UDC 517.9

E. Alaidarous (King Abdulaziz Univ., Saudi Arabia),
M. Benchohra, I. Medjadj (Djillali Liabes Univ. Sidi Bel-Abbès, Algeria)

GLOBAL EXISTENCE RESULTS FOR NEUTRAL FUNCTIONAL DIFFERENTIAL INCLUSIONS WITH STATE-DEPENDENT DELAY

РЕЗУЛЬТАТИ ПРО ГЛОБАЛЬНЕ ІСНУВАННЯ РОЗВ'ЯЗКІВ НЕЙТРАЛЬНИХ ФУНКЦІОНАЛЬНИХ ДИФЕРЕНЦІАЛЬНИХ ВКЛЮЧЕНЬ ІЗ ЗАТРИМКОЮ, ЩО ЗАЛЕЖИТЬ ВІД СТАНУ

We consider the existence of global solutions for a class of neutral functional differential inclusions with state-dependent delay. The proof of the main result is based on the semigroup theory and the Bohnenblust – Karlin fixed point theorem.

Розглянуто питання про існування глобальних розв'язків одного класу нейтральних функціональних диференціальних включень із затримкою, що залежить від стану. Доведення основного результату базується на теорії напівгруп та теоремі про нерухому точку Боненблюста та Карліна.

1. Introduction. Neutral functional differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last decades. The literature relative to ordinary neutral functional differential equations is very extensive and refer to [8, 9, 11, 22, 32, 33]. Partial neutral differential equation with finite delay arise, for instance, from the transmission line theory [38]. Wu and Xia have shown in [39] that a ring array of identical resistibly coupled lossless transmission lines leads to a system of neutral functional differential equations with discrete diffusive coupling which exhibits various types of discrete waves. For more results on partial neutral functional differential equations and related issues we refer to Adimy and Ezzinbi [2], Hale [20], Wu and Xia [38, 39] for finite delay equations, and Hern'andez and Henriquez [24, 25] for unbounded delays. Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received a significant amount of attention in the last years, see for instance [1 – 3, 6, 10, 15, 35] and the references therein. We also cite [4, 5, 16, 19, 23, 30, 31, 40] for the case neutral differential equations with state-dependent delay. In [12, 13] Benchohra et al. considered the global existence of mild solutions for some classes of functional evolutions equations on unbounded intervals.

In this work we prove the existence of solutions of a neutral functional differential inclusion. Our investigations will be situated in the Banach space of real functions which are defined, continuous and bounded on the real axis $\mathbb{R}$. We will use Bohnenblust – Karlin fixed point theorem, combined with the Corduneanu's compactness criteria. More precisely we will consider the following problem:

$$\frac{d}{dt}\big[y(t) - g(t, y_{\rho(t,y_t)})\big] - A\big[y(t) - g(t, y_{\rho(t,y_t)})\big] \in F(t, y_{\rho(t,y_t)}) \quad \text{a.e.} \quad t \in J := [0, +\infty), \tag{1}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{2}$$

where $F\colon J\times\mathcal{B} \to \mathcal{P}(E)$ is a multivalued map with nonempty compact values, $\mathcal{P}(E)$ is the family of all nonempty subsets of E, $g\colon J\times\mathcal{B} \to E$ is given function, $A\colon D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, $\mathcal{B}$ is the phase space to be specified later, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$ and $(E, |\cdot|)$ is a real separable Banach space. For any function y defined on $(-\infty, +\infty)$ and any $t \in J$ we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0]$. Here $y_t(\cdot)$ represents the history of the state from time $-\infty$, up to the present time t. We assume that the histories y_t belongs to some abstract phases $\mathcal{B}$, to be specified later.

To our knowledge the literature on the global existence of neutral evolution inclusions is very limited. Some of the exiting ones are obtained in the Fréchet space setting. The present results are given in the Banach space setting, and hence are considered as a contribution of this class of problems.

2. Preliminaries. In this section we present briefly some notations and definition, and theorem which are used throughout this work.

In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [21] and follow the terminology used in [27]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into E, and satisfying the following axioms:

(A_1) If $y\colon (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

(i) $y_t \in \mathcal{B}$;

(ii) there exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) there exist two functions $L(\cdot), M(\cdot)\colon \mathbb{R}_+ \to \mathbb{R}_+$ independent of y with L continuous and bounded, and M locally bounded such that

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\big\{|y(s)| : 0 \leq s \leq t\big\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) For the function y in (A_1), y_t is a $\mathcal{B}$-valued continuous function on J.

(A_3) The space $\mathcal{B}$ is complete.

Assume that

$$l = \sup\big\{L(t) : t \in J\big\}, \quad m = \sup\big\{M(t) : t \in J\big\}.$$

***Remark* 2.1.** 1. Condition (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.

3. From the equivalence of in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$: we necessarily have that $\phi(0) = \psi(0)$.

By BUC we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to E.

Let $BC := BC([0, +\infty))$ be the Banach space of all bounded and continuous functions from $[0, +\infty)$ into E equipped with the standard norm

$$\|y\|_{BC} = \sup_{t \in [0, +\infty)} |y(t)|.$$

Let (E, d) be a metric space. We use the following notations:

$$\mathcal{P}_{cl}(E) = \big\{Y \in \mathcal{P}(E) \colon Y \text{ closed}\big\}, \qquad \mathcal{P}_{cv}(E) = \big\{Y \in \mathcal{P}(E) \colon Y \text{ convex}\big\},$$
$$\mathcal{P}_b(E) = \big\{Y \in \mathcal{P}(E) \colon Y \text{ bounded}\big\}.$$

Consider $H_d \colon \mathcal{P}(E) \times \mathcal{P}(E) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$, given by

$$H_d(\mathcal{A},\mathcal{B}) = \max\left\{\sup_{a\in\mathcal{A}} d(a,\mathcal{B}),\ \sup_{b\in\mathcal{B}} d(\mathcal{A},b)\right\},$$

where $d(\mathcal{A},b) = \inf_{a\in\mathcal{A}} d(a,b),\ d(a,\mathcal{B}) = \inf_{b\in\mathcal{B}} d(a,b)$.

Definition 2.1. *Let X, Y be Hausdorff topological spaces and $F\colon X \to \mathcal{P}(Y)$ is called upper semicontinuous* (*u.s.c.*) *on X if for each $x_0 \in X$, the set $F(x_0)$ is a nonempty closed subset of Y and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood N_0 of x_0 such that $F(N_0) \subseteq N$.*

Let $(E, \|\cdot\|)$ be a Banach space. A multivalued map $A\colon E \to \mathcal{P}(E)$ has *convex (closed) values* if $A(x)$ is convex (closed) for all $x \in E$. We say that A is *bounded* on bounded sets if $A(B)$ is bounded in E for each bounded set B of E, i.e.,

$$\sup_{x\in B}\big\{\sup\{\|y\| : y \in A(x)\}\big\} < \infty.$$

F is said to be completely continuous if $F(B)$ is relatively compact for every $B \in \mathcal{P}_b(E)$. If the multivalued map F is completely continuous with non empty values, then F is u.s.c. if an only if F has a closed graph (i.e., $x_n \to x_*,\ y_n \to y_*,\ y_n \in F(x_n)$ implies $y_* \in F(x_*)$).

Definition 2.2. *A function $F\colon J \times \mathcal{B} \longrightarrow \mathcal{P}(E)$ is said to be an L^1-Carathéodory multivalued map if it satisfies*:

(i) *$y \mapsto F(t,y)$ is upper semicontinuous for almost all $t \in J$*;

(ii) *$t \mapsto F(t,y)$ is measurable for each $y \in \mathcal{B}$*;

(iii) *for every positive constant l there exists $h_l \in L^1(J,\mathbb{R}^+)$*

$$\big\|F(t,y)\big\| = \sup\big\{|v| : v \in F(t,y)\big\} \le h_l$$

for all $|y| \le l$ for almost all $t \in J$.

Definition 2.3. *A function $F\colon J \times \mathcal{B} \longrightarrow \mathcal{P}(E)$ is said to be an Carathéodory multivalued map if it satisfies* (i) *and* (ii).

The following two results are easily deduced from the limit properties.

Lemma 2.1 (see, e.g., [7], Theorem 1.4.13)**.** *If $G\colon X \to \mathcal{P}(X)$ is u.s.c., then, for any $x_0 \in X$,*

$$\lim_{x\to x_0}\sup G(x) = G(x_0).$$

Lemma 2.2 (see, e.g., [7], Lemma 1.1.9)**.** *Let $(K_n)_{n\in\mathbb{N}} \subset K \subset X$ be a sequence of subsets where K is compact in the separable Banach space X. Then*

$$\overline{\mathrm{co}}\left(\lim_{n\to\infty}\sup K_n\right) = \bigcap_{N>0}\overline{\mathrm{co}}\left(\bigcup_{n\ge N} K_n\right),$$

where $\overline{\mathrm{co}}\,A$ refers to the closure of the convex hull of A.

The second one is due to Mazur (1933).

Lemma 2.3 (Mazur's lemma [41])**.** *Let E be a normed space and $\{x_k\}_{k\in\mathbb{N}} \subset E$ be a sequence weakly converging to a limit $x \in E$. Then there exists a sequence of convex combinations $y_m = \sum_{k=1}^{m}\alpha_{mk}x_k$ with $\alpha_{mk} > 0$ for $k = 1,2,\ldots,m$ and $\sum_{k=1}^{m}\alpha_{mk} = 1$, which converges strongly to x.*

Lemma 2.4 [29]. *Let E be a Banach space. Let $F\colon J\times E\to\mathcal{P}_{cl,cv}(E)$ be a L^1-Carathéodory multivalued map, and let Γ be a linear continuous from $L^1(J;E)$ into $C(J;E)$, then the operator*

$$\Gamma\circ S_F\colon C(J,E)\longrightarrow\mathcal{P}_{cp,cv}(C(J,X)),\qquad y\longmapsto(\Gamma\circ S_F)(y):=\Gamma(S_{F,y})$$

is a closed graph operator in $C(J;X)\times C(J;X)$.

Finally, we say that A has a *fixed point* if there exists $x\in E$ such that $x\in A(x)$.

For each $y\colon(-\infty,+\infty)\to E$ let the set $S_{F,y}$ known as the set of selectors from F defined by

$$S_{F,y}=\big\{v\in L^1(J;E)\colon v(t)\in F(t,y_{\rho(t,y_t)})\ \text{ a.e. } t\in J\big\}.$$

For more details on multivalued maps we refer to the books of Deimling [18], Hu and Papageorgiou [28], Górniewicz [34], and Perestyuk et al. [37].

Theorem 2.1 (Bohnenblust–Karlin fixed point [14]). *Let $B\in\mathcal{P}_{cl,cv}(E)$, $N\colon B\to\mathcal{P}_{cl,cv}(B)$ be a upper semicontinuous operator and $N(B)$ is a relatively compact subset of E. Then N has at least one fixed point in B.*

Lemma 2.5 (Corduneanu [17]). *Let $D\subset BC([0,+\infty),E)$. Then D is relatively compact if the following conditions hold*:

(a) *D is bounded in BC.*

(b) *The function belonging to D is almost equicontinuous on $[0,+\infty)$, i.e., equicontinuous on every compact of $[0,+\infty)$.*

(c) *The set $D(t):=\{y(t)\colon y\in D\}$ is relatively compact on every compact of $[0,+\infty)$.*

(d) *The function from D is equiconvergent, that is, given $\epsilon>0$, responds $T(\epsilon)>0$ such that $\big|u(t)-\lim_{t\to+\infty}u(t)\big|<\epsilon$, for any $t\geq T(\epsilon)$ and $u\in D$.*

3. Existence of mild solutions. Now we give our main existence result for problem (1), (2). Before starting and proving this result, we give the definition of the mild solution.

Definition 3.1. *We say that a continuous function $y\colon(-\infty,+\infty)\to E$ is a mild solution of problem* (1), (2) *if $y(t)=\phi(t)$ for all $t\in(-\infty,0]$, and the restriction of $y(\cdot)$ to the interval J is continuous and there exists $f(\cdot)\in L^1(J;E)\colon f(t)\in F(t,y_{\rho(t,y_t)})$ a.e. in J such that y satisfies the integral equation*

$$y(t)=T(t)\big[\phi(0)-g(0,\phi(0))\big]+g(t,y_{\rho(t,y_t)})+\int\limits_0^t T(t-s)f(s)\,ds,\quad t\in J. \tag{3}$$

Set

$$\mathcal{R}(\rho^-)=\big\{\rho(s,\phi)\colon(s,\phi)\in J\times\mathcal{B},\ \rho(s,\phi)\leq 0\big\}.$$

We always assume that $\rho:J\times\mathcal{B}\to\mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

(H_ϕ) The function $t\to\phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi\colon\mathcal{R}(\rho^-)\to(0,\infty)$ such that

$$\|\phi_t\|\leq\mathcal{L}^\phi(t)\|\phi\|\quad\text{for every}\quad t\in\mathcal{R}(\rho^-).$$

Remark **3.1.** The condition (H_ϕ), is frequently verified by functions continuous and bounded. For more details, see, for instance, [27].

Lemma 3.1 [26]. *If* $y\colon (-\infty, +\infty) \to E$ *is a function such that* $y_0 = \phi$, *then*

$$\|y_s\|_{\mathcal{B}} \leq (M + \mathcal{L}^{\phi})\|\phi\|_{\mathcal{B}} + l \sup\big\{|y(\theta)|; \theta \in [0, \max\{0, s\}]\big\}, \quad s \in \mathcal{R}(\rho^-) \cup J,$$

where $\mathcal{L}^{\phi} = \sup_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^{\phi}(t)$.

Let us introduce the following hypotheses:

(H_1) The semigroup $T(t)$ is compact for $t > 0$, and there is a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

(H_2) The multifunction $F\colon J \times \mathcal{B} \longrightarrow \mathcal{P}(E)$ is Carathéodory with compact, closed and convex values.

(H_3) There exists a continuous function $k\colon J \to [0, +\infty)$ such that

$$H_d(F(t, u), F(t, v)) \leq k(t)\|u - v\|_{\mathcal{B}}$$

for each $t \in J$ and for all $u, v \in \mathcal{B}$ and

$$d(0, F(t, 0)) \leq k(t)$$

with

$$k^* := \sup_{t \in J} \int\limits_0^t k(s)\, ds < \infty. \tag{4}$$

(H_4) The function $g(t, \cdot)$ is continuous on J and there exists a constant $k_g > 0$ such that

$$\big|g(t, u) - g(t, v)\big| \leq k_g\|u - v\|_{\mathcal{B}} \quad \text{for each} \quad u, v \in \mathcal{B}$$

and

$$g^* := \sup_{t \in J} |g(t, 0)| < \infty.$$

(H_5) For each $t \in J$ and any bounded set $B \subset \mathcal{B}$, the set $\{g(t, u)\colon u \in B\}$ is relatively compact in E.

(H_6) For any bounded set $B \subset \mathcal{B}$, the function $\{t \to g(t, u)\colon u \in B\}$ is equicontinuous on each compact interval of $[0, +\infty)$.

Set

$$\Omega = \big\{y\colon (-\infty, +\infty) \to E : y|_{(-\infty, 0]} \in \mathcal{B} \ \text{ and } \ y|_{[0, +\infty)} \in BC\big\}.$$

Remark **3.2.** By the condition (H_4) we deduce that

$$|g(t, u)| \leq k_g\|u\|_{\mathcal{B}} + g^*, \qquad t \in J, \quad u \in \mathcal{B}.$$

Theorem 3.1. *Assume that* (H_1)–(H_6) *and* (H_ϕ) *hold. If* $l(Mk^* + k_g) < 1$, *then the problem* (1), (2) *has at least one mild solution.*

Proof. Transform the problem (1), (2) into a fixed point problem. Consider the operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(y) := \left\{ h \in \Omega : h(t) = \begin{cases} \phi(t), & \text{if} \quad t \in (-\infty, 0], \\ T(t)\big[\phi(0) - g(0, \phi(0))\big] + \\ \qquad + g(t, y_{\rho(t,y_t)}) + \displaystyle\int_0^t T(t-s) f(s)\, ds, & \text{if} \quad t \in J \end{cases} \right\}$$

where $f \in S_{F, y_{\rho(t,y_t)}}$.

Let $x(\cdot) : (-\infty, +\infty) \to E$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if} \quad t \in (-\infty, 0], \\ T(t)\phi(0), & \text{if} \quad t \in J. \end{cases}$$

Then $x_0 = \phi$. For each $z \in \Omega$ with $z(0) = 0$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0, & \text{if} \quad t \in \quad (-\infty, 0], \\ z(t), & \text{if} \quad t \in J, \end{cases}$$

if $y(\cdot)$ satisfies (3) we can decompose it as $y(t) = z(t) + x(t),\ t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0, \phi(0)) + \int\limits_0^t T(t-s) f(s)\, ds, \quad t \in J,$$

where $f \in S_{F, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}}$. Set

$$\Omega_0 = \{ z \in \Omega : z(0) = 0 \}$$

and let

$$\|z\|_{\Omega_0} = \sup \big\{ |z(t)| : t \in J \big\} \quad z \in \Omega_0.$$

Ω_0 is a Banach space with the norm $\| \cdot \|_{\Omega_0}$.

We define the operator $\mathcal{A} : \Omega_0 \to \mathcal{P}(\Omega_0)$ by

$$\mathcal{A}(z) := \left\{ h \in \Omega_0 : h(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - \\ \qquad - T(t)g(0, \phi(0)) + \displaystyle\int_0^t T(t-s) f(s)\, ds, & \text{if} \quad t \in J \end{cases} \right\}$$

where $f \in S_{F, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}}$.

The operator A maps Ω_0 into Ω_0, indeed the map $\mathcal{A}(z)$ is continuous on $[0, +\infty)$ for any $z \in \Omega_0$, $h \in \mathcal{A}(z)$ and for each $t \in J$ we have

$$|h(t)| \leq \left|g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})\right| + M\left|g(0,\phi(0))\right| + M\int\limits_0^t |f(s)|\,ds \leq$$

$$\leq M(k_g\|\phi\|_{\mathcal{B}} + g^*) + k_g\|z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\|_{\mathcal{B}} + g^* +$$

$$+M\int\limits_0^t |F(s,0)|ds + M\int\limits_0^t k(s)\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\,ds \leq$$

$$\leq M\big(k_g\|\phi\|_{\mathcal{B}} + g^*\big) + k_g\big(l|z(t)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}}\big) + g^* +$$

$$+Mk^* + M\int\limits_0^t k(s)\big(l|z(s)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}}\big)\,ds.$$

Set

$$C_1 := \big(m + \mathcal{L}^\phi + lMH\big)\|\phi\|_{\mathcal{B}},$$
$$C_2 := M\big(k_g\|\phi\|_{\mathcal{B}} + g^*\big) + k_gC_1 + g^* + Mk^*.$$

Then we have

$$|h(t)| \leq C_2 + k_gl|z(t)| + MC_1\int\limits_0^t k(s)\,ds + M\int\limits_0^t l|z(s)|k(s)\,ds \leq$$

$$\leq C_2 + k_gl\|z\|_{\Omega_0} + MC_1k^* + Ml\|z\|_{\Omega_0}k^*.$$

Hence, $\mathcal{A}(z) \in \Omega_0$.

Moreover, let $r > 0$ be such that

$$r \geq \frac{C_2 + MC_1k^*}{1 - l(Mk^* + k_g)},$$

and B_r be the closed ball in Ω_0 centered at the origin and of radius r. Let $z \in B_r$ and $t \in [0, +\infty)$. Then

$$|h(t)| \leq C_2 + k_glr + MC_1k^* + Mk^*lr.$$

Thus

$$\|h\|_{\Omega_0} \leq r,$$

which means that the operator $\mathcal{A}$ transforms the ball B_r into itself.

Now we prove that $\mathcal{A}\colon B_r \to \mathcal{P}(B_r)$ satisfies the assumptions of Bohnenblust – Karlin fixed point theorem. The proof will be given in several steps.

***Step* 1.** We shall show that the operator $\mathcal{A}$ is closed and convex valued. This will be given in several claims.

***Claim* 1.** $\mathcal{A}(z)$ is closed for each $z \in B_r$.

Let $(h_n)_{n\geq 0} \in \mathcal{A}(z)$ such that $h_n \to \tilde{h}$ in B_r. Then for $h_n \in B_r$ there exists $f_n \in$ $\in S_{F, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}}$ such that for each $t \in J$,

$$h_n(t) = g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t)g(0,\phi(0)) + \int\limits_0^t T(t-s)f_n(s)\,ds.$$

We shall use the fact that F has compact values and from hypotheses (H_2), (H_3), and Mazur's lemma we may pass a subsequence if necessary to get that f_n converges to $f \in L^1(J,E)$ and hence $f \in S_{F,y}$. Indeed, Lemma 2.3 yields the existence of $\alpha_i^n \geq 0$, $i = n, \ldots, k-n)$, such that $\sum_{i=1}^{k(n)} \alpha_i^n = 1$ and the sequence of convex combinations $g_n(\cdot) = \sum_{i=1}^{k(n)} \alpha_i^n f_i(\cdot)$ converges strongly to $f \in L^1$. Since F takes convex values, using Lemma 2.2, we obtain that for a.e. $t \in J$

$$f(t) \in \bigcap_{n\geq 1} \overline{\{g_n(t)\}} \subset$$

$$\subset \bigcap_{n\geq 1} \overline{\mathrm{co}}\,\{f_k(t), k \geq n\} \subset \bigcap_{n\geq 1} \overline{\mathrm{co}} \left\{ \bigcup_{k\geq n} F(t, z^k_{\rho(z^k_t+x_t)} + x_{\rho(t,z^k_t+x_t)}) \right\} =$$

$$= \overline{\mathrm{co}}\Big(\lim_{k\to\infty} \sup F\big(t, z^k_{\rho(z^k_t+x_t)} + x_{\rho(t,z^k_t+x_t)}\big)\Big).$$

Since F is u.s.c. with compact values, then by Lemma 2.1, we have

$$\lim_{n\to\infty} \sup F\big(t, z^n_{\rho(z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}\big) = F\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) \quad \text{for a.e.} \quad t \in J.$$

This implies that $f(t) \in \overline{\mathrm{co}}\, F(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})$. Since $F(\cdot,\cdot)$ has closed, convex values, we deduce that $f(t) \in F(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})$ for a.e. $t \in J$.

Let $f \in S_{F, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}}$. Then, for each $t \in J$,

$$h_n(t) \to \tilde{h}(t) = g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) - T(t)g(0,\phi(0)) + \int\limits_0^t T(t-s)f(s)\,ds.$$

So, $\tilde{h} \in \mathcal{A}(z)$.

***Claim* 2.** $\mathcal{A}(z)$ is convex for each $z \in B_r$.

Let $h_1, h_2 \in \mathcal{A}(z)$, the there exists $f_1, f_2 \in S_{F, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}}$ such that, for each $t \in J$ we have

$$h_i(t) = g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t)g\big(0,\phi(0)\big) + \int\limits_0^t T(t-s)f_i(s)\,ds, \quad i = 1, 2.$$

Let $0 \leq \delta \leq 1$. Then we have, for each $t \in J$,

$$\big(\delta h_1 + (1-\delta)h_2\big)(t) = g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t)g\big(0,\phi(0)\big) +$$

$$+ \int\limits_0^t T(t-s)\big[\delta f_1(s) + (1-\delta)f_2(s)\big]\,ds.$$

Since F has convex values, one has

$$\delta h_1 + (1-\delta)h_2 \in \mathcal{A}(z).$$

***Step* 2.** $\mathcal{A}(B_r) \subset B_r$ this is clear.

***Step* 3.** $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0,b]$ of $[0,+\infty)$ for $b>0$. Let $\tau_1, \tau_2 \in [0,b],\ h \in \mathcal{A}(z)$ with $\tau_2 > \tau_1$, we have

$$\big|h(\tau_2) - h(\tau_1)\big| \le$$

$$\le \left|g\big(\tau_2, z_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})}\big) - g\big(\tau_1, z_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})}\big)\right| +$$

$$+\big\|T(\tau_2) - T(\tau_1)\big\|_{B(E)} |g(0,\phi(0))| +$$

$$+\int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} |f(s)|\, ds +$$

$$+\int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} |f(s)|\, ds \le$$

$$\le \left|g\big(\tau_2, z_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})}\big) - g\big(\tau_1, z_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})}\big)\right| +$$

$$+\big\|T(\tau_2) - T(\tau_1)\big\|_{B(E)} (k_g \|\phi\|_{\mathcal{B}} + g^*) +$$

$$+\int_0^{\tau_1} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(E)} (k(s) \|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\|_{\mathcal{B}} + |F(s,0)|)\, ds +$$

$$+\int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} (k(s) \|z_{\rho(s, z_s + x_s)} + x_{\rho(s, z_s + x_s)}\|_{\mathcal{B}} + |F(s,0)|)\, ds \le$$

$$\le \left|g\big(\tau_2, z_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})} + x_{\rho(\tau_2, z_{\tau_2}+x_{\tau_2})}\big) - g\big(\tau_1, z_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})} + x_{\rho(\tau_1, z_{\tau_1}+x_{\tau_1})}\big)\right| +$$

$$+C_1 \int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} k(s)\, ds +$$

$$+rl \int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} k(s)\, ds +$$

$$+\int_0^{\tau_1} \big\|T(\tau_2 - s) - T(\tau_1 - s)\big\|_{B(E)} k(s)\, ds + C_1 \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s)\, ds +$$

$$+\, rl \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s)\, ds + \int_{\tau_1}^{\tau_2} \|T(\tau_2 - s)\|_{B(E)} k(s)\, ds.$$

When $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero, since (H_6) and $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$, implies the continuity in the uniform operator topology (see [36]), this proves the equicontinuity.

***Step* 4.** $\mathcal{A}(B_r)$ is relatively compact on every compact interval of $[0,\infty)$.

Let $t \in [0,b]$ for $b>0$ and let ε be a real number satisfying $0<\varepsilon<t$. For $z \in B_r$ we define

$$h_\varepsilon(t) = g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(\varepsilon)\big(T(t-\varepsilon)g(0,\phi(0))\big) + \\ + T(\varepsilon)\int\limits_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)\,ds.$$

Note that the set

$$\Bigg\{ g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t-\varepsilon)g(0,\phi(0)) + \\ + \int\limits_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)\,ds : z \in B_r \Bigg\}$$

is bounded,

$$\Bigg| g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t-\varepsilon)g(0,\phi(0)) + \\ + \int\limits_0^{t-\varepsilon} T(t-s-\varepsilon)f(s)\,ds \Bigg| \le r.$$

Since $T(t)$ is a compact operator for $t > 0$, and (H_5) we have that the set

$$\big\{h_\varepsilon(t) : z \in B_r\big\}$$

is precompact in E for every ε, $0 < \varepsilon < t$. Moreover, for every $z \in B_r$ we have

$$\big|h(t) - h_\varepsilon(t)\big| \le M \int\limits_{t-\varepsilon}^{t} |f(s)|\,ds \le$$

$$\le M \int\limits_{t-\varepsilon}^{t} k(s)\,ds + MC_1 \int\limits_{t-\varepsilon}^{t} k(s)\,ds + rM \int\limits_{t-\varepsilon}^{t} lk(s)\,ds \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

Therefore, the set $\{h(t) : z \in B_r\}$ is precompact, i.e., relatively compact.

***Step* 5.** $\mathcal{A}$ has closed graph.

Let $\{z_n\}$ be a sequence such that $z_n \to z_*$, $h_n \in \mathcal{A}(z_n)$ and $h_n \to h_*$. We shall show that $h_* \in \mathcal{A}(z_*)$.

$h_n \in \mathcal{A}(z_n)$ means that there exists $f_n \in S_{F, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}}$ such that

$$h_n(t) = g\big(t, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}\big) - T(t)g(0,\phi(0)) + \int\limits_0^t T(t-s)f_n(s)\,ds,$$

we must prove that there exists f_*

$$h_*(t) = g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t)g(0,\phi(0)) + \int\limits_0^t T(t-s)f_*(s)\,ds.$$

Consider the linear and continuous operator $K: L^1(J,E) \to B_r$ defined by

$$K(v)(t) = g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t)g(0,\phi(0)) + \int\limits_0^t T(t-s)v(s)\,ds.$$

We have

$$\big|K(f_n)(t) - K(f_*)(t)\big| = \big|\big(h_n(t) - g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) + T(t)g(0,\phi(0))\big) -$$
$$-\big(h_*(t) - g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}) + T(t)g(0,\phi(0))\big)\big| \le$$
$$\le \|h_n - h_*\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

From Lemma 2.2 it follows that $K \circ S_F$ is a closed graph operator and from the definition of K has

$$h_n(t) \in K \circ S_{F, z^n_{\rho(t,z^n_t+x_t)} + x_{\rho(t,z^n_t+x_t)}}.$$

As $z_n \to z_*$ and $h_n \to h_*$, there exist $f_* \in S_{F, z^*_{\rho(t,z^*_t+x_t)} + x_{\rho(t,z^*+x_t)}}$ such that

$$h_*(t) = g\big(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)}\big) - T(t)g(0,\phi(0)) + \int\limits_0^t T(t-s)f_*(s)\,ds.$$

Hence the mutivalued operator $\mathcal{A}$ is upper semicontinuous.

***Step* 6.** $\mathcal{A}(B_r)$ is equiconvergent.

Let $z \in B_r$, we have, for $h \in \mathcal{A}(z)$,

$$|h(t)| \le \big|g(t, z_{\rho(t,z_t+x_t)} + x_{\rho(t,z_t+x_t)})\big| + M|g(0,\phi(0))| + M\int\limits_0^t |f(s)|\,ds \le$$
$$\le M\big(k_g\|\phi\|_{\mathcal{B}} + g^*\big) + k_g\big(l|z(t)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}}\big) + g^* +$$
$$+ Mk^* + M\int\limits_0^t k(s)\big(l|z(s)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}}\big)\,ds \le$$
$$\le C_2 + k_g l\|z\|_{BC_0'} + MC_1k^* + Ml\|z\|_{BC_0'}k^*.$$

Then we obtain

$$|h(t)| \to l \le C_2 + k_g lr + MC_1k^* + Mlrk^* \quad \text{as} \quad t \to +\infty.$$

Hence,

$$\big|h(t) - h(+\infty)\big| \to 0 \quad \text{as} \quad t \to +\infty.$$

As a consequence of Steps 1 – 4, with Lemma 2.5, we can conclude that $\mathcal{A}\colon B_r \to \mathcal{P}(B_r)$ is continuous and compact. From Schauder's theorem, we deduce that $\mathcal{A}$ has a fixed point z^*. Then $y^* = z^* + x$ is a fixed point of the operators N, which is a mild solution of the problem (1), (2).

4. An example. Consider the following neutral functional partial differential inclusion

$$\frac{\partial}{\partial t}\big[z(t,x) - g(t, z(t-\sigma(t,z(t,0)),x))\big] - \frac{\partial^2}{\partial x^2}\big[z(t,x) - g(t, z(t-\sigma(t,z(t,0)),x))\big] \in$$

$$\in \int\limits_{-\infty}^{t} f(s, z(t-\sigma(s,z(s,0)),x))\,ds, \quad x\in[0,\pi], \quad t\in J := [0,+\infty), \tag{5}$$

$$z(t,0) = z(t,\pi) = 0, \quad t\in J, \tag{6}$$

$$z(\theta,x) = \tilde{z}(\theta,x), \qquad t\in(-\infty,0] \quad x\in[0,\pi], \tag{7}$$

where f is a given multivalued map, g a given function, and $\sigma\colon \mathbb{R}\to\mathbb{R}^+$. Take $E = L^2[0,\pi]$ and define $A\colon E\to E$ by $A\omega = \omega''$ with domain

$$D(A) = \big\{\omega\in E,\ \omega,\ \omega' \text{ are absolutely continuous } \omega''\in E,\ \omega(0)=\omega(\pi)=0\big\}.$$

Then

$$A\omega = \sum_{n=1}^{\infty} n^2(\omega,\omega_n)\omega_n, \quad \omega\in D(A),$$

where $\omega_n(s) = \sqrt{\dfrac{2}{\pi}}\sin ns,\ n=1,2,\ldots,$ is the orthogonal set of eigenvectors in A. It is well know (see [36]) that A is the infinitesimal generator of an analytic semigroup $T(t),\ t\geq 0,$ in E and is given by

$$T(t)\omega = \sum_{n=1}^{\infty}\exp(-n^2t)(\omega,\omega_n)\omega_n, \quad \omega\in E.$$

Since the analytic semigroup $T(t)$ is compact for $t>0$, there exists a positive constant M such that

$$\|T(t)\|_{B(E)} \leq M.$$

Let $\mathcal{B} = BCU(\mathbb{R}^-;E)$ and $\phi\in\mathcal{B}$, then (H_ϕ) is satisfied with

$$\rho(t,\varphi) = t - \sigma(\varphi), \quad t\in J.$$

Set

$$y(t)(x) = z(t,x), \quad (t,x)\in J\times[0,\pi],$$

$$F(t,\varphi)(x) = \int\limits_{-\infty}^{t} f(s,\varphi)\,ds, \quad (t,x)\in J\times[0,\pi],$$

$$\phi(t)(x) = \tilde{z}(t,x), \quad (t,x)\in(-\infty,0]\times[0,\pi].$$

Hence, the problem (1), (2) in an abstract formulation of the problem (5) – (7), and if the conditions (H_1) – (H_6) are satisfied, Theorem 3.1 implies that the problem (5) – (7) has at least one mild solutions.

References

1. *Abada N., Agarwal R. P., Benchohra M., Hammouche H.* Existence results for nondensely defined impulsive semilinear functional differential equations with state-dependent delay // Asian-Eur. Math J. – 2008. – **1**. – P. 449 – 468.
2. *Adimy M., Ezzinbi K.* A class of linear partial neutral functional-differential equations with nondense domain // J. Different. Equat. – 1998. – **147**. – P. 285 – 332.
3. *Aiello W. G., Freedman H. I., Wu J.* Analysis of a model representing stagestructured population growth with state-dependent time delay // SIAM J. Appl. Math. – 1992. – **52**. – P. 855 – 869.
4. *Ait Dads E., Ezzinbi K.* Boundedness and almost periodicity for some state-dependent delay differential equations // Electron. J. Different. Equat. – 2002. – **2002**, № 67. – P. 1 – 13.
5. *Anguraj A., Arjunan M. M., Hernàndez E. M.* Existence results for an impulsive neutral functional differential equation with state-dependent delay // Appl. Anal. – 2007. – **86**. – P. 861 – 872.
6. *Arino O., Boushaba K., Boussouar A.* A mathematical model of the dynamics of the phytoplankton-nutrient system: spatial heterogeneity in ecological models (Alcala de Henares, 1998) // Nonlinear Anal. Real World Appl. – 2000. – **1**. – P. 69 – 87.
7. *Aubin J. P., Frankowska H.* Set-valued analysis. – Boston: Birkhäuser, 1990.
8. *Baghli S., Benchohra M.* Existence results for semilinear neutral functional differential equations involving evolution operators in Fréchet spaces // Georg. Math. J. – 2010. – **17**. – P. 423 – 436.
9. *Baghli S., Benchohra M.* Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay // Different. Integral Equat. – 2010. – **23**. – P. 31–50.
10. *Belair J.* Population models with state-dependent delays // Lect. Notes Pure and Appl. Math. – 1990. – **131**. – P. 156 – 176.
11. *Benchohra M., Gatsori E., Ntouyas S. K.* Existence results for functional and neutral for functional integrodifferential inclusions with lower semicontinous right-hand side // J. Math Anal. and Appl. – 2003. – **281**. – P. 525 – 538.
12. *Benchohra M., Medjadj I.* Global existence results for functional differential equations with delay // Commun. Appl. Anal. – 2013. – **17**. – P. 213 – 220.
13. *Benchohra M., Medjadj I., Nieto J. J., Prakash P.* Global existence for functional differential equations with state-dependent delay // J. Funct. Spaces Appl. – 2013. – **2013**. – Article ID 863561. – 7 p.
14. *Bohnenblust H. F., Karlin S.* On a theorem of Ville. Contribution to the theory of games // Ann. Math. Stud. – 1950. – № 24. – P. 155 – 160.
15. *Cao Y., Fan J., Card T. C.* The effects of state-dependent time delay on a stage-structured population growth model // Nonlinear Anal. TMA. – 1992. – **19**. – P. 95 – 105.
16. *Chen F., Sun D., Shi J.* Periodicity in a food-limited population model with toxicants and state-dependent delays // J. Math Anal. and Appl. – 2003. – **288**. – P. 136 – 146.
17. *Corduneanu C.* Integral equations and stability of feedback systems. – New York: Acad. Press, 1973.
18. *Deimling K.* Multivalued differential equations. – Berlin; New York: Walter de Gruyter, 1992.
19. *Driver R. D.* A neutral system with state-dependent delays // J. Different. Equat. – 1984. – **54**, № 1. – P. 73 – 86.
20. *Hale J. K.* Partial neutral functional-differential equations // Rev. Roum. Math. Pures et Appl. – 1994. – **39**, № 4. – P. 339 – 344.
21. *Hale J., Kato J.* Phase space for retarded equations with infinite delay // Funkc. Ekvacioj. – 1978. – **21**. – P. 11 – 41.
22. *Hale J. K., Verduyn Lunel S. M.* Introduction to functional differential equations. – New York: Springer-Verlag, 1993.
23. *Hartung F., Krisztin T., Walther H. O., Wu J.* Functional differential equations with state-dependent delays: theory and applications // Handbook Different. Equat.: Ordinary Different. Equat. / Eds A. Canada, P. Drabek, A. Fonda. – 2006. – Vol. 3.
24. *Hernandez E., Henriquez H.* Existence results for partial neutral functional differential equations with unbounded delay // J. Math. Anal. and Appl. – 1998. – **221**. – P. 452 – 475.
25. *Hernandez E., Henriquez H.* Existence of periodic solutions of partial neutral functional differential equations with unbounded delay // J. Math. and Anal. Appl. – 1998. – **221**. – P. 499 – 522.
26. *Hernandez E., Prokopezyk A., Ladeira L.* A note on partial functional differential equation with state-dependent delay // Nonlinear Anal. Real World Appl. – 2006. – **7**. – P. 511 – 519.
27. *Hino Y., Murakami S., Naito T.* Functional differential equations with unbounded delay. – Berlin: Springer-Verlag, 1991.
28. *Hu Sh., Papageorgiou N.* Handbook of multivalued analysis. Vol. I: Theory. – Dordrecht: Kluwer Acad. Publ., 1997.
29. *Lasota A., Opial Z.* An application of the Kakutani – Ky Fan theorem in the theory of ordinary differential equations // Bull. Acad. Pol. Sci. Sér. Sci. Math., Astron. et Phys. – 1965. – **13**. – P. 781 – 786.
30. *Li W. S., Chang Y. K., Nieto J. J.* Solvability of impulsive neutral evolution differential inclusions with state-dependent delay // Math. Comput. Model. – 2009. – **49**. – P. 1920 – 1927.

31. *Nandakumar K., Wiercigroch M.* Galerkin projections for state-dependent delay differential equations with applications to drilling // Appl. Math. Model. – 2013. – **37**. – P. 1705 – 1722.
32. *Ntouyas S. K.* Global existence for neutral functional integrodifferential equations // Proc. Sond World Congr. Nonlinear Anal. Pt 4 (Athens, 1996). Nonlinear Anal. – 1997. – **30**. – P. 2133 – 2142.
33. *Ntouyas S. K., Sficas Y. G., Tsamatos C. P.* Existence results for initial value problems for neutral functional-differential equations // J. Different. Equat. – 1994. – **114**. – P. 527 – 537.
34. *Górniewicz L.* Topological fixed point theory of multivalued mappings // Math. and Appl. – 1999. – **495**.
35. *Kuang Y., Smith H. L.* Slowly oscillating periodic solutions of autonomous state-dependent delay equations // Nonlinear Anal. TMA. – 1992. – **19**. – P. 855 – 1318.
36. *Pazy A.* Semigroups of linear operators and applications to partial differential equations. – New York: Springer-Verlag, 1983.
37. *Perestyuk N. A., Plotnikov V. A., Samoilenko A. M., Skripnik N. V.* Differential equation with impulse effects // Multivalued Right-Hand Sides with Discontinuities. – Berlin; Boston: Walter de Gruyter, 2011.
38. *Wu J., Xia H.* Self-sustained oscillations in a ring array of coupled losseless transmission lines // J. Different. Equat. – 1996. – **124**. – P. 247 – 278.
39. *Wu J.* Theory and applications of partial functional-differential equations. – New York: Springer-Verlag, 1996.
40. *Yang Z., Cao J.* Existence of periodic solutions in neutral state-dependent delays equations and models // J. Comput. and Appl. Math. – 2005. – **174**. – P. 179 – 199.
41. *Yosida K.* Functional analysis. – 6th ed. – Berlin: Springer-Verlag, 1980.

Received 20.08.14,
after revision — 11.07.18