschke sequence and $(r_n)$ in $\mathcal{R}$ verifies*

$$r_n \leq c \, (1 - |z_n|) \, |B_n(z_n)| \quad \forall n \in \mathbb{N}, \tag{2}$$

*then $\sigma$ is $[r_n]$-reducing and furthermore, any $[r_n]$-reducing function $f$ is Lipschitz on $\sigma$, that is,*

$$|f(z_n) - f(z_m)| \leq c \, |z_n - z_m| \quad \forall n, \, m \in \mathbb{N}. \tag{3}$$

***Proof.*** For a sequence $(a_n)$ in $l^\infty$, the function $f$ defined in (1) is in $H^\infty$, because

$$|f(z)| \leq c \sum_{n=1}^{\infty} \frac{r_n}{|B_n(z_n)|} \leq c \sum_{n=1}^{\infty} (1 - |z_n|) < \infty \quad \forall z \in \mathbb{D}.$$

On the other hand,

$$|f(z_n) - f(z_m)| \leq |f(z_n)| + |f(z_m)| = r_n \, |a_n| + r_m \, |a_m| \leq$$

$$\leq c \, \|(a_n)\|_\infty \, [(1 - |z_n|) \, |B_n(z_n)| + (1 - |z_m) \, |B_m(z_m)|] \leq$$

$$\leq c \, [(1 - |z_n|) + (1 - |z_m|)] \, \rho(z_n, z_m) \leq c \, |z_n - z_m| \quad \forall n, \, m \in \mathbb{N}.$$

If $f \in H^\infty$, it is well known that

$$|f(z) - f(w)| \leq c \, \rho(z, w) \quad \forall z, \, w \in \mathbb{D}. \tag{4}$$

Then, estimate (3) is an improvement of (4) on the sequence $\sigma$.

We are interested in having some control of the derivative of the $[r_n]$-reducing function. Taking a sequence $(r_n)$ in $\mathcal{R}$ slightly more reductive than (3), we can obtain such control on the sequence $\sigma$:

**Theorem 3.** *If $\sigma = (z_n)$ is a Blaschke sequence and $(r_n)$ in $\mathcal{R}$ verifies*

$$r_n \le c\,(1-|z_n|)\,|B_n(z_n)|^2 \quad \forall n \in \mathbb{N},$$

*then there is a $[r_n]$-reducing function $f$ such that*

$$|(1-|z_n|)\,f'(z_n) - (1-|z_m|)\,f'(z_m)| \le c\,|z_n - z_m| \quad \forall n,\, m \in \mathbb{N}. \tag{5}$$

***Proof.*** Given a sequence $(a_n)$ in $l^\infty$, we define the function $f$ in $\mathbb{D}$ by

$$f(z) = \sum_{n=1}^{\infty} r_n\, a_n\, \frac{B_n(z)^2}{B_n(z_n)^2}.$$

Thus, $f(z_k) = r_k\, a_k\ \forall k \in \mathbb{N}$ and it is immediate that $f \in H^\infty$ (it is proved as in Theorem 2). Since

$$f'(z) = 2\sum_{n=1}^{\infty} r_n\, a_n \frac{B_n(z)\, B_n'(z)}{B_n(z_n)^2},$$

then

$$f'(z_k) = 2\, r_k\, a_k\, \frac{B_k'(z_k)}{B_k(z_k)} \quad \forall k \in \mathbb{N}.$$

Taking into account that

$$|B_k'(z_k)| \le c\,\frac{1}{1-|z_k|} \quad \forall k \in \mathbb{N},$$

we have $|f'(z_k)| \le c\,|B_k(z_k)|\ \forall k \in \mathbb{N}$, and

$$|(1-|z_n|)\,f'(z_n) - (1-|z_m|)\,f'(z_m)| \le (1-|z_n|)\,|f'(z_n)| + (1-|z_m|)\,|f'(z_m)| \le$$
$$\le c\,[(1-|z_n|)\,|B_n(z_n)| + (1-|z_m)\,|B_m(z_m)|] \le c\,|z_n - z_m| \quad \forall n,\, m \in \mathbb{N}.$$

If $f \in H^\infty$, it is proved in [1] that

$$\big|(1-|z|)\,f'(z) - (1-|w|)\,f'(w)\big| \le c\,\rho(z,w) \quad \forall z,\, w \in \mathbb{D}. \tag{6}$$

Then, estimate in (5) is an improvement of (6) on the sequence $\sigma$.

Adding one condition to the sequence $\sigma$, we can obtain a global control of the derivative of the $[r_n]$-reducing function:

**Theorem 4.** *If $\sigma = (z_n)$ is a Blaschke sequence located in a Stolz angle and $(r_n)$ in $\mathcal{R}$ verifies the condition in* (2), *then there is a $[r_n]$-reducing function $f$ such that $f'$ belongs to $A^1$.*

***Proof.*** Given a sequence $(a_n)$ in $l^\infty$, we consider the function $f$ defined in (1). Since

$$f'(z) = \sum_{n=1}^{\infty} r_n\, a_n \frac{B_n'(z)}{B_n(z_n)},$$

we have

$$\int\limits_{\mathbb{D}} |f'(z)|\, dm(z) \le c \int\limits_{\mathbb{D}} \left[\sum_{n=1}^{\infty} (1-|z_n|)\,|B_n'(z)|\right] dm(z) =$$
$$= c\sum_{n=1}^{\infty}(1-|z_n|) \int\limits_{\mathbb{D}} |B_n'(z)|\, dm(z).$$

It is proved in [4] that if $\sigma$ is a Blaschke sequence located in a Stolz angle, then $B' \in A^1$ and it follows from there that

$$\|B_n'\|_1 < \|B'\|_1 \leq c \quad \forall n \in \mathbb{N}.$$

Thus,

$$\int\limits_{\mathbb{D}} |f'(z)|\, dm(z) \leq c \sum_{n=1}^{\infty} (1 - |z_n|) < \infty.$$

***Remark.*** We think that it would be interesting to pose reducing sequences in other spaces of analytic functions, especially in the Bergman and Bloch spaces, to compare their results with those obtained here.

## References


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