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CONTACT CR-WARPED PRODUCT OF SUBMANIFOLDS OF THE GENERALIZED SASAKIAN SPACE FORMS ADMITTING THE NEARLY TRANS-SASAKIAN STRUCTURE

КОНТАКТНИЙ CR-ВИКРИВЛЕНИЙ ДОБУТОК ПІДМНОГОВИДІВ УЗАГАЛЬНЕНИХ ПРОСТОРОВИХ ФОРМ САСАКІ, ЩО ДОПУСКАЮТЬ МАЙЖЕ ТРАНССТРУКТУРУ САСАКІ

In the present paper, we apply Hopf's lemma to the contact CR-warped product of submanifolds of the generalized Sasakian space forms admitting nearly trans-Sasakian structure and establish a characterization inequality for the existence of these types of warped products. This inequality generalizes the inequalities obtained in [M. Atceken, Bull. Iran. Math. Soc. – 2013. – **39**, № 3. – P. 415–429; M. Atceken, Collect. Math. – 2011. – **62**, № 1. – P. 17–26, and Sibel Sular, Cihan Özgür, Turkish J. Math. – 2012. – **36**. – P. 485–497]. Moreover, we also compute another inequality for the squared norm of the second fundamental form in terms of warping functions. This inequality is a generalization of the inequalities acquired in [I. Mihai, Geom. Dedicata. – 2004. – **109**. – P. 165–173 and K. Arslan, R. Ezentas, I. Mihai, C. Murathan, J. Korean Math. Soc. – 2005. – **42**, № 5. – P. 1101–1110]. The inequalities proved in the paper either generalize or improve all inequalities available in the literature and related to the squared norm of the second fundamental form for contact CR-warped product of submanifolds of any almost contact metric manifold.

У роботі лему Хопфа застосовано до контактного CR-викривленого добутку підмноговидів узагальнених просторових форм Сасакі, що допускають майже трансструктуру Сасакі, та встановлено характеризаційну нерівність щодо існування викривлених добутків такого типу. Ця нерівність узагальнює нерівності, що були встановлені в [M. Atceken, Bull. Iran. Math. Soc. – 2013. – **39**, № 3. – P. 415–429; M. Atceken, Collect. Math. – 2011. – **62**, № 1. – P. 17–26 та Sibel Sular, Cihan Özgür, Turkish J. Math. – 2012. – **36**. – P. 485–497]. Крім того, отримано іншу нерівність для квадрата норми другої фундаментальної форми в термінах викривляючих функцій. Ця нерівність узагальнює нерівності, що були встановлені в [I. Mihai, Geom. Dedicata. – 2004. – **109**. – P. 165–173 та K. Arslan, R. Ezentas, I. Mihai, C. Murathan, J. Korean Math. Soc. – 2005. – **42**, № 5. – P. 1101–1110]. Нерівності, що доведені в роботі, узагальнюють або поліпшують усі нерівності, доступні в літературі, що відносяться до квадрата норми другої фундаментальної форми для контактного CR-викривленого добутку підмноговидів будь-якого майже контактного метричного многовиду.

1. Introduction. Gray and Hervella classified the almost Hermitian manifolds [2], in this classification there exists a class $\mathcal{W}_4$ of almost Hermitian manifolds, which is closely related to a locally conformal Kaehler manifold. An almost contact metric structure on a manifold $\bar{M}$ is called a trans-Sasakian structure if the product manifold $\bar{M} \times R$ belongs to class $\mathcal{W}_4$ [17]. The class $\mathcal{C}_6 \oplus \mathcal{C}_5$ coincides with the class of trans-Sasakian structure of the type (α, β). This trans-Sasakian structure is cosymplectic or Sasakian or Kenmotsu if $\alpha = 0, \beta = 0$, or $\beta = 0$ or $\alpha = 0$. Later on, D. Chinea and C. Gonzalez [11] generalized these structures actually they divided almost contact structure into twelve different classes. An almost contact metric manifold is nearly trans-Sasakian manifold if it is associated to the class $\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6$. Recently, C. Gherghe [7] introduced a nearly trans-Sasakian structure of the type (α, β) which is the generalization of the trans-Sasakian manifold of the type (α, β). Moreover, if $\beta = 0$ or $\alpha = 0$ or $\alpha = \beta = 0$, then nearly trans-Sasakian structure of the type (α, β) becomes nearly Sasakian [9] or nearly Kenmotsu [18] or nearly cosymplectic [8], respectively.

On the other hand the notion of CR-warped product submanifolds as a natural generalization of CR-products was introduced by B. Y. Chen (see [3, 5]). Basically, Chen obtained some basic results for CR-warped product submanifolds of Kaehler manifolds and established a sharp relationship between the warping function f and squared norm of the second fundamental form. Later, I. Hesigawa and I. Mihai proved a similar inequality for contact CR-warped product submanifolds of Sasakian manifolds [14]. Moreover, I. Mihai in [15] improved same inequality for contact CR-warped product submanifolds of Sasakian space form. Furthermore, in [19] K. Arslan et al. established a sharp estimation for contact CR-warped product submanifolds in the setting of Kenmotsu space forms. Many geometers obtained similar estimation for different setting of almost contact metric manifolds (see references).

In the other direction, M. Ateceken [20, 21], Sibel Sular and Cihan Özgür [26] proved the characterizing inequalities for existence the contact CR-warped product submanifolds of cosymplectic space forms, Kenmotsu space forms and generalized Sasakian space forms admitting a trans-Sasakian structure.

In the present study, we consider contact CR-warped product submanifolds of nearly trans-Sasakian generalized Sasakian space forms and obtain a characterizing inequality for the squared norm of the second fundamental form. Finally, we also establish a sharp inequality for squared norm of the second fundamental form in terms of warping function. Our inequalities generalize or improve all the inequalities for contact CR-warped products in any contact metric manifold.

2. Preliminaries. A $(2n+1)$-dimensional C^∞-manifold $\bar{M}$ is said to have an almost contact structure if there exist on $\bar{M}$ a tensor field ϕ of type $(1,1)$ a vector field ξ and a 1-form η satisfying [10]

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.$$

There always exists a Riemannian metric g on an almost contact metric manifold $\bar{M}$ satisfying the conditions

$$\eta(X) = g(X,\xi), \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for all $X, Y \in T\bar{M}$.

An almost contact structure (ϕ, ξ, η) is said to be normal if the almost complex structure J on the product manifold $\bar{M} \times R$ given by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right),$$

where f is a C^∞-function on $\bar{M} \times R$, has no torsion, that is J is integrable and the condition for normality in terms of ϕ, ξ and η is $[\phi,\phi] + 2d\eta \otimes \xi$ on $\bar{M}$, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. Finally, the *fundamental* 2*-form* Φ is defined by $\Phi(X,Y) = g(X, \phi Y)$.

An almost contact metric manifold is said to be trans-Sasakian manifold if [7]

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.1}$$

for all $X, Y \in T\bar{M}$.

An almost contact metric manifold is said to be nearly trans-Sasakian manifold if

$$(\bar{\nabla}_X \phi)Y + (\bar{\nabla}_Y \phi)X = \alpha(2g(X,Y) - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)\phi X + \eta(X)\phi Y \tag{2.2}$$

for all $X, Y \in T\bar{M}$.

Given an almost contact metric manifold $\bar{M}$, it is said to be a generalized Sasakian space form [23] if there exist three functions $f_1,\ f_2$ and f_3 on $\bar{M}$ such that

$$\bar{R}(X,Y)Z = f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X+$$

$$+2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\} \tag{2.3}$$

for any vector fields X, Y, Z on $\bar{M}$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$. If $f_1 = \dfrac{c+3}{4}$, $f_2 = f_3 = \dfrac{c-1}{4}$, then $\bar{M}$ is Sasakian space form [10]; if $f_1 = \dfrac{c-3}{4}$, $f_2 = f_3 = \dfrac{c+1}{4}$, then $\bar{M}$ is a Kenmotsu space form [18]; if $f_1 = f_2 = f_3 = \dfrac{c}{4}$, then $\bar{M}$ is a cosymplectic space form [23].

Let M be a submanifold of an almost contact metric manifold $\bar{M}$ with induced metric g, and if ∇ and $\nabla^{\perp}$ are the induced connection on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively, then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.5}$$

for each $X, Y \in TM$ and $N \in T^{\perp}M$, where h and A_N are the second fundamental form and the shape operator, respectively, for the immersion of M in $\bar{M}$, they are related as

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.6}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as on M.

The mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i),$$

where n is the dimension of M and $\{e_1, e_2, \ldots, e_n\}$ is a local orthonormal frame of vector fields on M. The squared norm of the second fundamental form is defined as

$$\|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i,e_j), h(e_i,e_j)). \tag{2.7}$$

A submanifold M of $\bar{M}$ is said to be a totally geodesic submanifold if $h(X,Y) = 0$, for each $X, Y \in TM$, and totally umbilical submanifold if $h(X,Y) = g(X,Y)H$.

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.8}$$

where PX is the tangential component and FX is the normal component of ϕX.

Similarly, for $N \in T^{\perp}M$, we can write

$$\phi N = tN + fN, \tag{2.9}$$

where tN and fN are the tangential and normal components of ϕN, respectively.

The covariant differentiation of the tensors ϕ, P, F, t and f are defined as respectively

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \tag{2.10}$$

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \tag{2.11}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \tag{2.12}$$

$$(\bar{\nabla}_X t)N = \nabla_X tN - t\nabla^{\perp}_X N, \tag{2.13}$$

$$(\bar{\nabla}_X f)N = \nabla^{\perp}_X fN - f\nabla^{\perp}_X N. \tag{2.14}$$

Furthermore, for any $X, Y \in TM$, the tangential and normal parts of $(\bar{\nabla}_X \phi)Y$ are denoted by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$, i.e.,

$$(\bar{\nabla}_X \phi)Y = \mathcal{P}_X Y + \mathcal{Q}_X Y. \tag{2.15}$$

On using equations (2.4) – (2.12) and (2.15), we may obtain

$$\mathcal{P}_X Y = (\bar{\nabla}_X P)Y - A_{FY}X - th(X,Y),$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X F)Y + h(X,TY) - fh(X,Y).$$

Similarly, for $N \in T^{\perp}M$, denoting by $\mathcal{P}_X N$ and $\mathcal{Q}_X N$ respectively the tangential and normal parts of $(\bar{\nabla}_X \phi)N$, we find

$$\mathcal{P}_X N = (\bar{\nabla}_X t)N + PA_N X - A_{fN}X,$$

$$\mathcal{Q}_X N = (\bar{\nabla}_X f)N + h(tN, X) + FA_N X.$$

On a submanifold M of a nearly trans-Sasakian manifold by (2.1) and (2.15)

$$\mathcal{P}_X Y + \mathcal{P}_Y X = \alpha(2g(X,Y)\xi - \eta(Y)X - \eta(X)Y) - \beta(\eta(Y)PX + \eta(X)PY), \tag{2.16a}$$

$$\mathcal{Q}_X Y + \mathcal{Q}_Y X = -\beta(\eta(Y)FX + \eta(X)FY) \tag{2.16b}$$

for any $X, Y \in TM$.

An m-dimensional Riemannian submanifold M of an almost contact metric manifold $\bar{M}$, where ξ is tangent to M, is called *contact CR-submanifold* if it admits an invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti invariant, that is

$$TM = D \oplus D^{\perp} \oplus \langle \xi \rangle,$$

where $\phi D \subseteq D$, $\phi D^{\perp} \subseteq T^{\perp}M$ and $\langle \xi \rangle$ denotes 1-dimensional distribution which is spanned by ξ.

If μ is the invariant subspace of the normal bundle $T^{\perp}M$, then in the case of contact CR-submanifold, the normal bundle $T^{\perp}M$ can be decomposed as follows:

$$T^{\perp}M = \mu \oplus \phi D^{\perp}.$$

A contact CR-submanifold M is called the contact CR-product submanifold if the distributions D and $D^{\perp}$ are parallel on M. In this case M is foliated by the leaves of these distributions. In general, if N_1 and N_2 are Riemannian manifolds with Riemannian metrics g_1 and g_2 respectively, then the product manifold $(N_1 \times N_2, g)$ is a Riemannian manifold with Riemannian metric g defined as

$$g(X, Y) = g_1(d\pi_1 X, d\pi_1 Y) + g_2(d\pi_2 X, d\pi_2 Y),$$

where π_1 and π_2 are the projection maps of M onto N_1 and N_2, respectively, and $d\pi_1$, $d\pi_2$ are their differentials.

As a generalization of the product manifold and in particular of contact CR-product submanifold, one can consider the warped product of manifolds which are defined as follows.

Definition 2.1. *Let (B, g_B) and (C, g_C) be two Riemannian manifolds with Riemannian metric g_B and g_C, respectively, and f be a positive differentiable function on B. The warped product of B and C is the Riemannian manifold $(B \times C, g)$, where*

$$g = g_B + f^2 g_C.$$

For a warped product manifold $N_1 \times_f N_2$, we denote by D_1 and D_2 the distributions defined by the vectors tangent to the leaves and fibers, respectively. In other words, D_1 is obtained by the tangent vectors of N_1 via the horizontal lift and D_2 is obtained by the tangent vectors of N_2 via vertical lift. In case of contact CR-warped product submanifolds D_1 and D_2 are replaced by D and $D^{\perp}$, respectively.

The warped product manifold $(B \times C, g)$ is denoted by $B \times_f C$. If X is the tangent vector field to $M = B \times_f C$ at (p, q), then

$$\|X\|^2 = \|d\pi_1 X\|^2 + f^2(p)\|d\pi_2 X\|^2.$$

R. L. Bishop and B. O'Neill [24] proved the following theorem.

Theorem 2.1. *Let $M = B \times_f C$ be warped product manifolds. If $X, Y \in TB$ and $V, W \in TC$, then*:

(i) $\nabla_X Y \in TB$,

(ii) $\nabla_X V = \nabla_V X = \left(\dfrac{Xf}{f}\right) V$,

(iii) $\nabla_V W = \nabla^C_V W - g(V, W)\nabla \ln f$.

From above theorem, for the warped product $M = B \times_f C$ it is easy to conclude that

$$\nabla_X V = \nabla_V X = (X \ln f)V \tag{2.17}$$

for any $X \in TB$ and $V \in TC$.

∇f is the gradient of f and is defined as

$$g(\nabla f, X) = Xf \tag{2.18}$$

for all $X \in TM$.

***Corollary* 2.1.** *On a warped product manifold $M = N_1 \times_f N_2$, the following statements hold*:

(i) N_1 *is totally geodesic in* M,

(ii) N_2 *is totally umbilical in* M.

In what follows, $N_\perp$ and N_T will denote a anti-invariant and invariant submanifold, respectively, of an almost contact metric manifold $\bar{M}$.

A warped product manifold is said to be trivial if its warping function f is constant. More generally, a trivial warped product manifold $M = N_1 \times N_2$ is a Riemannian product $N_1 \times N_2^f$, where N_2^f is the manifold with the Riemannian metric $f^2 g_2$ which is homothetic to the original metric g_2 of N_2. For example, a trivial contact CR-warped product is contact CR-product.

Let M be a m-dimensional Riemannian manifold with Riemannian metric g and let $\{e_1, \ldots, e_m\}$ be an orthogonal basis of TM. As a consequence of (2.18), we have

$$\|\nabla f\|^2 = \sum_{i=1}^{m} (e_i(f))^2.$$

The Laplacian of f is defined by

$$\Delta f = \sum_{i=1}^{m} \{(\nabla_{e_i} e_i) f - e_i e_i f\}. \tag{2.19}$$

Now, we state the Hopf's lemma.

Hopf's lemma [5]. *Let M be a n-dimensional connected compact Riemannian manifold. If ψ is differentiable function on M such that $\Delta\psi \geq 0$ everywhere on M (or $\Delta\psi \leq 0$ everywhere on M), then ψ is a constant function.*

3. Contact CR-warped product submanifolds. In this section we consider contact CR-warped product of the type $N_T \times_f N_\perp$ of the nearly trans-Sasakian manifolds $\bar{M}$, where N_T and $N_\perp$ are the invariant and anti-invariant submanifolds respectively of $\bar{M}$. Throughout, this section, we consider ξ tangent to N_T.

Now we have some fundamental results in the following lemma for later use.

Lemma 3.1 [1]. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold $\bar{M}$ such that N_T and $N_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. Then we have*:

(i) $\xi \ln f = \beta$,

(ii) $g(h(X, Y), \phi Z) = 0$,

(iii) $g(h(X, Z), \phi Z) = -\{(\phi X \ln f) + \alpha\eta(X)\}\|Z\|^2$,

(iv) $g(h(\xi, Z), \phi Z) = -\alpha\|Z\|^2$,

(v) $g(h(\phi X, Z), \phi Z) = (X \ln f - \beta\eta(X))\|Z\|^2$

for any $X \in TN_T$ and $Z \in TN_\perp$.

Now, we have the following lemma.

Lemma 3.2. *Let $M = N_T \times_f N_\perp$ be a contact CR-warped product submanifold of a nearly trans-Sasakian manifold $\bar{M}$. Then*

$$g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 - g(\phi h(X, Z), \mathcal{Q}_X Z)$$

for any $X \in TN_T$ and $Z \in TN_\perp$.

Proof. By (2.4) and (2.10)

$$h(\phi X, Z) = (\bar{\nabla}_Z \phi)X + \phi\nabla_Z X + \phi h(X, Z) - \nabla_Z \phi X.$$

Thus by using (2.15) and (2.17)

$$h(\phi X, Z) = \mathcal{P}_Z X + \mathcal{Q}_Z X + X \ln f \phi Z + \phi h(X, Z) - \phi X \ln f Z.$$

Comparing normal parts

$$h(\phi X, Z) = \mathcal{Q}_Z X + X \ln f \phi Z + \phi h_\mu(X, Z)$$

or

$$g(h(\phi X, Z), \phi h(X, Z)) = g(\mathcal{Q}_Z X, \phi h(X, Z)) + \|h_\mu(X, Z)\|^2.$$

By using (2.16b), we get

$$g(h(\phi X, Z), \phi h(X, Z)) = \|h_\mu(X, Z)\|^2 - g(\phi h(X, Z), \mathcal{Q}_X Z).$$

Next we prove the following characterization theorem.

Theorem 3.1. *Let $M = N_T \times_f N_\perp$ be a contact* CR-*warped product submanifold of generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting the nearly trans-Sasakian structure such that N_T is connected and compact. Then M is contact CR-product submanifold if either one of the following inequality holds*:

(i) $\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \geq f_2 pq + \sum_{i=1}^{p}\sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2,$

(ii) $\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \leq f_2 pq,$

where h_μ denotes the component of h in μ, $p+1$ and q are the dimensions of N_T and $N_\perp$.

Proof. For any unit vector fields X tangent to N_T and orthogonal to ξ and Z tangent to $N_\perp$. Then from (2.3) we have

$$\bar{R}(X, \phi X, Z, \phi Z) = -2 f_2 g(X, X) g(Z, Z). \tag{3.1}$$

On the other hand by Coddazi equation

$$\begin{aligned}\bar{R}(X, \phi X, Z, \phi Z) = g(\nabla^\perp_X h(\phi X, Z), \phi Z) - g(h(\nabla_X \phi X, Z), \phi Z) - \\ - g(h(\phi X, \nabla_X Z), \phi Z) - g(\nabla^\perp_{\phi X} h(X, Z), \phi Z) + \\ + g(h(\nabla_{\phi X} X, Z), \phi Z) + g(h(X, \nabla_{\phi X} Z), \phi Z).\end{aligned} \tag{3.2}$$

By using part (iii) of Lemma 3.1, (2.10), (2.4) and (2.15), we get

$$\begin{aligned}g(\nabla^\perp_X h(\phi X, Z), \phi Z) = X g(h(\phi X, Z), \phi Z) - g(h(\phi X, Z), \bar{\nabla}_X \phi Z) = \\ = X(X \ln f g(Z, Z)) - g(h(\phi X, Z), (\bar{\nabla}_X \phi) Z + \phi \bar{\nabla}_X Z).\end{aligned}$$

On further simplification above equation yields

$$\begin{aligned}g(\nabla^\perp_X h(\phi X, Z), \phi Z) = X^2 \ln f g(Z, Z) + 2(X \ln f)^2 g(Z, Z) - g(h(\phi X, Z), \mathcal{Q}_X Z) - \\ - g(h(\phi X, Z), \phi h(X, Z)) - X \ln f g(h(\phi X, Z), \phi Z).\end{aligned}$$

Utilizing the part (v) of Lemma 3.1 and Lemma 3.2, we have

$$g(\nabla^{\perp}_X h(\phi X, Z), \phi Z) = X^2 \ln f g(Z,Z) + (X \ln f)^2 g(Z,Z) - \|h_\mu(X,Z)\|^2 -$$
$$-g(\phi h(X,Z) - h(\phi X, Z), \mathcal{Q}_X Z).$$

Further, using (2.4), (2.15), (2.16b) and (2.17) in the last term of above equation, we obtain

$$g(\nabla^{\perp}_X h(\phi X, Z), \phi Z) = X^2 \ln f g(Z,Z) + (X \ln f)^2 g(Z,Z) - \|h_\mu(X,Z)\|^2 + \|\mathcal{Q}_X Z\|^2. \quad (3.3)$$

In the same way, we can calculate

$$-g(\nabla^{\perp}_{\phi X} h(X, Z), \phi Z) = (\phi X)^2 \ln f g(Z,Z) + (\phi X \ln f)^2 g(Z,Z) -$$
$$-\|h_\mu(\phi X,Z)\|^2 + \|\mathcal{Q}_{\phi X} Z\|^2. \quad (3.4)$$

From part (iii) of Lemma 3.1, we get

$$g(A_{\phi Z} Z, \phi X) = X \ln f,$$

replacing X by $\nabla_X X$

$$g(A_{\phi Z} Z, \phi \nabla_X X) = \nabla_X X \ln f.$$

By using the Gauss formula in preceding equation, we have

$$g(A_{\phi Z} Z, \phi(\bar{\nabla}_X X - h(X,X)) = \nabla_X X \ln f. \quad (3.5)$$

By use of (2.4), (2.10), (2.2) and (2.17), it is straightforward to see that $h(X,X) \in \mu$, applying this fact in (3.5), we obtain

$$g(A_{\phi Z} Z, \bar{\nabla}_X \phi X - (\bar{\nabla}_X \phi) X) = \nabla_X X \ln f.$$

In view of (2.2) the previous equation abridged to

$$g(h(\nabla_X \phi X, Z), \phi Z) = \nabla_X X \ln f g(Z,Z). \quad (3.6)$$

Correspondingly,

$$g(h(\nabla_{\phi X} X, Z), \phi Z) = -\nabla_{\phi X} \phi X \ln f g(Z,Z). \quad (3.7)$$

By use of (2.17) and part (iii) of Lemma 3.1, it is simple to see the following:

$$g(h(\phi X, \nabla_X Z), \phi Z) = (X \ln f)^2 g(Z,Z) \quad (3.8)$$

and

$$g(h(X, \nabla_{\phi X} Z), \phi Z) = -(\phi X \ln f)^2 g(Z,Z). \quad (3.9)$$

Substituting (3.3), (3.4), (3.6), (3.7), (3.8) and (3.9) in (3.2), we find

$$\bar{R}(X, \phi X, Z, \phi Z) = X^2 \ln f g(Z,Z) + (\phi X)^2 \ln f g(Z,Z) - \nabla_X X \ln f g(Z,Z) -$$
$$-\nabla_{\phi X} \phi X g(Z,Z) - \|h_\mu(X,Z)\|^2 - \|h(\phi X, Z)\|^2 + \|\mathcal{Q}_X Z\|^2 + \|\mathcal{Q}_{\phi X} Z\|^2. \quad (3.10)$$

Let $\{e_0 = \xi, e_1, e_2, \ldots, e_{p/2}, \phi e_1, \phi e_2, \ldots, e_p = \phi e_{p/2}, e^1, e^2, \ldots, e^q\}$ be an orthonormal frame of TM such that $\{e_0, e_1, \ldots, e_{p/2}, \phi e_1, \phi e_2, \ldots, \phi e_{p/2}\}$ are tangent to TN_T and $\{e^1, e^2, \ldots, e^q\}$ are tangent to $TN_\perp$.

By using (3.1) and (2.19) in (3.10) and summing over $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, we get

$$q\Delta \ln f = f_2 pq - \sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 + \sum_{i=1}^{p}\sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2. \tag{3.11}$$

From Hopf's lemma and (3.11), if

$$\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \geq f_2 pq + \sum_{i=1}^{p} \|\mathcal{Q}_{e_i} e^j\|^2$$

or

$$\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \leq f_2 pq,$$

then the warping function f is constant on M, i.e., M is simply a contact CR-product submanifold, which proves the theorem completely.

Now we have the following corollary, which can be confirmed straightforwardly.

***Corollary* 3.1.** *Let $M = N_T \times_f N_\perp$ be a contact* CR-*warped product submanifolds of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting the nearly trans-Sasakian structure such that N_T is connected and compact. Then M is contact* CR-*product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 = f_2 pq + \sum_{i=1}^{p}\sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2,$$

where h_μ denotes the component of h in μ, $p+1$ and q are the dimensions of N_T and $N_\perp$.

Furthermore, if the ambient manifold $\bar{M}$ is a generalized Sasakian manifolds with trans-Sasakian structure, then from above findings we have the following corollary.

***Corollary* 3.2.** *Let $M = N_T \times_f N_\perp$ be a contact* CR-*warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting the trans-Sasakian structure such that N_T is connected and compact. Then M is contact* CR-*product submanifold if either one of the inequality*

$$\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \geq f_2 pq,$$

or

$$\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 \leq f_2 pq$$

holds, where h_μ denotes the component of h in μ, $p+1$ and q are the dimensions of N_T and $N_\perp$.

***Remark* 3.1.** In the above corollary, the first characterizing inequality was also proved by Sibel Sular and Cihan Özgür in [26]. In particualr the above inequalities also generalize the results obtained in [20, 21].

***Corollary* 3.3.** *Let $M = N_T \times_f N_\perp$ be a contact* CR-*warped product submanifolds of a generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ admitting the trans-Sasakian structure such that N_T is compact. Then M is contact* CR-*product if and only if*

$$\sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 = f_2 pq,$$

where h_μ denotes the component of h in μ, $p+1$ and q are the dimensions of N_T and $N_\perp$.

***Remark* 3.2.** Similar consequence in Corollary 3.3 was also proved in [26].

4. Another inequality. In the present section, we estimate the squared norm of the second fundamental form in terms of warping function.

Theorem 4.1. *Let $\bar{M}(f_1, f_2, f_3)$ be a $(2n+1)$-dimensional generalized Sasakian space form admitting the nearly trans-Sasakian structure and $M = N_T \times_f N_\perp$ be an m-dimensional contact* CR-*warped product submanifold, such that N_1 is $(p+1)$-dimensional invariant submanifold tangent to ξ, where p is an even number and $N_\perp$ be a q-dimensional anti-invariant submanifold of $\bar{M}(f_1, f_2, f_3)$. Then*:

(i) *The squared norm of the second fundamental form h satisfies*

$$\|h\|^2 \geq q[\|\nabla \ln f\|^2 - \Delta \ln f - \alpha - \beta] + f_2 pq + \|\mathcal{Q}_D D^\perp\|^2, \tag{4.1}$$

where Δ denotes the Laplace operator on N_T.

(ii) *The equality sign of* (4.1) *holds identically if and only if we have*

(a) *N_T is totally geodesic invariant submanifold of $\bar{M}(f_1, f_2, f_3)$; hence, N_T is a generalized Sasakian space form admitting the nearly trans-Sasakian structure,*

(b) *$N_\perp$ is a totally umbilical anti-invariant submanifold of $\bar{M}(f_1, f_2, f_3)$.*

Proof. For any $X \in TN_T - \langle \xi \rangle$ and $Z \in TN_\perp$, from Lemma 3.1 we have

$$g(h(\xi, Z), \phi Z) = -\alpha \|Z\|^2$$

and

$$g(h(\phi X, Z), \phi Z) = X \ln f \|Z\|^2.$$

Since $\xi \ln f = \beta$, then combining this with above two equations, we get

$$\sum_{i=0}^{p} \sum_{j=1}^{q} \|h_{\phi D^\perp}(e_i, e^j)\|^2 = q[\|\nabla \ln f\|^2 - \alpha - \beta]. \tag{4.2}$$

Once more from (3.11)

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \|h_\mu(e_i, e^j)\|^2 = f_2 pq - q\Delta \ln f + \sum_{i=1}^{p} \sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2. \tag{4.3}$$

We use the following notation:

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \|\mathcal{Q}_{e_i} e^j\|^2 = \|\mathcal{Q}_D D^\perp\|^2.$$

Substituting above notation in (4.3) and combining it with (4.2), we acquire the inequality (4.1).

Let h'' be the second fundamental form of $N_\perp$ in M. Then we have

$$g(h''(Z, W), X) = g(\nabla_Z W, X) = -X \ln f g(Z, W),$$

on using (2.18), we get

$$h''(Z, W) = -g(Z, W) \nabla \ln f. \tag{4.4}$$

If the equality sign of (4.1) holds identically, then we achieve

$$h(D,D)=0, \quad h(D^{\perp},D^{\perp})=0. \tag{4.5}$$

The first condition of (4.5) implies that N_T is totally geodesic in M. On the other hand, one has

$$g(h(X,\phi Y),\phi Z)=g(\bar{\nabla}_X\phi Y,\phi Z)=-g(\phi Y,(\bar{\nabla}_X\phi)Z). \tag{4.6}$$

By use of (2.10) and (2.4) we get the following equation:

$$g(\phi Y,(\bar{\nabla}_Z\phi)X)=g(\phi Y,\nabla_Z\phi X)-g(Y,\nabla_Z X),$$

in view of (2.17) the above equation reduced to

$$g(\phi Y,(\bar{\nabla}_Z\phi)X)=0. \tag{4.7}$$

From (4.6), (4.7) and (2.16a) we have

$$g(h(X,\phi Y),\phi Z)=-g(\phi Y,(\bar{\nabla}_X\phi)Z+(\bar{\nabla}_Z\phi)X)=0. \tag{4.8}$$

From (4.8), it is evident that N_T is totally geodesic in $\bar{M}(f_1,f_2,f_3)$ and hence is a generalized Sasakian space form admitting the nearly trans-Sasakian structure.

The second condition of (4.5) and (4.4) imply that $N_{\perp}$ is totally umbilical in $\bar{M}(f_1,f_2,f_3)$.

In the last we have the following corollary which can be deduced from inequality (4.1).

***Corollary* 4.1.** *Let $M=N_T\times_f N_{\perp}$ be a contact CR-warped product submanifold of a generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ admitting the trans-Sasakian structure, then squared norm of the second fundamental form satisfies*

$$\|h\|^2\geq q[\|\nabla\ln f\|^2-\Delta\ln f-\alpha-\beta]+f_2pq,$$

where Δ is the Laplace operator on N_T, and $p+1$ and q are the dimensions of N_T and $N_{\perp}$, respectively.

***Remark* 4.1.** For the contact CR-warped product submanifolds of generalized Sasakian space forms $\bar{M}(f_1,f_2,f_3)$, admitting nearly trans-Sasakian structure if we consider the tensorial equation as follows:

$$(\bar{\nabla}_X\phi)Y+(\bar{\nabla}_Y\phi)X=\alpha(2g(X,Y)-\eta(Y)X-\eta(X)Y)-\beta(\eta(Y)\phi X+\eta(X)\phi Y$$

and dimension of invariant submanifold N_T as $2p+1$. Then by similar calculations the inequality (4.1) will be change as follows:

$$\|h\|^2\geq q[\|\nabla\ln f\|^2-\Delta\ln f-\alpha-\beta]+2f_2pq+\|\mathcal{Q}_D D^{\perp}\|^2. \tag{4.9}$$

Now we have the following conclusions:

(i) If the ambient manifold $\bar{M}(f_1,f_2,f_3)$ is Kenmotsu space form, i.e., $\alpha=0,\beta=1,f_2=\dfrac{c+1}{4}$ and $\mathcal{Q}_D D^{\perp}=0$, then the inequality (4.9) reduced to the inequality obtained in Theorem 4.1 of [19].

(ii) Moreover, if the ambient manifold $\bar{M}(f_1,f_2,f_3)$ is Sasakian space form, i.e., $\alpha=0,\beta=1,f_2=\dfrac{c+1}{4}$ and $\mathcal{Q}_D D^{\perp}=0$, then the inequality (4.9) will be very much similar to the inequality (3.1) of Theorem 3.1 in [15].

***Remark* 4.2.** The inequality (4.1) of Theorem 4.1 is also an improved version of the inequality 4.1 proved in [1].

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