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B. He (College Sci., Northwest A&F Univ., Yangling, Shaanxi, China)

ON q-CONGRUENCES INVOLVING HARMONIC NUMBERS*

ПРО q-КОНГРУЕНЦІЇ, ЩО ВКЛЮЧАЮТЬ ГАРМОНІЧНІ ЧИСЛА

We give some congruences involving q-harmonic numbers and alternating q-harmonic numbers of order m. Some of them are q-analogues of several known congruences.

Наведено деякі конгруенції, що включають q-гармонічні числа та знакозмінні q-гармонічні числа m-го порядку. Деякі з цих конгруенцій є q-аналогами кількох відомих конгруенцій.

1. Introduction. For arbitrary positive integer n, the q-integer can be defined by

$$[n]_q = \frac{1-q^n}{1-q}.$$

It is easy to see that $\lim_{q\to 1}[n]_q = n$. Supposing that $a \equiv b \pmod p$, we have

$$[a]_q = \frac{1-q^a}{1-q} = \frac{1-q^b+q^b(1-q^{a-b})}{1-q} \equiv \frac{1-q^b}{1-q} = [b]_q \pmod{[p]_q}.$$

Here and in what follows, each congruence is considered over the polynomial ring $\mathbb{Z}[q]$ in the variable q with integral coefficients.

For $m = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$, we define

$$H_0^{(m)} = 0, \qquad H_n^{(m)} = \sum_{j=1}^{n} \frac{1}{j^m} \qquad \text{for} \quad n \geq 1$$

and call it a harmonic number of order m. Those $H_n = H_n^{(1)}$ are usually called the classical harmonic numbers. Similarly, the alternating harmonic numbers of order m are given by

$$I_0^{(m)} = 0, \qquad I_n^{(m)} = \sum_{j=1}^{n} \frac{(-1)^j}{j^m} \qquad \text{for} \quad n \geq 1.$$

In this paper, we define

$$H_n(q) = \sum_{j=1}^{n} \frac{1}{[j]_q}, \qquad \widetilde{H}_n(q) = \sum_{j=1}^{n} \frac{q^j}{[j]_q},$$

$$H_n^{(2)}(q) = \sum_{j=1}^{n} \frac{1}{[j]_q^2}, \qquad \widetilde{H}_n^{(2)}(q) = \sum_{j=1}^{n} \frac{q^j}{[j]_q^2},$$

$$H_n^{(3)}(q) = \sum_{j=1}^{n} \frac{1}{[j]_q^3}, \qquad \widetilde{H}_n^{(3)}(q) = \sum_{j=1}^{n} \frac{q^j}{[j]_q^3}$$

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and

$$I_n(q)=\sum_{j=1}^{n}\frac{(-1)^j}{[j]_q},$$

$$I_n^{(2)}(q)=\sum_{j=1}^{n}\frac{(-1)^j}{[j]_q^2},$$

where

$$H_0(q)=\widetilde{H}_0(q)=H_0^{(2)}(q)=\widetilde{H}_0^{(2)}(q)=H_0^{(3)}(q)=\widetilde{H}_0^{(3)}(q)=I_0(q)=I_0^{(2)}(q)=0.$$

They are q-analogues of harmonic numbers of order m. So we call them q-harmonic numbers and alternating q-harmonic numbers of order m.

In view of the q-analogue of Glaishers congruence, Andrews [1] (Theorem 4) showed that

$$H_{p-1}(q)\equiv\frac{p-1}{2}(1-q)\pmod{[p]_q}$$

and

$$\widetilde{H}_{p-1}(q)\equiv\frac{p-1}{2}(q-1)\pmod{[p]_q}.$$

L. L. Shi and H. Pan obtained (see [6], Theorem 1)

$$H_{p-1}(q)\equiv\frac{p-1}{2}(1-q)+\frac{p^2-1}{24}(1-q)^2[p]_q\pmod{[p]_q^2}\tag{1.1}$$

for each $p\geq 5$, which is equivalent to

$$\widetilde{H}_{p-1}(q)\equiv\frac{1-p}{2}(1-q)+\frac{p^2-1}{24}(1-q)^2[p]_q\pmod{[p]_q^2}.$$

Pan established (see [5], Theorem 1.1) that for each odd prime p, there holds

$$2\sum_{j=1}^{\frac{p-1}{2}}\frac{1}{[2j]_q}+2Q_p(2,q)-Q_p(2,q)^2[p]_q\equiv$$

$$\equiv\left(Q_p(2,q)(1-q)+\frac{p^2-1}{8}(1-q)^2\right)[p]_q\pmod{[p]_q^2},\tag{1.2}$$

where $Q_p(2,q)=\dfrac{(-q;q)_{p-1}-1}{[p]_q}$ and $(x;q)_n=\prod_{k=0}^{n-1}(1-xq^k)$. For some material on congruences of q-harmonic numbers, see, for example, [3]. Some other q-congruences were obtained by different authors, see, for example [2, 4, 9].

Our aim of this paper is to give some congruences involving q-harmonic numbers and alternating q-harmonic numbers of order m which are q-analogues of several known identities.

Theorem 1.1. *Let $p \geq 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} q^k H_k^{(2)}(q) \equiv -\frac{p-1}{2}(1-q) - \frac{(p-1)(p-3)}{8}(1-q)^2[p]_q \pmod{[p]_q^2}, \tag{1.3}$$

$$\sum_{k=1}^{p-1} q^k \widetilde{H}_k^{(2)}(q) \equiv \frac{p-1}{2}(1-q) - \frac{p^2-1}{8}(1-q)^2[p]_q \pmod{[p]_q^2}, \tag{1.4}$$

$$\sum_{k=1}^{p-1} q^k H_k^{(3)}(q) \equiv \frac{(p-1)(p-5)}{12}(1-q)^2 \pmod{[p]_q}, \tag{1.5}$$

$$\sum_{k=1}^{p-1} q^k \widetilde{H}_k^{(3)}(q) \equiv \frac{p^2-1}{12}(1-q)^2 \pmod{[p]_q}. \tag{1.6}$$

When $q \to 1$, the first two q-congruences in Theorem 1.1 reduce to the following result [7] (Lemma 2.1):

$$\sum_{k=1}^{p-1} H_k^{(2)} \equiv 0 \pmod{p^2}$$

while the last two q-congruences reduce to

$$\sum_{k=1}^{p-1} H_k^{(3)} \equiv 0 \pmod{p}.$$

Theorem 1.2. *Let $p \geq 5$ be a prime. Then*

$$I_{p-1}(q) \equiv -2Q_p(2,q) - \frac{p-1}{2}(1-q) + \\ + \left(Q_p(2,q)^2 + Q_p(2,q)(1-q) + \frac{p^2-1}{12}(1-q)^2\right)[p]_q \pmod{[p]_q^2} \tag{1.7}$$

and

$$I_{p-1}^{(2)}(q) \equiv \frac{1-p}{2}(1-q)^2 - 2Q_p(2,q)(1-q) \pmod{[p]_q}. \tag{1.8}$$

It is clear that (1.7) and (1.8) are respectively q-analogues of

$$I_{p-1} \equiv -2q_p(2) + q_p(2)^2 p \pmod{p^2}$$

and

$$I_{p-1}^{(2)} \equiv 0 \pmod{p}$$

(see [8], Lemma 2.1), where $q_p(2) = \dfrac{2^{p-1}-1}{p}$.

Theorem 1.3. *Let $p \geq 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} q^k I_k(q) \equiv \left(-2Q_p(2,q) - \frac{p-1}{2}(1-q)\right)[p]_q +$$

$$+\left(Q_p(2,q)^2 + Q_p(2,q)(1-q) + \frac{p^2-1}{12}(1-q)^2\right)[p]_q^2 \pmod{[p]_q^3},$$

$$\sum_{k=1}^{p-1} q^k I_k^{(2)}(q) \equiv 2Q_p(2,q) + \frac{p-1}{2}(1-q) -$$

$$-\left(Q_p(2,q)^2 + 3Q_p(2,q)(1-q) + \frac{(7+p)(p-1)}{12}(1-q)^2\right)[p]_q \pmod{[p]_q^2}.$$

When $q \to 1$, the two q-congruences in Theorem 1.3 reduce respectively to the following two congruences:

$$\sum_{k=1}^{p-1} I_k \equiv -2q_p(2)p + q_p(2)^2 p^2 \pmod{p^3},$$

$$\sum_{k=1}^{p-1} I_k^{(2)} \equiv 2q_p(2) - q_p(2)^2 p \pmod{p^2}.$$

Theorem 1.4. *Let $p \geq 5$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{[k]_q} I_k(q) \equiv 2Q_p^2(2,q) + (p-1)(1-q) + \frac{p^2-1}{12}(1-q)^2 \pmod{[p]_q}.$$

The congruence in Theorem 1.4 is a q-analogue of

$$\sum_{1 \leq i \leq j \leq p-1} \frac{(-1)^{i+j}}{ij} \equiv 2q_p^2(2) \pmod{p}.$$

Our method of proving Theorems 1.1 – 1.4 is to write the finite sums involving (alternating) q-harmonic numbers into a linear combination of at most two (alternating) q-harmonic sums. We will provide one lemma in the next section. Section 3 is devoted to our proof of Theorems 1.1 – 1.4.

2. Auxiliary result. To prove Theorems 1.1 – 1.4, we need the following auxiliary result.

Lemma 2.1. *For any prime $p \geq 5$, there hold*

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(1-q)^2 \pmod{[p]_q}, \tag{2.1}$$

$$\sum_{j=1}^{p-1} \frac{q^j}{[j]_q^2} \equiv -\frac{p^2-1}{12}(1-q)^2 \pmod{[p]_q}, \tag{2.2}$$

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{[2j]_q^2} \equiv -\frac{p^2-1}{24}(1-q)^2 - Q_p(2,q)(1-q) \pmod{[p]_q}. \tag{2.3}$$

Proof. See [6] (Lemma 2) for the proof of (2.1) and (2.2).

We now prove (2.3). It is obvious that

$$\frac{1}{[p-2j]_q} \equiv -\frac{q^{2j}}{[2j]_q} \pmod{[p]_q}. \tag{2.4}$$

Then by (2.2) and (2.4), we get

$$-\frac{p^2-1}{12}(1-q)^2 \equiv \sum_{j=1}^{p-1} \frac{q^j}{[j]_q^2} = \sum_{j=1}^{\frac{p-1}{2}} \frac{q^{2j}}{[2j]_q^2} + \sum_{j=1}^{\frac{p-1}{2}} \frac{q^{p-2j}}{[p-2j]_q^2} \equiv$$

$$\equiv 2\sum_{j=1}^{\frac{p-1}{2}} \frac{q^{2j}}{[2j]_q^2} \pmod{[p]_q},$$

namely,

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{q^{2j}}{[2j]_q^2} \equiv -\frac{p^2-1}{24}(1-q)^2 \pmod{[p]_q}. \tag{2.5}$$

By (1.2), we have

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{[2j]_q} \equiv -Q_p(2,q) \pmod{[p]_q}. \tag{2.6}$$

Hence, with the help of

$$\frac{q^{2j}}{[2j]_q^2} = \frac{1}{[2j]_q^2} - (1-q)\frac{1}{[2j]_q},$$

(2.5) and (2.6), we obtain

$$\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{[2j]_q^2} = \sum_{j=1}^{\frac{p-1}{2}} \frac{q^{2j}}{[2j]_q^2} + (1-q)\sum_{j=1}^{\frac{p-1}{2}} \frac{1}{[2j]_q} \equiv$$

$$\equiv -\frac{p^2-1}{24}(1-q)^2 - Q_p(2,q)(1-q) \pmod{[p]_q}.$$

Lemma 2.1 is proved.

3. Proofs of Theorems 1.1 – 1.4. ***Proof of Theorem* 1.1.** We first prove (1.3) and (1.4). Observe that

$$\sum_{k=1}^{p-1} q^k H_k^{(2)}(q) = \sum_{j=1}^{p-1} \frac{1}{[j]_q^2} \sum_{k=j}^{p-1} q^k = \sum_{j=1}^{p-1} \frac{1}{[j]_q^2} \frac{q^j - q^p}{1-q} =$$

$$= \sum_{j=1}^{p-1} \frac{1}{[j]_q^2} \left(\frac{1-q^p}{1-q} - \frac{1-q^j}{1-q} \right) =$$

$$= [p]_q H_{p-1}^{(2)}(q) - H_{p-1}(q).$$

In view of (1.1) and (2.1), we obtain

$$\sum_{k=1}^{p-1} q^k H_k^{(2)}(q) \equiv -\frac{p-1}{2}(1-q) - \frac{(p-1)(p-3)}{8}(1-q)^2[p]_q \pmod{[p]_q^2}.$$

This proves (1.3).

By [3] (Theorem 1.2),

$$\sum_{k=1}^{p-1} q^k H_k(q) \equiv 1-p+\frac{p-1}{2}(1-q)[p]_q + \frac{p^2-1}{24}(1-q)^2[p]_q^2 \pmod{[p]_q^3}$$

which implies that

$$\sum_{k=1}^{p-1} q^k H_k(q) \equiv 1-p+\frac{p-1}{2}(1-q)[p]_q \pmod{[p]_q^2}. \tag{3.1}$$

Then (1.4) follows from (1.3), (3.1) and the fact $H_k^{(2)}(q) - \widetilde{H}_k^{(2)}(q) = (1-q)H_k(q)$.

We now show (1.5) and (1.6). Similarly, we can arrive at

$$\sum_{k=1}^{p-1} q^k H_k^{(3)}(q) = [p]_q H_{p-1}^{(3)}(q) - H_{p-1}^{(2)}(q) \equiv$$

$$\equiv -H_{p-1}^{(2)}(q) \pmod{[p]_q}.$$

We use the above and (2.1) to get

$$\sum_{k=1}^{p-1} q^k H_k^{(3)}(q) \equiv \frac{(p-1)(p-5)}{12}(1-q)^2 \pmod{[p]_q},$$

which proves (1.5). Then (1.6) follows from (1.5) and the fact $H_k^{(3)}(q) - \widetilde{H}_k^{(3)}(q) = (1-q)H_k^{(2)}(q)$.

Theorem 1.1 is proved.

***Proof of Theorem* 1.2.** We first prove (1.7). Notice that

$$I_{p-1}(q) = \sum_{k=1}^{p-1} \frac{(-1)^k+1}{[k]_q} - \sum_{k=1}^{p-1} \frac{1}{[k]_q} =$$

$$= 2\sum_{k=1}^{\frac{p-1}{2}} \frac{1}{[2k]_q} - \sum_{k=1}^{p-1} \frac{1}{[k]_q}.$$

By (1.1) and (1.2), we obtain

$$I_{p-1}(q) \equiv -2Q_p(2,q) - \frac{p-1}{2}(1-q) +$$

$$+\left(Q_p(2,q)^2 + Q_p(2,q)(1-q) + \frac{p^2-1}{12}(1-q)^2\right)[p]_q \pmod{[p]_q^2}.$$

We now show (1.8). Note that

$$I^{(2)}_{p-1}(q)=\sum_{k=1}^{p-1}\frac{(-1)^k+1}{[k]_q^2}-\sum_{k=1}^{p-1}\frac{1}{[k]_q^2}=$$

$$=2\sum_{k=1}^{\frac{p-1}{2}}\frac{1}{[2k]_q^2}-\sum_{k=1}^{p-1}\frac{1}{[k]_q^2}.$$

With the help of (2.1) and (2.3), we get

$$I^{(2)}_{p-1}(q)\equiv\frac{1-p}{2}(1-q)^2-2Q_p(2,q)(1-q)\pmod{[p]_q}.$$

Theorem 1.2 is proved.

***Proof of Theorem* 1.3.** Observe that

$$\sum_{k=1}^{p-1}q^kI_k(q)=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\sum_{k=j}^{p-1}q^k=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\frac{q^j-q^p}{1-q}=$$

$$=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\left(\frac{1-q^p}{1-q}-\frac{1-q^j}{1-q}\right)=[p]_qI_{p-1}(q)$$

and

$$\sum_{k=1}^{p-1}q^kI^{(2)}_k(q)=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q^2}\sum_{k=j}^{p-1}q^k=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q^2}\frac{q^j-q^p}{1-q}=$$

$$=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q^2}\left(\frac{1-q^p}{1-q}-\frac{1-q^j}{1-q}\right)=$$

$$=[p]_qI^{(2)}_{p-1}(q)-I_{p-1}(q).$$

By (1.7) and (1.8), we arrive at

$$\sum_{k=1}^{p-1}q^kI_k(q)\equiv\left(-2Q_p(2,q)-\frac{p-1}{2}(1-q)\right)[p]_q+$$

$$+\left(Q_p(2,q)^2+Q_p(2,q)(1-q)+\frac{p^2-1}{12}(1-q)^2\right)[p]_q^2\pmod{[p]_q^3}$$

and

$$\sum_{k=1}^{p-1}q^kI^{(2)}_k(q)\equiv 2Q_p(2,q)+\frac{p-1}{2}(1-q)-$$

$$-\left(Q_p(2,q)^2+3Q_p(2,q)(1-q)+\frac{(7+p)(p-1)}{12}(1-q)^2\right)[p]_q\pmod{[p]_q^2},$$

which completes the proof of Theorem 1.3.

***Proof of Theorem* 1.4.** Note that

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{[k]_q}I_k(q)=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\sum_{k=j}^{p-1}\frac{(-1)^k}{[k]_q}=$$

$$=\sum_{j=1}^{p-1}\frac{(-1)^j}{[j]_q}\left(\sum_{k=1}^{p-1}\frac{(-1)^k}{[k]_q}-\sum_{k=1}^{j}\frac{(-1)^k}{[k]_q}+\frac{(-1)^j}{[j]_q}\right).$$

Hence,

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{[k]_q}I_k(q)=\frac{1}{2}\left(I_{p-1}^2(q)+\sum_{j=1}^{p-1}\frac{1}{[j]_q^2}\right). \tag{3.2}$$

By (1.7), we have

$$I_{p-1}(q)\equiv -2Q_p(2,q)-\frac{p-1}{2}(1-q) \pmod{[p]_q},$$

which implies that

$$I_{p-1}^2(q)\equiv 4Q_p^2(2,q)+2(p-1)(1-q)+\frac{(p-1)^2}{4}(1-q)^2 \pmod{[p]_q}. \tag{3.3}$$

With the help of (2.1), (3.2) and (3.3), we obtain

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{[k]_q}I_k(q)\equiv 2Q_p^2(2,q)+(p-1)(1-q)+\frac{p^2-1}{12}(1-q)^2 \pmod{[p]_q},$$

which completes the proof of Theorem 1.4.

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