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THE DRAZIN INVERSES OF INFINITE TRIANGULAR MATRICES AND THEIR LINEAR PRESERVERS

ОБЕРНЕНІ МАТРИЦІ ДРАЗІНА ДЛЯ НЕСКІНЧЕННИХ ТРИКУТНИХ МАТРИЦЬ ТА ЇХ ЛІНІЙНИХ ЗБЕРІГАЧІВ

We consider the ring of all infinite ($\mathbb{N} \times \mathbb{N}$) upper triangular matrices over a field F. We give a description of elements that are Drazin invertible in this ring. In the case where F is such that $\mathrm{char}(F) \neq 2$ and $|F| > 4$, we find the form of linear preservers for the Drazin inverses.

Розглянуто кільце всіх нескінченних ($\mathbb{N} \times \mathbb{N}$) верхніх трикутних матриць над полем F. Наведено опис елементів, що є оборотними за Дразіним у цьому кільці. У випадку, коли F є таким, що $\mathrm{char}(F) \neq 2$ та $|F| > 4$, знайдено форму лінійних зберігачів для обернених матриць Дразіна.

1. Introduction. Let R be a ring with identity. One knows that an arbitrary element $x \in R$ does not have to be invertible in R. However, sometimes elements of a ring have some generalized inverses. The most commonly known (see, for instance, [1, 2, 7]) are the Moore–Penrose inverse and the Drazin inverse. It was proved (in [17] and [11]) that in the ring of all $n \times n$, $n \in \mathbb{N}$, matrices over a field such inverses always exist and are unique.

In this paper we wish to focus on the Drazin inverses.

We say that x^D is the Drazin inverse of x if the three following hold:

$$xx^D = x^D x, \tag{1a}$$

$$x^D x x^D = x^D, \tag{1b}$$

$$\exists k \in \mathbb{N} \cup \{0\} : x^{k+1} x^D = x^k. \tag{1c}$$

The existence and form of the Drazin inverse is particularly interesting for the case of matrix rings. Such inverses have applications, for example in Markov chains, singular differential equations and iterative methods.

The problem of expressing the Drazin inverse of a matrix was studied by many authors (to name only [10, 14, 21, 24]), especially in the case where the given matrix is a sum or a difference of some matrices whose Drazin inverses are known (see, for example, [9, 12, 13, 15]).

In this article we will develop the results about such inverses and consider the problem of existence of the Drazin inverse in the ring of upper triangular infinite ($\mathbb{N} \times \mathbb{N}$) matrices over a field. A similar problem and its connection to Markov chains was considered in [6]. Let us also note that the Moore–Penrose inverses of infinite matrices were investigated in [18, 19].

Our first result is the following.

Theorem 1. *Let F be a field. Then $x \in \mathcal{T}_\infty(F)$ is Drazin invertible if and only if there exists $k \in \mathbb{N}$ such that the following condition is fulfilled. If $x_{nn} = 0$ for some $n \geq 2$ and $i(n)$ is a minimal number for which $x_{i(n)j} \neq 0$ for some j, then either*

(1) $(x^k)_{pn} = 0$ *for all* $i(n) < p \leq n$,

or

(2) *there exists* $i(n) < p \leq n$ *such that* $(x^k)_{pn} \neq 0$, *but in this case* $(x^{k+1})_{p,n+1} \neq 0$.

Next, we will move to the problem connected to the Drazin inverses and some linear maps.

We call ϕ a Drazin inverses preserver if

$$x^D = y \quad \Rightarrow \quad (\phi(x))^D = \phi(y) \tag{2}$$

for all x, y from a given ring. For the case of finite dimensional matrices such preservers were studied in [5, 8, 23].

Before presenting the description of such preservers we need to introduce the following type of maps.

We will say that a map $\phi : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is a *separable sum*, if there exist nonzero maps $\phi_1, \phi_2, \ldots, \phi_n$, $n \geq 2$, or $\phi_1, \phi_2, \phi_3, \ldots$, such that

$$\phi(x) = \sum_i \phi_i(x) \quad \text{for all} \quad x \in \mathcal{T}_\infty(F)$$

and

$$\phi_i(x)\phi_j(y) = \phi_j(y)\phi_i(x) = 0 \quad \text{for all} \quad i, j \quad \text{and} \quad x, y \in \mathcal{T}_\infty(F).$$

Now we can formulate the following theorem.

Theorem 2. *Assume that* F *is a field such that* $\operatorname{char}(F) \neq 2$ *and* $|F| > 4$. *If* $\phi \colon \mathcal{T}_\infty(F) \to$ $\to \mathcal{T}_\infty(F)$ *is a linear map satisfying* (2), *then* ϕ *is a separable map of* ϕ_1 *and* ϕ_2, *where* ϕ_1 *and* $-\phi_2$ *are idempotent preservers.*

2. Proofs of results. We start with presenting the notation used in the paper.

We denote by $\mathcal{M}_{n\times m}(F)$ the ring of all $n \times m$ matrices over F.

We write e_{nm} for the infinite matrix such that

$$(e_{nm})_{kl} = \begin{cases} 1, & \text{if} \quad k = n \quad \text{and} \quad l = m, \\ 0, & \text{otherwise.} \end{cases}$$

The symbols e_n and e_∞ are used for, respectively, $n \times n$ and infinite, identity matrices.

By x^T we denote the matrix transposed to x.

By x^y, where y is an invertible matrix, we mean $y^{-1}xy$.

If an infinite matrix x may have nonzero coefficients only in the positions (i, j) with $i \in I$, $j \in J$, then we will write

$$x = \sum_{i \in I, \, : \, j \in J} x_{ij} e_{ij}.$$

In our considerations there will also appear some subrings of $\mathcal{T}_\infty(F)$.

By $\mathcal{NT}_\infty(F)$ we understand the subring of $\mathcal{T}_\infty(F)$ of all matrices x such that $x_{nn} = 0$ for all $n \in \mathbb{N}$, whereas by $\mathcal{NT}^k_\infty(F)$ (where $k \in \mathbb{N}$) we mean a subring of $\mathcal{T}_\infty(F)$ of all x such that $x_{n,n+i} = 0$ holds for all $n \in \mathbb{N}$ and $0 \leq i \leq k$. Additionally we put $\mathcal{NT}^0_\infty(F) = \mathcal{NT}_\infty(F)$.

2.1. The Drazin invertibility. In this section we study the Drazin invertibility of elements of $\mathcal{T}_\infty(F)$.

The Drazin inverses of upper triangular finite matrices were studied in [3, 16]. As we have already mentioned, it was proved that every (not only) triangular matrix posseses the Drazin inverse. More precisely, if x is of the form

$$\left(\begin{array}{c|c} a & b \\ \hline 0 & c \end{array}\right),$$

then

$$x^D = \left(\begin{array}{c|c} a^D & B \\ \hline 0 & c^D \end{array}\right)$$

with

$$B = \left(a^D\right)^2 \left(\sum_{i=0}^{\mathrm{ind}(c)-1} (a^D)^i bc^i\right) \left(e_{n-k} - cc^D\right) + \left(e_k - aa^D\right) \left(\sum_{i=0}^{\mathrm{ind}(a)-1} a^i b\left(c^D\right)^i\right) \left(c^D\right)^2 - a^D bc^D,$$

where $\mathrm{ind}(x)$ denotes the index of x, which is a finite number associated with x and will be considered further later.

In the case of infinite matrices, it can be checked that this formula also holds, but, clearly, under the assumption that the infinite matrix c is Drazin invertible. Obviously, this criterion does not solve our problem.

First we will prove some results about the matrices from $\mathcal{NT}_\infty(F)$. We notice the following remark.

***Remark* 1.** *Let F be any field and let $k_1, k_2 \in \mathbb{N} \cup \{0\}$. Suppose that $x \in \mathcal{NT}_\infty^{k_1}(F) \setminus \mathcal{NT}_\infty^{k_1+1}(F)$, $y \in \mathcal{NT}_\infty^{k_2}(F) \setminus \mathcal{NT}_\infty^{k_2+1}(F)$, $z \in \mathcal{T}_\infty(F) \setminus \mathcal{NT}_\infty(F)$. Then*:

(1) $xy \in \mathcal{NT}_\infty^{k_1+k_2+1}(F)$,

(2) $xz, zx \in \mathcal{NT}_\infty^{k_1}(F)$.

Proof. From our assumptions we have $x_{nm} = 0$, if $m - n \leq k_1$, and $y_{nm} = 0$, if $m - n \leq k_2$.

To prove the first point, notice that for $n \in \mathbb{N}$ and $0 \leq m \leq k_1 + k_2 + 1$, we have

$$(xy)_{n,n+m} = \sum_{i=0}^{m} x_{n,n+i} y_{n+i,n+m} =$$

$$= \sum_{i=0}^{k_1} x_{n,n+i} y_{n+i,n+m} + \sum_{j=0}^{m-k_1-1} x_{n,n+m-j} y_{n+m-j,n+m} =$$

$$= \sum_{i=0}^{k_1} 0 \cdot y_{n+i,n+m} + \sum_{j=0}^{m-k_1-1} x_{n,n+m-j} \cdot 0 = 0.$$

Analogously, for the matrix xz we have

$$(xz)_{n,n+m} = \sum_{i=0}^{m} x_{n,n+i} z_{n+i,n+m} = \sum_{i=0}^{m} 0 \cdot z_{n+i,n+m} = 0$$

for $n \in \mathbb{N}$, $0 \leq m \leq k_1$. The case zx is almost the same, so we omit the calculations.

The above remark will be useful in the proof of the following proposition.

Proposition 1. *If F is any field and $x \in \mathcal{NT}_\infty(F)$, then x is Drazin invertible in $\mathcal{T}_\infty(F)$ if and only if it is nilpotent.*

Proof. It is known that if an element of a ring is nilpotent, then it is Drazin invertible and its Drazin inverse is 0. Therefore, it suffices to consider the elements that are not nilpotent.

Let $x \in \mathcal{NT}^k_\infty(F) \setminus \mathcal{NT}^{k+1}_\infty(F)$ $\big(k \in \mathbb{N} \cup \{0\}\big)$ be so. If x^D existed, then one of the three cases would hold:

(1) $x^D = 0$,

(2) $x^D \in \mathcal{NT}^l_\infty(F) \setminus \mathcal{NT}^{l+1}_\infty(F)$ for some $l \in \mathbb{N} \cup \{0\}$,

(3) $x^D \in \mathcal{T}_\infty(F) \setminus \mathcal{NT}_\infty(F)$.

If $x^D = 0$, then by condition (1c), we would have $x^n = 0$ for some $n \in \mathbb{N}$. However, since x is not nilpotent, this is impossible.

To discuss the second and third case we use condition (1b).

In the second case, from the first point of Remark 1, it follows that $x^D x x^D \in \mathcal{NT}^{2l+k+2}_\infty(F)$. As $x^D \in \mathcal{NT}^k_\infty(F) \setminus \mathcal{NT}^{k+1}_\infty(F)$ and (1b) holds, this is possible only if $k \geq 2l + k + 2$, i.e., $l \leq -1$ — a contradiction.

In the third case, by the second point of Remark 1, we would have $x^D x x^D \in \mathcal{NT}^k_\infty(F)$ and $x^D \in \mathcal{T}_\infty(F) \setminus \mathcal{NT}_\infty(F)$ — a contradiction again. Concluding, x is not Drazin invertible.

Proposition 1 is proved.

Now we will discuss the general case.

Before we prove Theorem 1 we need to introduce two notions.

The first of them is the index of a matrix. The index of a matrix $x \in \mathcal{M}_{n\times n}(F)$ is defined as the minimal number $n \in \mathbb{N} \cup \{0\}$ for which we have

$$\operatorname{rank}\left(a^{n+1}\right) = \operatorname{rank}(a^n). \tag{3}$$

We write then $\operatorname{ind}(x) = n$. In particular, every invertible matrix has index 0. It is known that $\operatorname{ind}(x)$ is the minimal number for which (1c) holds. (For more information about the index of triangular matrices we refer to [3].)

The case of infinite matrices is more complicated. See, for example, the matrix J_∞ defined by

$$J_\infty = \sum_{n=1}^{\infty} e_{n,n+1} = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 0 & \\ & & & & \ddots \end{pmatrix}. \tag{4}$$

As all rows of J_∞ are linearly independent, we have $\operatorname{rank}(J_\infty) = \infty$. (By the rank we mean the row rank.)

Notice that for every $k \in \mathbb{N} \cup \{0\}$ we have $J^k_\infty = \sum_{n=1}^{\infty} e_{n,n+k}$, so $\operatorname{rank}(J^k_\infty) = \infty$. Hence, according to the definition given above we should have $\operatorname{ind}(J_\infty) = 0$, i.e., J_∞ should be invertible — a contradiction.

For this reason we can not consider an index of infinite matrix defined as in (3). However, we let ourselves to introduce a notion of *the Drazin inverse of an infinite matrix* which will be denoted by ind_D. If an infinite matrix x is Drazin invertible, then we will say that its Drazin index is the minimal number for which (1c) holds. Otherwise, we will write $\operatorname{ind}_D(x) = \infty$.

The second notion we want to define is connected to matrices from $\mathcal{M}_{n\times 1}(F)$. For every such matrix x there exists the minimal number k such that $x_{l1}=0$ for all $l\geq k$. We will call this number a nonzero-length of x and denote it $nzl(x)$.

Now we will give a proof of our first main result.

***Proof of Theorem* 1.** Let $x\in\mathcal{T}_\infty(F)$ be of the form

$$\left(\begin{array}{c|c} a_n & b_n \\ \hline 0 & c_n \end{array}\right) \quad \text{for some} \quad a_n\in\mathcal{T}_\infty(F), \quad c_n\in\mathcal{T}_\infty(F).$$

If

$$x^D=\left(\begin{array}{c|c} a'_n & b'_n \\ \hline 0 & c'_n \end{array}\right) \quad \text{with} \quad a'_n\in\mathcal{T}_\infty(F), \quad c'_n\in\mathcal{T}_\infty(F),$$

then $a_n^D=a'_n$. Hence, if x^D exists, then it fulfills the condition $(x^D)_{ij}=a'_{ij}$ for all $1\leq i,j\leq n$. Notice that since $\operatorname{ind}(x)\geq\operatorname{ind}(a_n)$, this means that there must exist such n that for all $m\geq n$ we have $\operatorname{ind}(a_{m+1})=\operatorname{ind}(a_m)=\operatorname{ind}_D(x)$. Consider then the matrices of the form

$$a_{m+1}=\left(\begin{array}{c|c} a_m & b \\ \hline 0 & c \end{array}\right) \quad \text{where} \quad c\in F.$$

We need to investigate when $\operatorname{ind}(a_{m+1})=\operatorname{ind}(a_m)$. We divide this problem into two cases.

Case 1: $c\neq 0$.

Suppose that $\operatorname{ind}(a_m)=i$. As a_m is finite, for all $j\geq i$ we have $\operatorname{rank}\left(a_m^{j+1}\right)=\operatorname{rank}\left(a_m^j\right)$. Now observe that since $c\neq 0$, for all $j\geq i$ we get

$$\operatorname{rank}\left(\left(\begin{array}{c|c} a_m & b \\ \hline 0 & c \end{array}\right)^{j+1}\right)=$$

$$=\operatorname{rank}\left(\left(\begin{array}{c|c} a_m^{j+1} & b' \\ \hline 0 & c^{j+1} \end{array}\right)\right)=$$

$$=1+\operatorname{rank}(a_m^{j+1})=1+\operatorname{rank}(a_m^j)=$$

$$=\operatorname{rank}\left(\left(\begin{array}{c|c} a_m^j & b'' \\ \hline 0 & c^j \end{array}\right)\right)=\operatorname{rank}\left(\left(\begin{array}{c|c} a_m & b \\ \hline 0 & c \end{array}\right)^j\right)$$

for $b',b''\in\mathcal{M}_{n\times 1}(F)$. Thus, $\operatorname{ind}(a_{m+1})\leq\operatorname{ind}(a_m)$ and we are done.

Case 2: $c=0$.

First we notice that for any $n\in\mathbb{N}$

$$a_{m+1}^n=\left(\begin{array}{c|c} a_m^n & a_m^{n-1}b \\ \hline 0 & 0 \end{array}\right).$$

If $\operatorname{ind}(a_m)=i$, then, for all $j\geq i$, $\operatorname{rank}\left(a_m^{j+1}\right)=\operatorname{rank}\left(a_m^j\right)$. Therefore we have $\operatorname{ind}(a_{m+1})=$ $=\operatorname{ind}(a_m)$ if and only if

$$\operatorname{rank}\left(\left(\begin{array}{c|c} a_m^{j+1} & a_m^j \\ \hline 0 & 0 \end{array}\right)\right)=\operatorname{rank}\left(\left(\begin{array}{c|c} a_m^j & a_m^{j-1} \\ \hline 0 & 0 \end{array}\right)\right) \quad \text{for all} \quad j\geq i.$$

Since $\operatorname{rank}\left(a_m^{j+1}\right) = \operatorname{rank}\left(a_m^j\right)$, this happens if and only if $nzl\left(a_m^{j-1}b\right) \leq nzl\left(a_m^j b\right)$ or $\operatorname{rank}(a_{m+1}^j) = \operatorname{rank}(a_m^j)$. These two are the situations described in our theorem.

Theorem 1 is proved.

It can be easily observed that in particular we have the following corollary.

***Corollary* 1.** *If F is a field and $x \in \mathcal{T}_\infty(F)$ is a matrix such that $x_{nn} = 0$ holds only for finitely many $n \in \mathbb{N}$, then x is Drazin invertible in $\mathcal{T}_\infty(F)$.*

2.2. The Drazin inverses preservers. In this section we will prove Theorem 2. We will use some facts about tripotents, i.e., about elements x such that $x^3 = x$.

***Remark* 2.** *Let R be a ring. If $\phi\colon R \to R$ is a linear map satisfying* (2), *then ϕ preserves tripotents.*

Proof. Let ϕ satisfy our assumptions and let $x \in R$ be tripotent. As $x^3 = x$, we must have $x^D = x$. Hence, $\phi(x^D) = \phi(x)$ and from (2) we get $(\phi(x))^D = \phi(x)$. Substituing the latter into (1b), we get $(\phi(x))^3 = \phi(x)$.

It is well-known that if x is $n \times n$ tripotent matrix, then x^y is a diagonal matrix for some y. (Some more information about characterization of tripotents can be found in [22].) We wish to obtain an analogous result for elements of $\mathcal{T}_\infty(F)$. To do this, we first notice the following remark.

***Remark* 3.** *Let F be any field, $n \in \mathbb{N}$, and let*

$$x = \left(\begin{array}{c|c} a & b \\ \hline 0 & 0 \end{array}\right),$$

$$y = \left(\begin{array}{c|c} a & b \\ \hline 0 & 1 \end{array}\right),$$

$$z = \left(\begin{array}{c|c} a & b \\ \hline 0 & -1 \end{array}\right)$$

be block matrices from $\mathcal{T}_{n+1}(F)$.

1. *The matrix x is tripotent if and only if $a \in \mathcal{T}_n(F)$ is tripotent and $a^2 b = b$.*
2. *The matrix y is tripotent if and only if $a \in \mathcal{T}_n(F)$ is tripotent and $a^2 b + ab = 0$.*
3. *The matrix z is tripotent if and only if $a \in \mathcal{T}_n(F)$ is tripotent and $a^2 b = ab$.*

The proof is by elementary calculations, so we let ourselves to omit it.

We use the above remark in the proof of the following.

Lemma 1. *Suppose F is a field of characteristic different from* 2 *and that $n \in \mathbb{N}$. If $x \in \mathcal{T}_\infty(F)$ is tripotent, then there exists an invertible $y \in \mathcal{T}_\infty(F)$ such that x^y is a diagonal matrix.*

Proof. First we show that if

$$x = \left(\begin{array}{c|c} a & b \\ \hline 0 & c \end{array}\right) \in \mathcal{T}_{k+1}(F) \tag{5}$$

is tripotent matrix with $a \in \mathcal{T}_k(F)$ diagonal, then there exists y of the form

$$y = \left(\begin{array}{c|c} e_k & \hat{y} \\ \hline 0 & 1 \end{array}\right)$$

such that x^y is diagonal.

As $x^3 = x$, the coefficient c in (5) is equal to 0, 1 or -1. Let us discuss these three cases.

If $c=0$, then by taking $\hat{y}=-ab$ we obtain

$$x^y=\left(\begin{array}{c|c} e_k & ab \\ \hline 0 & 1 \end{array}\right)\left(\begin{array}{c|c} a & b \\ \hline 0 & 0 \end{array}\right)\left(\begin{array}{c|c} e_k & -ab \\ \hline 0 & 1 \end{array}\right)=\left(\begin{array}{c|c} a & -a^2b+b \\ \hline 0 & 0 \end{array}\right),$$

so by the first point of Remark 3

$$x^y=\left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right).$$

Let now $c=1$. Notice that from the second point of Remark 3 and the assumption $\operatorname{char}(F)\neq 2$, it follows that $a_i=1$ implies $b_i=0$. If

$$x^y=\left(\begin{array}{c|c} a & 0 \\ \hline 0 & 1 \end{array}\right),$$

then $\hat{y}=(\hat{y}_1,\hat{y}_2,\ldots,\hat{y}_k)^T$ must satisfy the system

$$\begin{gathered}(a_1-1)\hat{y}_1+b_1=0,\\ (a_2-1)\hat{y}_2+b_2=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ (a_k-1)\hat{y}_k+b_k=0.\end{gathered}$$

As for those i for which we have $a_i=1$, the coefficients b_i are equal to 0, we can choose $\hat{y}_i$ arbitrarily (say $\hat{y}_i=0$). For the case where $a_i\neq 0$, the coefficients a_i-1 are invertible, so we simply have $\hat{y}_i=(1-a_i)^{-1}b_i$. Thus, the desired y exist.

The case where $c=-1$ is analogous. Since $\operatorname{char}(F)\neq 2$, from the third point of Remark 3 we can deduce that if $a_i=-1$, then $b_i=0$. If

$$x^y=\left(\begin{array}{c|c} a & 0 \\ \hline 0 & -1 \end{array}\right),$$

then $\hat{y}$ must satisfy the system

$$\begin{gathered}(a_1+1)\hat{y}_1+b_1=0,\\ (a_2+1)\hat{y}_2+b_2=0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\\ (a_k+1)\hat{y}_k+b_k=0.\end{gathered}$$

Analogously to the previous case, if for some i we have $a_i=-1$, then $b_i=0$ and $\hat{y}_i$ may be arbitrarily chosen, whereas if $a_i\neq -1$, then $1+a_i$ is invertible and we have $\hat{y}_i=-(a_i+1)^{-1}b_i$.

Let $x\in\mathcal{T}_n(F)$ be tripotent. Then

$$x_2:=\left(\begin{array}{c|c} x_{11} & x_{12} \\ \hline 0 & x_{22} \end{array}\right)$$

is tripotent as well. According to what was done above, there exists $y_2\in\mathcal{T}_2(F)$ such that $x_2^{y_2}$ is a diagonal matrix. Hence

$$x^{\left(\begin{array}{c|c} y_2 & 0 \\ \hline 0 & e_{n-2} \end{array}\right)} = \left(\begin{array}{c|c} a_2 & b_2 \\ \hline 0 & c_2 \end{array}\right)$$

where $a_2 \in \mathcal{T}_2(F)$ is diagonal.

Suppose that we have proved that for some $2 \leq m < n$ we have

$$\left(\left(x^{\left(\begin{array}{c|c} y_2 & 0 \\ \hline 0 & e_{n-2} \end{array}\right)}\right)^{\left(\begin{array}{c|c} y_3 & 0 \\ \hline 0 & e_{n-3} \end{array}\right)} \cdots\right)^{\left(\begin{array}{c|c} y_m & 0 \\ \hline 0 & e_{n-m} \end{array}\right)} = \left(\begin{array}{c|c} a_m & b_m \\ \hline 0 & c_m \end{array}\right)$$

with $a_m \in \mathcal{T}_m(F)$ diagonal. Then, by the first part of the proof, there exists $y_{m+1} \in \mathcal{T}_{m+1}(F)$ such that

$$\left(\begin{array}{c|c} a_m & b_m \\ \hline 0 & c_m \end{array}\right)^{\left(\begin{array}{c|c} y_{m+1} & 0 \\ \hline 0 & e_{n-m-1} \end{array}\right)} = \left(\begin{array}{c|c} a_{m+1} & b_{m+1} \\ \hline 0 & c_{m+1} \end{array}\right),$$

where $a_{m+1} \in \mathcal{T}_{m+1}(F)$ is diagonal.

Lemma 1 is proved.

From the above lemma we can easily obtain the following corollary.

***Corollary* 2.** *Let F be a field such that $\operatorname{char}(F) \neq 2$. If $x \in \mathcal{T}_\infty(F)$ is tripotent, then there exist an invertible matrix $y \in \mathcal{T}_\infty(F)$ such that x^y is diagonal.*

Proof. For any $n \in \mathbb{N}$ define $x_n \in \mathcal{T}_n(F)$ by the following rule: $(x_n)_{ij} = x_{ij}$ for all $1 \leq i$, $j \leq n$.

As every x_n is tripotent, by Lemma 1, for every $n \in \mathbb{N}$ there exists $y_n \in \mathcal{T}_n(F)$ such that $x_n^{y_n}$ is diagonal. Moreover, we can choose $y_1,\ y_2,\ y_3, \ldots$ in such a way that $(y_n)_{ij} = (y_{n+1})_{ij}$ for all $1 \leq i,\ j \leq n$. Define now $y \in \mathcal{T}_\infty(F)$ according to the condition $y_{ij} = (y_n)_{ij}$ for $n \geq \max(i,j)$. For every $1 \leq i < j$ we have

$$(x^y)_{ij} = \sum_{i \leq k \leq l \leq j} (y^{-1})_{ik} x_{kl} y_{lj} = \sum_{i \leq k \leq l \leq j} (y_j^{-1})_{ik} (x_j)_{kl} (y_j)_{lj} = (y_j^{-1} x_j y_j)_{ij} = 0.$$

Thus, x^y is a diagonal matrix.

Now we will investigate maps satisfying (2) that fulfil some extra conditions.

In the following lemmas we make use of ideas from [4].

Lemma 2. *Let F be a field of characteristic different from 2 and containing more than 3 elements. If $\phi\colon \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is a linear map preserving the Drazin inverses, then $\phi(e_\infty)\phi(x) = \phi(x)\phi(e_\infty)$ for all $x \in \mathcal{T}_\infty(F)$.*

Proof. We give a proof in 4 steps.

Step 1. For all $n \in \mathbb{N}$ we have $\phi(e_\infty)\phi(e_{nn}) = \phi(e_{nn})\phi(e_\infty)$.

Fix $\alpha \in F^* \setminus \{-1, 1\}$. As $|F| > 3$ such element exists.

It can be easily noticed that $(e_\infty + (\alpha-1)e_{nn})^D = e_\infty + (\alpha^{-1}-1)e_{nn}$. Hence, $\phi(e_\infty + (\alpha-1)e_{nn})$ and $\phi\big(e_\infty + (\alpha^{-1}-1)e_{nn}\big)$ commute. From this and the linearity of ϕ we have

$$(\alpha^2 - 1)(\phi\big(e_\infty)\phi(e_{nn}) - \phi(e_{nn})\phi(e_\infty)\big) = 0.$$

As $\alpha \neq \pm 1$, we are done.

Step 2. For all $n, m \in \mathbb{N}$, $n < m$ we have $\phi(e_\infty)\phi(e_{nm}) = \phi(e_{nm})\phi(e_\infty)$.

This time notice that $(e_\infty + e_{nm})^D = e_\infty - e_{nm}$, so $\phi(e_\infty + e_{nm})$ and $\phi(e_\infty - e_{nm})$ commute. From this and the linearity of ϕ, we obtain

$$2(\phi\big(e_\infty)\phi(e_{nm}) - \phi(e_{nm})\phi(e_\infty)\big) = 0.$$

As $\operatorname{char}(F) \neq 2$, the result follows.

Step 3. For any $x \in \mathcal{T}_\infty(F)$ of the form

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{with} \quad a \in \mathcal{T}_n(F) \quad \text{for some } n \in \mathbb{N},$$

we have $\phi(e_\infty)\phi(x) = \phi(x)\phi(e_\infty)$.

This is a consequence of the linearity of ϕ.

Step 4. For all $x \in \mathcal{T}_\infty(F)$ we have $\phi(e_\infty)\phi(x) = \phi(x)\phi(e_\infty)$.

Suppose that the claim of this point is not true.

Define $x^{(n)}$ to be a matrix from $\mathcal{T}_\infty(F)$ such that

$$(x^{(n)})_{ij} = \begin{cases} x_{ij} & \text{for} \quad 1 \leq i, \quad j \leq n, \\ 0, & \text{otherwise}. \end{cases}$$

Clearly, by the preceding step, $\phi(e_\infty)$ commutes with $\phi\big(x^{(n)}\big)$ for all $n \in \mathbb{N}$. Hence, if $\phi(e_\infty)\phi(x) \neq$ $\neq \phi(x)\phi(e_\infty)$, then

$$\phi(e_\infty)\phi\big(x - x^{(n)}\big) \neq \phi\big(x - x^{(n)}\big)\phi(e_\infty)$$

for all $n \in \mathbb{N}$. However, this means that if there exists x such that $\phi(x)$ does not commute with $\phi(e_\infty)$, then there exists a matrix y that has an arbitrary number of first zero columns and such that $\phi(y)$ does not commute with $\phi(e_\infty)$. Yet, the only matrix with the latter property is the zero matrix, but ona can see that $\phi(0) = 0$ commutes with $\phi(e_\infty)$ — a contradiction. Concluding, $\phi(e_\infty)\phi(x) = \phi(x)\phi(e_\infty)$ for all $x \in \mathcal{T}_\infty(F)$.

Lemma 2 is proved.

Lemma 3. *Let F be a field of at least 4 elements. If $\phi\colon \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is a linear map such that ϕ preserves the Drazin inverses and there exist N, $N \subseteq \mathbb{N}$ for which the following conditions hold*:

if $i \notin N$ or $j \notin N$, then $(\phi(x))_{ij} = 0$ for all $x \in \mathcal{T}_\infty(F)$,

$\phi(e_\infty) = \sum_{n\in N} e_{nn}$,

then ϕ preserves idempotents.

Proof. Let $x \in \mathcal{T}_\infty(F)$ be an idempotent. Every idempotent is also a tripotent, so it is conjugate to some diagonal matrix. Clearly, the map ϕ preserves the Drazin inverses if and only if for every t the map $\phi \cdot \mathcal{I}nn_t$ preserves them. Therefore, with no loss of generality, we can assume that x is a diagonal matrix $\big($more precisely, it is of the form $\sum_{n\in M} e_{nn}$ for some $\varnothing \subseteq M \subseteq N\big)$.

Let us choose $\alpha \in F^* \setminus \{1, 2\}$. One can see that $\big(e_\infty + (\alpha^{-1} - 1)x\big) = e_\infty + (\alpha - 1)x$. As ϕ preserves the Drazin inverses, this means that $\big(\phi(e_\infty + (\alpha^{-1} - 1)x)\big)^D = \phi\big(e_\infty + (\alpha - 1)x\big)$. In particular, condition (1b) must be satisfied. Hence

$$\phi\big(e_\infty + (\alpha-1)x\big)\phi\big(e_\infty + (\alpha^{-1}-1)x\big)\phi\big(e_\infty + (\alpha-1)x\big) = \phi\big(e_\infty + (\alpha-1)x\big).$$

Using the linearity and fact that $\phi(e_\infty) = \sum_{n\in N} e_{nn}$ commutes with $\phi(y)$ for all $y \in \mathcal{T}_\infty(F)$, we get

$$\left[\sum_{n\in N} e_{nn} + (\alpha-1)\phi(x)\right]\left[\sum_{n\in N} e_{nn} + (\alpha^{-1}-1)\phi(x)\right]\left[\sum_{n\in N} e_{nn} + (\alpha-1)\phi(x)\right] =$$
$$= \sum_{n\in N} e_{nn} + (\alpha-1)\phi(x).$$

We multiply the above equality by α and write it as follows:

$$\left[\sum_{n\in N} e_{nn} + (\alpha-1)\phi(x)\right]\left[\sum_{n\in N} e_{nn} + (\alpha-1)\left(\sum_{n\in N} e_{nn} - \phi(x)\right)\right]\left[\sum_{n\in N} e_{nn} + (\alpha-1)\phi(x)\right] =$$
$$= \sum_{n\in N} e_{nn} + (\alpha-1)\left(\sum_{n\in N} e_{nn} + \phi(x)\right) + (\alpha-1)^2\phi(x).$$

After evaluating we obtain

$$\alpha\sum_{n\in N} e_{nn} + \left[\alpha - 1 + 2(\alpha-1)^2\right]\phi(x) + \left[(\alpha-1)^3 - (\alpha-1)^2\right]\big(\phi(x)\big)^2 - (\alpha-1)^3\big(\phi(x)\big)^3 =$$
$$= \alpha\sum_{n\in N} e_{nn} + (\alpha-1)\phi(x).$$

We simplify it and make use of the fact that $\big(\phi(x)\big)^3 = \phi(x)$ and have

$$\left[(\alpha-1)^3 - (\alpha-1)^2\right]\left[(\phi(x))^2 - \phi(x)\right] = 0.$$

This is possible only if $\alpha \in \setminus\{1,2\}$ or $(\phi(x))^2 - \phi(x) = 0$. As we have chosen $\alpha \neq 1, 2,$ $\phi(x)$ must be an idempotent.

Lemma 3 is proved.

Lemma 4. *Let F be a field of at least 5 elements. If $\phi\colon \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ is a linear map that preserve the Drazin inverses and $\phi(e_\infty) = 0$, then $\phi(x) = 0$ for all $x \in \mathcal{T}_\infty(F)$.*

Proof. Again we divide the proof into steps.

Step 1. For all $n \in \mathbb{N}$ we have $\phi(e_{nn}) = 0$.

Analogously, as it was done in the proof of Lemma 3, we choose $\alpha \in F^* \setminus \{1\}$ and make use of the fact that $\big(e_\infty + (\alpha^{-1}-1)e_{nn}\big)^D = e_\infty + (\alpha-1)e_{nn}$. Performing the same way and using the fact that $\phi(e_\infty) = 0$ we obtain $\left[(\alpha-1)^3 + (\alpha-1)^2 + \alpha - 1\right]\phi(e_{nn}) = 0$. As $\alpha \neq 1$, this implies

$$(\alpha^2 - \alpha + 1)\phi(e_{nn}) = 0. \tag{6}$$

Since $|F| > 4$, there exist $\alpha \neq 0, 1$ such that $\alpha^2 - \alpha + 1 \neq 0$. Substituing this α into (6) we get $\phi(e_{nn}) = 0$.

Step 2. For all $n < m$ we have $\phi(e_{nm}) = 0$.

Let $\alpha \in F^*$. From $(e_\infty + \alpha e_{nm})^D = e_\infty - \alpha e_{nm}$, $\phi(e_\infty) = 0$ and condition (1b) we get $\alpha\phi(e_{nm}) + \alpha^3(\phi(e_{nm}))^3 = 0$. Now we choose $\alpha_1 \in F^*$ and $\alpha_2 \in F^*$ such that $\alpha_1 \neq \pm\alpha_2$ (as $|F| > 4$, it is possible). For such coefficients the system of equations

$$\alpha_1\phi(e_{nm}) + \alpha_1^3\big(\phi(e_{nm})\big)^3 = 0,$$
$$\alpha_2\phi(e_{nm}) + \alpha_2^3\big(\phi(e_{nm})\big)^3 = 0$$

has unique solution and we have $\phi(e_{nm}) = 0$.

Step 3. For all $x \in \mathcal{T}_\infty(F)$ of the form

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{for some} \quad a \in \mathcal{T}_n(F) \quad \text{for some} \quad n \in \mathbb{N},$$

we have $\phi(x) = 0$.

This is a consequence of the two preceding steps and linearity.

Step 4. For all $x \in \mathcal{T}_\infty(F)$ we have $\phi(x) = 0$.

This can be proved by the same method as one used in step 4 in the proof of Lemma 2, so we do not repeat the arguments.

Lemma 4 is proved.

Now we will prove our second main result.

***Proof of Theorem* 2.** Suppose ϕ preserves the Drazin inverses and is linear. Then, by Remark 2, $\phi(e_\infty)$ is tripotent. From Corollary 2 it follows that there exist $t \in \mathcal{T}_\infty(F)$ such that $(\phi(e_\infty))^t$ is diagonal. Consider then $\phi' := \mathcal{I}nn_t \cdot \phi$ instead of ϕ. We have then $\phi'(e_\infty) = \sum_{n\in\mathbb{N}} a_n e_{nn}$, where $a_n \in \{-1, 0, 1\}$. By Lemma 2, we get $\phi'(e_\infty)\phi'(x) = \phi'(x)\phi'(e_\infty)$ for all $x \in \mathcal{T}_\infty(F)$. Let n, m be such natural numbers that $n < m$. We should obtain

$$\big(\phi'(e_\infty)\phi'(x)\big)_{nm} = a_n\big(\phi'(x)\big)_{nm} = \big(\phi'(x)\big)_{nm} a_m = \big(\phi'(x)\phi'(e_\infty)\big).$$

As $a_n \neq a_m$ and $\operatorname{char}(F) \neq 2$, this is possible only if $\big(\phi'(x)\big)_{nm} = 0$ for any x. Concluding, we have proved that if $\big(\phi'(e_\infty)\big)_{nn} \neq \big(\phi'(e_\infty)\big)_{mm}$, then $\big(\phi'(e_\infty)\big)_{nm} = 0$ for all $x \in \mathcal{T}_\infty(F)$. Hence, ϕ' is a separable sum of at most three 'types' of maps ψ_1, ψ_2, ψ_3 such that

all these maps preserve the Drazin inverses,

the maps of 'type' ψ_1 satisfy the condition $\psi_1(e_\infty) = \sum_{n\in N_1} e_{nn}$ for some $\varnothing \subsetneq N_1 \subseteq \mathbb{N}$,

the maps of 'type' ψ_2 satisfy the condition $\psi_2(e_\infty) = -\sum_{n\in N_2} e_{nn}$ for some $\varnothing \subsetneq N_2 \subseteq \mathbb{N}$,

the maps of 'type' ψ_3 satisfy the condition $\psi_3(e_\infty) = 0$.

Now we discuss these three 'types' of maps.

The map ψ_1 satisfies the assumptions of Lemma 3, so ψ_1 preserves idempotents.

The map $-\psi_2$ is of the same form as ψ_1, so $-\psi_2$ is an idempotent preserver.

The map ψ_3 fulfills the assumptions of Lemma 4, so $\psi_3(x) = 0$ for all $x \in \mathcal{T}_\infty(F)$.

Summing up, ϕ is a separable map of ψ_1 and ψ_2, where ψ_1 and $-\psi_2$ are idempotent preservers.

Theorem 2 is proved.

Since the problem of describing the Drazin inverses preservers is reduced to idempotent preservers, we give a characterization of such maps. To do this we need to present a few more sorts of linear maps.

If t is an invertible infinite upper triangular matrix, then $\phi\colon \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ such that $\phi(x) = t^{-1}xt$ is an inner automorphism of $\mathcal{T}_\infty(F)$ will and is denoted by $\mathcal{I}nn_t$.

For a family of infinite triangular matrices $\{b_{nm}\}_{n<m}$ such that

if $i \notin \mu(n)$ or $j \notin \mu(m)$, then $(b_{nm})_{ij} = 0$,

for all $n < p < m$ we have $b_{np}b_{pm} = b_{nm}$,

by $\mathcal{S}pl_{\mu,\{b_{nm}\}_{n<m}} : \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ we denote a map such that

$$\mathcal{S}pl_{\mu,\{b_{nm}\}_{n<m}}(x) = \sum_{n\in\mathbb{N}} \sum_{i\in\mu(n)} x_{nn}e_{ii} + \sum_{n<m} x_{nm}b_{nm}$$

for any $x = (x_{ij}) \in \mathcal{T}_\infty(F)$.

In particular, if τ is an increasing map on $\mathbb{N}$, then $\phi\colon \mathcal{T}_\infty(F) \to \mathcal{T}_\infty(F)$ defined by

$$\phi\left(\sum_{i\leq j} x_{ij}\right) = \sum_{i\leq j} x_{ij}e_{\tau(i)\tau(j)}$$

is a special case of $\mathcal{S}pl_{\mu,\{b_{nm}\}_{n<m}}$. We call it a splashing map and denote it by $\mathcal{S}pl_\tau$.

Suppose that N is a subset of $\mathbb{N}$ equal either to $\{1, 2, \ldots, n\}$ or $\mathbb{N} \setminus \{n, n+1, \ldots, m\}$, where $n, m \in \mathbb{N}$, $n < m$. By $\mathcal{C}ut_N$ we will mean the map sending $x \in \mathcal{T}_\infty(F)$ to

$$\begin{pmatrix}
0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
 & \ddots & \vdots & & & & \\
 & & 0 & 0 & 0 & 0 & \ldots \\
 & & & x_{n+1,n+1} & x_{n+1,n+2} & x_{n+1,n+3} & \ldots \\
 & & & & x_{n+2,n+2} & x_{n+2,n+3} & \\
 & & & & & x_{n+3,n+3} & \\
 & & & & & & \ddots
\end{pmatrix}$$

in the first case, or to

$$\begin{pmatrix}
x_{nn} & \ldots & x_{nm} & 0 & \ldots \\
 & \ddots & \vdots & & \\
 & & x_{mm} & 0 & \\
 & & & 0 & \\
 & & & & \ddots
\end{pmatrix}$$

in the second case.

If $x \in \mathcal{T}_\infty(F)$ is of the form

$$\left(\begin{array}{c|c} x & 0 \\ \hline 0 & 0 \end{array}\right), \quad \text{where} \quad x \in \mathcal{T}_n(F),$$

then by $\mathcal{J}(x)$ we will denote a matrix

$$\left(\begin{array}{c|c} y & 0 \\ \hline 0 & 0 \end{array}\right) \quad \text{with} \quad y \in \mathcal{T}_n(F),$$

where $y_{ij} = x_{n+1-j,n+1-i}$.

The defined above maps appear in the theorem below.

***Corollary* 3** [20]**.** *Let F be a field whose characteristic is different from 2. If $\phi\colon \mathcal{T}_\infty(F) \to \to \mathcal{T}_\infty(F)$ is a linear map satisfying the condition*

$$x \quad \textit{is an idempotent} \quad \Rightarrow \quad \phi(x) \quad \textit{is an idempotent},$$

then ϕ is a separable sum of any number of maps of one of the forms

$$\mathcal{I}nn_t \cdot \mathcal{S}pl_{\mu,\{b_{nm}\}_{n<m}},$$

$$\mathcal{I}nn_t \cdot \mathcal{S}pl_{\mu,\{b_{nm}\}_{n<m}} \cdot \mathcal{C}ut_N,$$

$$\mathcal{I}nn_t \cdot \mathcal{S}pl_{\mu,\{b_{nm}\}_{n<m}} \cdot \mathcal{J} \cdot \mathcal{C}ut_N,$$

where $N = \{1, 2, \ldots, n\}$ or $N = \mathbb{N} \setminus \{n, n+1, \ldots, m\}$ for some $n < m$.

2.3. Closing remarks. In this section we mention some problems and present some remarks related to the considered issues.

Suppose that ϕ has a following property: for every $n \in \mathbb{N}$ there exist $\big(\phi(e_\infty)\big)_{nn} \neq 0$. Then it can be observed that ϕ preserves inverses, i.e., if $x^{-1} = y$, then $(\phi(x))^{-1} = \phi(y)$.

In this article we have considered the ring of upper triangular infinite matrices. It is worth noting that this ring is contained in a more general one. More precisely, in $\mathcal{M}_{Cf}(F)$ — the ring of all $\mathbb{N} \times \mathbb{N}$ matrices over F whose all columns contain only a finite number of nonzero entries.

Let us get back to the matrix J_∞ defined in (4). Suppose that it has the Drazin inverse, but in $\mathcal{M}_{Cf}(F)$. Denote it by y. By (1a) we would then have

$$y = \begin{pmatrix} y_{11} & y_{22} & y_{33} & \ldots & y_{nn} & \ldots \\ 0 & y_{22} & y_{33} & & y_{nn} & \\ 0 & y_{22} & y_{33} & & y_{nn} & \\ 0 & y_{22} & y_{33} & & y_{nn} & \\ \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}.$$

The condition $y \in \mathcal{M}_{Cf}(F)$ forces $y_{22} = y_{33} = y_{44} = \ldots = 0$. Thus, $y = y_{11}e_{11}$, i.e., $y \in \mathcal{T}_\infty(F)$. However, we have already proved that J_∞ is not Drazin invertible in this ring. Summing up, J_∞ is not Drazin invertible not only in $\mathcal{T}_\infty(F)$, but also in $\mathcal{M}_{Cf}(F)$.

(3) At the end of the paper we prove the following result.

Proposition 2. *Let F be a field and let $n \in \mathbb{N}$. The matrix $x \in \mathcal{M}_{Cf}(F)$ of the form*

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array}\right) \quad \textit{with} \quad a \in \mathcal{M}_{n\times n}(F), \quad b \in \mathcal{M}_{Cf}(F)$$

is Drazin invertible if and only if b is so.

Proof. We use a result from [25] which says that if $AB = BA$, then $A + B$ is Drazin invertible if and only if $1 + A^D B$ is.

As

$$\left(\begin{array}{c|c} a & 0 \\ \hline 0 & b \end{array}\right) = \left(\begin{array}{c|c} a & 0 \\ \hline 0 & e_\infty \end{array}\right) + \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & b - e_\infty \end{array}\right),$$

our matrix is Drazin invertible if and only if

$$e_\infty + \left(\begin{array}{c|c} a & 0 \\ \hline 0 & e_\infty \end{array}\right)^D \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & b - e_\infty \end{array}\right) =$$

$$= e_\infty + \left(\begin{array}{c|c} a^D & 0 \\ \hline 0 & e_\infty \end{array}\right) \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & b - e_\infty \end{array}\right) = \left(\begin{array}{c|c} e_n & 0 \\ \hline 0 & b \end{array}\right)$$

is.

Therefore, it suffices to prove that

$$\left(\begin{array}{c|c} e_n & 0 \\ \hline 0 & b \end{array}\right) \tag{7}$$

is Drazin invertible if and only if b is.

Suppose that the Drazin inverse of matrix given by (7) exists. Denote it by

$$y = \left(\begin{array}{c|c} y_1 & y_2 \\ \hline y_3 & y_4 \end{array}\right).$$

From (1a) we get

$$by_3 = y_3, \tag{8}$$

whereas by (1c) we have

$$y_2 = 0, \tag{9}$$

$$b^{k+1} y_3 = 0. \tag{10}$$

Now using (8) and (10) we obtain

$$b^k(by_3) = 0 \;\Rightarrow\; b^k y_3 = 0 \;\Rightarrow\; b^{k-1}(by_3) = 0 \;\Rightarrow\; b^{k-1} y_3 = 0 \;\Rightarrow\; \ldots \;\Rightarrow\; y_3 = 0.$$

Hence, from the latter and (9)

$$y = \left(\begin{array}{c|c} y_1 & 0 \\ \hline 0 & y_4 \end{array}\right), \quad \text{i.e.,} \quad y_1 = e_n^D \quad \text{and} \quad y_4 = b^D.$$

Concluding, our matrix is Drazin invertible if and only if b is.

Proposition 2 is proved.

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