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ON THE GENERALIZATION OF SOME HERMITE – HADAMARD INEQUALITIES FOR FUNCTIONS WITH CONVEX ABSOLUTE VALUES OF THE SECOND DERIVATIVES VIA FRACTIONAL INTEGRALS*

ПРО УЗАГАЛЬНЕННЯ ДЕЯКИХ НЕРІВНОСТЕЙ ЕРМІТА – АДАМАРА ДЛЯ ФУНКЦІЙ З ОПУКЛИМИ АБСОЛЮТНИМИ ЗНАЧЕННЯМИ ДРУГИХ ПОХІДНИХ ЗА ДОПОМОГОЮ ІНТЕГРАЛІВ ДРОБОВОГО ПОРЯДКУ

We provide a unified approach to getting Hermite – Hadamard inequalities for functions with convex absolute values of the second derivatives via the Riemann – Liouville integrals. Some particular inequalities generalizing the classical results, such as the trapezoid inequality, Simpson's inequality, and midpoint inequality are also presented.

Запропоновано уніфікований підхід до отримання нерівностей Ерміта – Адамара для функцій з опуклими абсолютними значеннями других похідних за допомогою інтегралів Рімана – Ліувілля. Наведено деякі частинні нерівності, що узагальнюють класичні результати, такі як нерівність трапецій, нерівність Сімпсона та нерівність середньої точки.

1. Introduction. Let $f : I \subset R \to R$ be a convex function on the interval I, then for any $a,\ b \in I$ with $a \neq b$ we have the following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int\limits_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This remarkable result is well known in the literature as the Hermite – Hadamard inequality. Note that some of the the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave. Some refinements of the Hermite – Hadamard inequality on convex functions have been extensively investigated by a number of authors (see [2 – 7, 12]).

In [8], M. Z. Sarikaya et al. established some inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality, and they used the following lemma to prove their results.

Lemma 1.1. *Let $f : I \subseteq \mathbb{R} \to R$ be a twice differentiable mapping on I^0, $a,\ b \in I^0$ with $a < b$. If $f \in L_1[a,b]$, then*

$$\frac{1}{b-a}\int\limits_a^b f(x)dx - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2}\int\limits_0^1 s(t)[f''(ta+(1-t)b)+f''(tb+(1-t)a)]dt,$$

where

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$$s(t) := \begin{cases} t^2, & t \in \left[0, \dfrac{1}{2}\right], \\ (1-t)^2, & t \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

In [1], Alomari et al. obtained some inequalities for functions with quasiconvex absolute values of the second derivatives connecting with the Hermite – Hadamard inequality on the basis of the following lemma.

Lemma 1.2. *Let* $f: I \subseteq \mathbb{R} \to R$ *be a twice differentiable mapping on* I^0, $a, b \in I^0$ *with* $a < b$. *If* $f \in L_1[a, b]$, *then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int\limits_a^b f(x)dx = \frac{(b-a)^2}{2}\int\limits_0^1 t(1-t)f''(ta+(1-t)b)dt.$$

The following general integral identity for functions with convex absolute values of the second derivatives is proposed by M. Z. Sarikaya in [9].

Lemma 1.3. *Let* $I \subseteq \mathbb{R}$ *be an open interval,* $a, b \in I$ *with* $a < b$. *If* $f: I \to R$ *is a twice differentiable mapping such that* f'' *is integrable and* $0 \le \lambda \le 1$, *then the following equality holds*:

$$(\lambda - 1)f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2} + \frac{1}{b-a}\int\limits_a^b f(x)dx =$$

$$= \frac{(b-a)^2}{2}\int\limits_0^1 Q(t)f''(ta+(1-t)b)dt,$$

where

$$Q(t) := \begin{cases} t(t-\lambda), & t \in \left[0, \dfrac{1}{2}\right], \\ (1-t)(1-\lambda-t), & t \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

It is remarkable that M. Z. Sarikaya et al. [10] proved the following interesting inequalities of Hermite – Hadamard type involving Riemann – Liouville fractional integrals.

Theorem 1.1. *Let* $f: [a, b] \to R$ *be a positive function with* $a < b$ *and* $f \in L_1[a, b]$. *If* f *is a convex function on* $[a, b]$, *then the following inequalities for fractional integrals hold*:

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)] \le \frac{f(a)+f(b)}{2} \tag{2}$$

with $\alpha > 0$.

We remark that the symbol $J^\alpha_{a^+}$ and $J^\alpha_{b^-}f$ denote the left-hand and right-hand Riemann – Liouville fractional integrals of the order $\alpha \ge 0$ with $a \ge 0$ which are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a,$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < b,$$

respectively. Here $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt.$

J. R. Wang et al. [11] established the following fundamental integral identity including the second order derivatives of a given function via Riemann – Liouville integrals.

Lemma 1.4. *Let $f\colon [a,b] \to R$ be a twice differentiable mapping on (a,b) with $a < b$. If $f \in L_1[a,b]$, then the following equality for fractional integrals holds*:

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] - \frac{f(a)+f(b)}{2} =$$

$$= \frac{(b-a)^2}{2} \int\limits_0^1 \frac{(1-t)^{\alpha+1} + t^{\alpha+1} - 1}{\alpha+1} f''(ta+(1-t)b)dt. \tag{3}$$

Another integral identity including the second order derivatives of a given function via Riemann – Liouville integrals is obtained by Y. R. Zhang and J. R. Wang in [13] as follows.

Lemma 1.5. *Let $f\colon [a,b] \to R$ be a twice differentiable mapping on (a,b) with $a < b$. If $f \in L_1[a,b]$, then*

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] - f\left(\frac{a+b}{2}\right) = \frac{(b-a)^2}{2} \int\limits_0^1 m(t) f''(ta+(1-t)b)dt, \tag{4}$$

where

$$m(t) := \begin{cases} t - \dfrac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}, & t \in \left[0, \dfrac{1}{2}\right], \\ 1 - t - \dfrac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}, & t \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

In this paper, we generalize the results (3) and (4) for functions with convex absolute values of the second derivatives via Riemann – Liouville integrals.

2. Main results. In order to prove our main theorems, we need the following lemma.

Lemma 2.1. *Let $f\colon [a,b] \to R$ be a twice differentiable mapping on (a,b) with $a < b$ such that f'' is integrable and $0 \leq \lambda \leq 1$, then the following equality for fractional integrals holds*:

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} =$$

$$= \frac{(b-a)^2}{2} \int\limits_0^1 k(t) f''(ta+(1-t)b)dt, \tag{5}$$

where

$$k(t) := \begin{cases} t(1-\lambda) - \dfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left[0, \dfrac{1}{2}\right], \\ (1-t)(1-\lambda) - \dfrac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}, & t \in \left(\dfrac{1}{2}, 1\right]. \end{cases}$$

Proof. Multiplying (3) by λ, (4) by $1-\lambda$ on both sides, respectively, and adding the resulting inequalities, we get (5). Then we get the desired result.

By using this lemma, we can obtain the following general integral inequalities.

Theorem 2.1. *Let $f : [a, b] \to R$ be a twice differentiable mapping on (a, b) with $a < b$ such that f'' is integrable and $0 \le \lambda \le 1$. If $|f''|$ is convex on $[a, b]$, then the following inequality holds*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)] - (1-\lambda)f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2}\right| \le$$

$$\le \frac{(b-a)^2}{2}\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)} + \frac{1-\lambda}{8}\right](|f''(a)| + |f''(b)|).$$

Proof. From Lemma 2.1 and the definition of $k(t)$, we have

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)] - (1-\lambda)f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2} \le$$

$$\le \frac{(b-a)^2}{2}\left\{\int\limits_0^{\frac{1}{2}}\left|t(1-\lambda) - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right||f''(ta+(1-t)b)|dt+\right.$$

$$\left.+\int\limits_{\frac{1}{2}}^{1}\left|(1-t)(1-\lambda) - \frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right||f''(ta+(1-t)b)|dt\right\} \le$$

$$\le \frac{(b-a)^2}{2}\left\{\int\limits_0^{\frac{1}{2}} t(1-\lambda)|f''(ta+(1-t)b)|dt + \int\limits_{\frac{1}{2}}^{1}(1-t)(1-\lambda)|f''(ta+(1-t)b)|dt+\right.$$

$$\left.+\int\limits_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right||f''(ta+(1-t)b)|dt\right\}. \qquad (6)$$

Because $(1-t)^{\alpha+1} + t^{\alpha+1} \le 1$ for any $t \in [0, 1]$ and $|f''|$ is convex on $[a, b]$, we get

$$\int\limits_0^1\left|\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\right||f''(ta+(1-t)b)|dt \le$$

$$\le \int\limits_0^1\frac{1-(1-t)^{\alpha+1}-t^{\alpha+1}}{\alpha+1}\left(t|f''(a)| + (1-t)|f''(b)|\right)dt =$$

$$= \frac{\alpha}{(\alpha+1)(\alpha+2)} \frac{|f''(a)|+|f''(b)|}{2}. \tag{7}$$

On the other hand,

$$\int_0^{\frac{1}{2}} t(1-\lambda)|f''(ta+(1-t)b)|dt + \int_{\frac{1}{2}}^1 (1-t)(1-\lambda)|f''(ta+(1-t)b)|dt =$$

$$= \frac{(1-\lambda)(|f''(a)|+|f''(b)|)}{8}. \tag{8}$$

Now by (6) – (8), we can obtain the desired result which completes the proof.

Theorem 2.2. *Let* $f : [a,b] \to R$ *be a twice differentiable mapping on* (a,b) *with* $a < b$ *such that* $|f''|$ *is integrable and* $0 \le \lambda \le 1$. *If* $|f''|^q$ *is convex on* $[a,b]$ *with* $q > 1$, *then the following inequality holds*:

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} \right| \le$$

$$\le \frac{(b-a)^2(1-\lambda)}{2} \left(\frac{2}{p+1}\right)^{\frac{1}{p}} \left[\left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1} \right]^{\frac{1}{p}} \times$$

$$\times \left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}},$$

where $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Proof. From Lemma 2.1 and the definition of $k(t)$, by using the Hölder inequality, we have

$$\left| \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} [J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} \right| \le$$

$$\le \frac{(b-a)^2}{2} \int_0^1 |k(t)||f''(ta+(1-t)b)|dt \le$$

$$\le \frac{(b-a)^2}{2} \left(\int_0^1 |k(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}} =$$

$$= \frac{(b-a)^2}{2} \left(\int_0^{\frac{1}{2}} |k_1(t)|^p dt + \int_{\frac{1}{2}}^1 |k_2(t)|^p dt \right)^{\frac{1}{p}} \left(\int_0^1 |f''(ta+(1-t)b)|^q dt \right)^{\frac{1}{q}}, \tag{9}$$

where

$$k_1(t) = t(1-\lambda) - \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1},$$

$$k_2(t) = (1-t)(1-\lambda) - \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}.$$

Because $(1-t)^{\alpha+1} + t^{\alpha+1} \leq 1$ for any $t \in [0,1]$, using one skill of shrinking about inequality, we get

$$\int\limits_0^{\frac{1}{2}} |k_1(t)|^p dt \leq \int\limits_0^{\frac{1}{2}} \left(t(1-\lambda) + \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right)^p dt \leq$$

$$\leq \int\limits_0^{\frac{1}{2}} \left(t(1-\lambda) + \frac{1}{\alpha+1} \right)^p dt = (1-\lambda)^p \int\limits_0^{\frac{1}{2}} \left(t + \frac{1}{(\alpha+1)(1-\lambda)} \right)^p dt,$$

$$\int\limits_0^{\frac{1}{2}} \left(t + \frac{1}{(\alpha+1)(1-\lambda)} \right)^p dt =$$

$$= \frac{1}{p+1} \left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)} \right)^{p+1} - \frac{1}{p+1} \left(\frac{1}{(\alpha+1)(1-\lambda)} \right)^{p+1},$$

and

$$\int\limits_{\frac{1}{2}}^1 |k_2(t)|^p dt \leq \int\limits_{\frac{1}{2}}^1 \left((1-t)(1-\lambda) + \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1} \right)^p dt \leq$$

$$\leq \int\limits_{\frac{1}{2}}^1 \left((1-t)(1-\lambda) + \frac{1}{\alpha+1} \right)^p dt = (1-\lambda)^p \int\limits_{\frac{1}{2}}^1 \left((1-t) + \frac{1}{(\alpha+1)(1-\lambda)} \right)^p dt,$$

$$\int\limits_{\frac{1}{2}}^1 \left((1-t) + \frac{1}{(\alpha+1)(1-\lambda)} \right)^p dt =$$

$$= \frac{1}{p+1} \left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)} \right)^{p+1} - \frac{1}{p+1} \left(\frac{1}{(\alpha+1)(1-\lambda)} \right)^{p+1}.$$

Thus,

$$\int\limits_0^1 |k(t)|^p dt = (1-\lambda)^p \frac{2}{p+1} \left[\left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)} \right)^{p+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)} \right)^{p+1} \right]. \tag{10}$$

Moreover, because $|f''|^q$ is convex on $[a,b]$, we obtain

$$\int_0^1 |f''(ta+(1-t)b)|^q dt \leq \frac{|f''(a)|^q + |f''(b)|^q}{2}. \tag{11}$$

Thus, submitting (9) and (10) to (11), we can derive the desired result.

***Corollary* 2.1.** *With the assumptions as in Theorem* 2.2, *if* $|f''(x)| \leq M$ *on* $[a,b]$, *we have the following inequality*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2}\right| \leq$$

$$\leq \frac{M(b-a)^2(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}}.$$

Another Hermite – Hadamard inequalities for powers in terms of the second derivatives are given as follows.

Theorem 2.3. *Let* $f\colon [a,b] \to R$ *be a twice differentiable mapping on* (a,b) *with* $a<b$ *such that* $|f''|$ *is integrable and* $0 \leq \lambda \leq 1$. *If* $|f''|^q$ *is convex on* $[a,b]$ *with* $q>1$, *then the following inequality holds*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2}\right| \leq$$

$$\leq \frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1}\right]^{\frac{1}{q}} \times$$

$$\times \left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Proof. From Lemma 2.1 and using the Hölder inequality, we have

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2}\right| \leq$$

$$\leq \frac{(b-a)^2}{2}\left(\int_0^1 1 dt\right)^{\frac{1}{p}}\left(\int_0^1 |k(t) f''(ta+(1-t)b)|^q dt\right)^{\frac{1}{q}} \leq$$

$$\leq \frac{(b-a)^2}{2}\left(|f''(a)|^q \int_0^1 t|k(t)|^q dt + |f''(b)|^q \int_0^1 (1-t)|k(t)|^q dt\right)^{\frac{1}{q}}. \tag{12}$$

Calculating by parts, we get

$$\int_0^1 t|k(t)|^q dt = \int_0^{\frac{1}{2}} t|k_1(t)|^q dt + \int_{\frac{1}{2}}^1 t|k_2(t)|^q dt$$

with

$$\int_0^{\frac{1}{2}} t|k_1(t)|^q dt \leq \int_0^{\frac{1}{2}} t\left(t(1-\lambda) + \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}\right)^q dt \leq$$

$$\leq \int_0^{\frac{1}{2}} t\left(t(1-\lambda) + \frac{1}{\alpha+1}\right)^q dt = (1-\lambda)^q \int_0^{\frac{1}{2}} t\left(t + \frac{1}{(\alpha+1)(1-\lambda)}\right)^q dt$$

and

$$\int_{\frac{1}{2}}^1 t|k_2(t)|^q dt \leq \int_{\frac{1}{2}}^1 t\left((1-t)(1-\lambda) + \frac{1-(1-t)^{\alpha+1} - t^{\alpha+1}}{\alpha+1}\right)^q dt \leq$$

$$\leq \int_{\frac{1}{2}}^1 t\left((1-t)(1-\lambda) + \frac{1}{\alpha+1}\right)^q dt = (1-\lambda)^q \int_{\frac{1}{2}}^1 t\left((1-t) + \frac{1}{(\alpha+1)(1-\lambda)}\right)^q dt,$$

where

$$\int_0^{\frac{1}{2}} t\left(t + \frac{1}{(\alpha+1)(1-\lambda)}\right)^q dt = \frac{1}{2(q+1)}\left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} -$$

$$-\frac{1}{(q+1)(q+2)}\left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+2} + \frac{1}{(q+1)(q+2)}\left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+2}$$

and

$$\int_{\frac{1}{2}}^1 t\left((1-t) + \frac{1}{(\alpha+1)(1-\lambda)}\right)^q dt = -\frac{1}{(q+1)}\left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} +$$

$$+\frac{1}{2(q+1)}\left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} - \frac{1}{(q+1)(q+2)}\left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+2} +$$

$$+\frac{1}{(q+1)(q+2)}\left(\frac{1}{2} + \frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+2}.$$

Thus,

$$\int_0^1 t|k(t)|^q dt \le (1-\lambda)^q \frac{1}{q+1}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1}\right]. \tag{13}$$

Similarly,

$$\int_0^1 (1-t)|k(t)|^q dt \le (1-\lambda)^q \frac{1}{q+1}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1}\right]. \tag{14}$$

Thus, using (12)–(14), we get the desired result.

***Corollary* 2.2.** *With the assumptions as in Theorem* 2.3, *if* $|f''(x)| \le M$ *on* $[a,b]$, *we have the following inequality*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2}\right| \le$$

$$\le \frac{M(b-a)^2(1-\lambda)}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1}\right]^{\frac{1}{q}}.$$

***Corollary* 2.3.** *With the assumptions as in Theorems* 2.2 *and* 2.3, *we have*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2}\right| \le \min\{N_1, N_2\},$$

where

$$N_1 = \frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}} \times$$

$$\times \left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}$$

and

$$N_2 = \frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1} - \left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{q+1}\right]^{\frac{1}{q}} \times$$

$$\times \left(\frac{|f''(a)|^q + |f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Theorem 2.4. *Let* $f\colon [a,b] \to R$ *be a twice differentiable mapping on* (a,b) *with* $a < b$ *such that* $|f''|$ *is integrable and* $0 \le \lambda \le 1$. *If* $|f''|^q$ *is concave on* $[a,b]$ *with* $q > 1$, *then the following inequality holds*:

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] - (1-\lambda) f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2}\right| \le$$

$$\leq \frac{(b-a)^2(1-\lambda)}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1}-\left(\frac{1}{(\alpha+1)(1-\lambda)}\right)^{p+1}\right]^{\frac{1}{p}}\times$$

$$\times\left|f''\left(\frac{a+b}{2}\right)\right|,$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1.$

Proof. Similarly as in Theorem 2.2, but now $|f''|^q$ is concave on $[a,b]$, we have

$$\int_0^1 |f''(ta+(1-t)b)|^q dt \leq \left|f''\left(\frac{a+b}{2}\right)\right|^q,$$

so the desired result immediately follows.

3. Applications to quadrature formulas. In this section, we point out some particular inequalities generalizing the classical results, such as the trapezoid inequality, Simpson's inequality, and midpoint inequality.

Proposition 3.1 (trapezoid inequality)**.** *Under the assumptions in Theorem* 2.1 *with* $\lambda=1$, *we get*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]-\frac{f(a)+f(b)}{2}\right|\leq\frac{\alpha(b-a)^2}{4(\alpha+1)(\alpha+2)}(|f''(a)|+|f''(b)|).$$

Proposition 3.2 (midpoint inequality)**.** *Under the assumptions in Theorem* 2.1 *with* $\lambda=0$, *we have*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\leq$$

$$\leq\frac{(b-a)^2}{2}\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{1}{8}\right](|f''(a)|+|f''(b)|).$$

Proposition 3.3 (Simpson's inequality)**.** *Under the assumptions in Theorem* 2.1 *with* $\lambda=\dfrac{1}{3}$, *we obtain*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]\right]\right|\leq$$

$$\leq\frac{(b-a)^2}{2}\left[\frac{\alpha}{2(\alpha+1)(\alpha+2)}+\frac{1}{12}\right](|f''(a)|+|f''(b)|).$$

Proposition 3.4 (midpoint inequality)**.** *Under the assumptions in Theorem* 2.2 *with* $\lambda=0$, *we get*

$$\left|\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]-f\left(\frac{a+b}{2}\right)\right|\leq$$

$$\leq \frac{(b-a)^2}{2}\left(\frac{2}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{1}{2}+\frac{1}{(\alpha+1)}\right)^{p+1}-\left(\frac{1}{(\alpha+1)}\right)^{p+1}\right]^{\frac{1}{p}}\times$$

$$\times\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

Proposition 3.5 (Simpson's inequality). *Under the assumptions in Theorem* 2.3 *with* $\lambda=\dfrac{1}{3}$, *we have*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b)+J^\alpha_{b-}f(a)]\right]\right|\leq$$

$$\leq \frac{(b-a)^2}{3}\left(\frac{2}{q+1}\right)^{\frac{1}{q}}\left[\left(\frac{1}{2}+\frac{3}{2(\alpha+1)}\right)^{q+1}-\left(\frac{3}{2(\alpha+1)}\right)^{q+1}\right]^{\frac{1}{q}}\times$$

$$\times\left(\frac{|f''(a)|^q+|f''(b)|^q}{2}\right)^{\frac{1}{q}}.$$

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