К О Р О Т К І П О В І Д О М Л Е Н Н Я

UDC 519.21, 517.5

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ON THE LACUNARY (A, φ)-STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES

ПРО ЛАКУНАРНУ (A, φ)-СТАТИСТИЧНУ ЗБІЖНІСТЬ ПОДВІЙНИХ ПОСЛІДОВНОСТЕЙ

We extend some results known from the literature for ordinary (single) sequences to multiple sequences of real numbers. Further, we introduce a concept of double lacunary strong (A, φ)-convergence with respect to a modulus function. In addition, we also study some relationships between double lacunary strong (A, φ)-convergence with respect to a modulus and double lacunary statistical convergence.

Деякі відомі результати для звичайних (одинарних) послідовностей поширено на багатократні послідовності дійсних чисел. Крім того, введено поняття подвійної лакунарної сильної (A, φ)-збіжності відносно функції модуля, а також вивчено деякі співвідношення між подвійною лакунарною сильною (A, φ)-збіжністю відносно модуля та подвійною лакунарною статистичною збіжністю.

1. Introduction. A notion of a modulus function was introduced by Nakano [10]. We recall that a modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at zero.

A modulus may be bounded or unbounded. For example, $f(x) = x^p$, for $0 < p \leq 1$ is unbounded, but $f(x) = \dfrac{x}{1+x}$ is bounded (see [13]).

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [6] as an extension of the definition of strongly Cesàro summable sequences. Connor [1] further extended this definition by replacing the Cesàro matrix with an abritrary nonnegative regular matrix summability A and established some elementary connections between strong A-summability with respect to a modulus and A-statistical convergence. Recently E. Savas [14] generalized the concept of strong almost convergence by using a modulus function and examined some properties of the corresponding new sequence spaces. Malkowsky and Savas [8] introduced and studied some sequence spaces which arise from the notation of generalized de la Vallée Poussin means and the concept of a modulus function. Furthermore, the four dimensional matrix transformation $(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}$ was studied extensively by Robison [12] and Hamilton [5], respectively. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise.

In [11] the notion of convergence for double sequences was presented by A. Pringsheim.

Before continuing with this paper we present a few definitions and preliminaries.

A lacunary sequence $\theta = (k_r),\ r = 0, 1, 2, \ldots,$ where $k_0 = 0,$ is an increasing sequence of nonnegative integers such that $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $q_r = \dfrac{k_r}{k_{r-1}}$.

The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [3] as follows:

$$N_\theta = \left\{ x = (x_k) \colon \lim_r \frac{1}{h_r} \sum_{k \in I_r} |x_k - L|) = 0 \ \text{ for some } L \right\}.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called **double lacunary** if there exist two increasing sequences of integers such that

$$k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty$$

and

$$l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty.$$

Let us denote $k_{r,s} = k_r l_s,\ h_{r,s} = h_r \bar{h}_s$ and $\theta_{r,s}$ is determine by $I_{r,s} = \{(k,l) \colon k_{r-1} < k \leq$ $\leq k_r$ and $l_{s-1} < l \leq l_s\},\ q_r = \dfrac{k_r}{k_{r-1}},\ \bar{q}_s = \dfrac{l_s}{l_{s-1}},$ and $q_{r,s} = q_r \bar{q}_s$.

For more recent developments on double sequences one can consult the papers (see [17 – 24]), where more references can be found.

By a φ-function we understand a continuous nondecreasing function $\varphi(u)$ defined for $u \geq 0$ and such that $\varphi(0) = 0,\ \varphi(u) > 0$ for $u > 0$ and $\varphi(u) \to \infty$ as $u \to \infty$ (see [25]).

A φ-function φ is called non weaker than a φ-function ψ if there are constants $c,\ b,\ k,\ l > 0$ such that $c\psi(lu) \leq b\varphi(ku)$ (for all large u) and we write $\psi \prec \varphi$. A φ-functions φ and ψ are called equivalent if there are positive constants $b_1,\ b_2,\ c,\ k_1,\ k_2,\ l$ such that $b_1\varphi(k_1 u) \leq c\psi(lu) \leq$ $\leq b_2 \varphi(k_2 u)$ (for all large u) and we write $\varphi \sim \psi$.

In the present paper, we introduce and study an idea of double lacunary strong (A, φ)-convergence with respect to a modulus function. We also investigate the relationship between double lacunary strong (A, φ)-convergence with respect to a modulus and double lacunary (A, φ)-statistical convergence.

2. Main results. Throughout this paper we shall examine our sequence spaces using the following type of transformation:

Definition 2.1. *Let $A = (a_{m,n,k,l})$ denote a four dimensional summability method that maps the real double sequences x into the double sequence Ax where the (mn)th term to Ax is as follows:*

$$(Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}.$$

Such transformation is said to be nonnegative if all $a_{m,n,k,l}$ is nonnegative.

By the convergence of a double sequence we mean the convergence in the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has **Pringsheim limit** L (denoted by $\mathrm{P}-\lim x = L$) provided that given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ whenever $k,\ l > \mathbb{N}$ [11]. We shall describe such an x more briefly as "**P-convergent**".

Let φ and f be given φ-function and modulus function, respectively. Moreover, let $A = (a_{m,n,k,l})$ be a nonnegative four dimensional matrix of real entries and double lacunary sequence θ be given. Then we define the following:

$$N_\theta^2(A,\varphi,f) =$$

$$= \left\{ x = (x_{k,l}) : \mathrm{P}-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}} f\left(\left| \sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l} - L|) \right| \right) = 0 \ \text{ for some } L \right\}$$

and

$$N_\theta^2(A,\varphi,f)_0 = \left\{ x = (x_{k,l}) : \mathrm{P}-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}} f\left(\left| \sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}|) \right| \right) = 0 \right\}.$$

If $x \in N_\theta^2(A,\varphi,f)_0$, the sequence x is said to be double lacunary strong (A,φ)-convergent to zero with respect to a modulus f.

If $f(x) = x$, we write

$$N_\theta^2(A,\varphi) = \left\{ x = (x_{k,l}) : \mathrm{P}-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}} \left| \sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l} - L|) \right| = 0 \ \text{ for some } L \right\}.$$

If we take $A = I$ and $\varphi(x) = x$ respectively, then we have

$$N_\theta^2(f) = \left\{ x = (x_{k,l}) : \mathrm{P}-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} f\left(|x_{k,l} - L|\right) = 0 \ \text{ for some } L \right\}.$$

If we take $A = I$, $\varphi(x) = x$ and $f(x) = x$ respectively, then we obtain

$$N_\theta^2 = \left\{ x = (x_{k,l}) : \mathrm{P}-\lim_{r,s} \frac{1}{h_{r,s}} \sum_{(k,l)\in I_{r,s}} |x_{k,l} - L| = 0 \ \text{ for some } L \right\},$$

which was defined and studied in [18].

In the next theorem we establish inclusion relations between $w^2(A,\varphi,f)_0$ and $N_\theta^2(A,\varphi,f)_0$. We now have the following theorem.

Theorem 2.1. *Let f be any modulus function, φ-function φ, and let $A = (a_{m,n,k,l})$ be a nonnegative four dimensional matrix of real entries and double lacunary sequence θ be given. If*

$$w^2(A,\varphi,f)_0 = \left\{ x = (x_{k,l}) : \mathrm{P}-\lim_{i,j} \frac{1}{ij} \sum_{m,n=1}^{ij} f\left(\left| \sum_{k,l=1}^{\infty} a_{m,n,k,l}\varphi(|x_{kl}|) \right| \right) = 0 \right\},$$

then the following relation is true:

if $\liminf q_r > 1$ *and* $\liminf \bar{q}_s > 1$, *then we have* $w^2(A,\varphi,f)_0 \subseteq N_\theta^2(A,\varphi,f)_0$.

Proof. Let us suppose that $x \in w^2(A,\varphi,f)_0$. There exists $\delta > 0$ such that $q_r > 1+\delta$ for sufficiently large r and $\liminf \bar{q}_s > 1+\delta$ for sufficiently large s we get $h_r/k_r \geq \delta/(1+\delta)$ for sufficiently large r and $\dfrac{\bar{h}_s}{l_s} \geq \dfrac{\delta}{1+\delta}$ for sufficiently large s. Then

$$\frac{1}{k_{r,s}}\sum_{n,m=1}^{k_{r,s}} f\left(\left|\sum_{k,l=1}^{\infty} a_{m,n,k,l}\varphi(|x_{kl}|)\right|\right) \geq$$

$$\geq \frac{1}{k_{r,s}}\sum_{(m,n)\in I_{r,s}} f\left(\left|\sum_{k,l=1}^{\infty} a_{m,n,k,l}\varphi(|x_{kl}|)\right|\right) =$$

$$= \frac{h_{r,s}}{k_{r,s}}\frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}} f\left(\left|\sum_{k,l=1}^{\infty} a_{m,n,k,l}\varphi(|x_{kl}|)\right|\right) \geq$$

$$\geq \left(\frac{\delta}{1+\delta}\right)^2 \frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}} f\left(\left|\sum_{k,l=1}^{\infty} a_{m,n,k,l}\varphi(|x_{kl}|)\right|\right).$$

Hence, $x \in N_\theta^2(A,\varphi,f)_0$.

Theorem 2.1 is proved.

We now have the following theorem.

Theorem 2.2. $N_\theta^2(A,\varphi) \subset N_\theta^2(A,\varphi,f)$.

Proof. Let $x \in N_\theta^2(A,\varphi)$. For a given $\varepsilon > 0$ we choose $0 < \delta < 1$ such that $f(x) < \varepsilon$ for every $x \in [0,\delta]$. We have

$$\frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}} f\left(\left|\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|)\right|\right) = S_{11} + S_{22},$$

where

$$S_{11} = \frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}} f\left(\left|\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|)\right|\right)$$

and this sum is taken over

$$\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|) \leq \delta,$$

and

$$S_{22} = \frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}} f\left(\left|\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|)\right|\right)$$

and this sum is taken over

$$\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|) > \delta.$$

By definition of the modulus f we obtain

$$S_{11} = \frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}} f(\delta) = f(\delta) < \varepsilon$$

and further

$$S_{22} = f(1)\frac{1}{\delta}\frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}} \sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|).$$

Finally, we have $x \in N_\theta^2(A,\varphi,f)$.

Theorem 2.2 is proved.

3. Double A-statistical convergence. The concept of statistical convergence was introduced by Fast [2] in 1951. A real number sequence x is said to be statistically convergent to the number L if for every $\varepsilon > 0$

$$\lim_n \frac{1}{n}|\{k \le n \colon |x_k - L| \ge \varepsilon\}| = 0,$$

where by $k \le n$ we mean that $k = 0,1,2,\ldots,n$ and the vertical bars indicate the number of elements in the enclosed set. In this case we write $st_1 - \lim x = L$ or $x_k \to L(st_1)$.

We first recall the definition of lacunary statistical convergence of a sequence of real numbers which is defined by Friday and Orhan [4] as follows. Let θ be a lacunary sequence; the number sequence x is S_θ-convergent to L provided that for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r}|\{k \in I_r \colon |x_k - L| \ge \varepsilon\}| = 0.$$

In this case we write $S_\theta - \lim x = L$ or $x_k \to L(S_\theta)$.

Let $K \subseteq \mathbb{N}\times\mathbb{N}$ be a two dimensional set of natural numbers and let $K_{m,n}$ be the numbers of (i,j) in K such that $i \le n$ and $j \le m$. Then the lower asymptotic density of K is defined as

$$\mathrm{P}-\liminf_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

In the case when the sequence $\left\{\dfrac{K_{m,n}}{mn}\right\}_{m,n=1}^{\infty}$ has a limit then we say that K has a natural density and is defined

$$\mathrm{P}-\lim_{m,n} \frac{K_{m,n}}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2,j^2) \colon i,j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \mathrm{P}-\lim_{m,n} \frac{K_{m,n}}{mn} \le \mathrm{P}-\lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$$

(i.e., the set K has double natural density zero).

Recently, Mursaleen and Edely [9], defined the statistical analogue for double sequences $x = (x_{k,l})$ as follows: a real double sequences $x = (x_{k,l})$ is said to be P-statistically convergent to L provided that for each $\varepsilon > 0$

$$\mathrm{P} - \lim_{k,l} \frac{1}{kl} \{\text{number of } (m,n) : k \le m \text{ and } l \le n, |x_{k,l} - L| \ge \varepsilon\} = 0.$$

In this case we write $st_2 - \lim_{kl} x_{k,l} = L$ and we denote the set of all statistical convergent double sequences by st_2.

Furthermore, Savas and Patterson [15] studied double lacunary sequence spaces as follows:

Definition 3.1. *Let $\theta_{r,s}$ be a double lacunary sequence; the double sequence x is $S_{\theta_{r,s}}$-convergent to L provided that for every $\varepsilon > 0$,*

$$\mathrm{P} - \lim_{r,s} \frac{1}{h_{r,s}} |\{(k,l) \in I_{r,s} : |x_{k,l} - L| \ge \varepsilon\}| = 0.$$

In this case we write $S_\theta^2 - \lim_{k,l} x_{k,l} = L$ and we denote the set of all statistical convergent double sequences by S_θ^2.

We now define the following: Let θ be a double lacunary sequence, and let the nonnegative matrix $A = (a_{m,n,k,l})$, the sequence $x = (x_{kl})$, the φ- function $\varphi(x)$ and a positive number $\varepsilon > 0$ be given. We write

$$K_\theta^2(A, \varphi, \varepsilon) = \left\{ (n,m) \in I_{r,s} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} \varphi(|x_{k,l} - L|) \ge \varepsilon \right\}.$$

The sequence x is said to be double lacunary (A, φ)-statistically convergent to a number zero if for every $\varepsilon > 0$

$$\mathrm{P} - \lim_{r,s} \frac{1}{h_{r,s}} \mu(K_\theta^2(A, \varphi, \varepsilon)) = 0,$$

where $\mu(K_\theta^2(A, \varphi, \varepsilon))$ denotes the number of elements belonging to $K_\theta^2(A, \varphi, \varepsilon)$. We denote by $S_\theta^2(A, \varphi)$, the set of sequences $x = (x_{k,l})$ which are double lacunary (A, φ)-statistical convergent to zero. We write

$$S_\theta^2(A, \varphi) = \left\{ x = (x_{k,l}) : \mathrm{P} - \lim_{r,s} \frac{1}{h_{r,s}} \mu(K_\theta^2(A, \varphi, \varepsilon)) = 0 \right\}.$$

If we take $A = I$ and $\varphi(x) = x$ respectively, then $S_\theta^2(A, \varphi)$ reduce to S_θ^2 (see [16]).

In the next theorem we prove the following inclusion.

Theorem 3.1. *If $\psi \prec \varphi$, then $S_\theta^2(A, \psi) \subset S_\theta^2(A, \varphi)$.*

Proof. By assumption we have $\psi(|x_{k,l} - L|) \le b\varphi(c|x_{k,l} - L|)$ and

$$\sum_{k,l=0}^{\infty} a_{m,n,k,l} \psi(|x_{k,l} - L|) \le b \sum_{k,l=0}^{\infty} a_{m,n,k,l} \varphi(c|x_{k,l} - L|) \le M \sum_{k,l=0}^{\infty} a_{m,n,k,l} \varphi(|x_{k,l} - L|)$$

for b, $c > 0$, where the constant M is connected with properties of φ. Thus, the condition

$$\sum_{k,l=0}^{\infty} a_{m,n,k,l} \psi(|x_{k,l} - L|) \ge \varepsilon$$

implies the condition

$$\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|) \geq \varepsilon$$

and in consequence we get

$$\mu(K_\theta^2(A,\varphi,\varepsilon)) \subset \mu(K_\theta^2(A,\psi,\varepsilon))$$

and

$$\lim_{r,s}\frac{1}{h_{r,s}}\mu(K_\theta^2(A,\varphi,\varepsilon)) \leq \lim_{r,s}\frac{1}{h_{r,s}}\mu(K_\theta^2(A,\psi,\varepsilon)).$$

Theorem 3.1 is proved.

We establish a relation between the sets $N_\theta^2(A,\varphi,f)$ and $S_\theta^2(A,\varphi)$ as follows:

Theorem 3.2. (i) *If the nonnegative double matrix* A, *the lacunary sequence* θ *and functions* f *and* φ *are given, then*

$$N_\theta^2(A,\varphi,f) \subset S_\theta^2(A,\varphi).$$

(ii) *If the* φ*-function* $\varphi(u)$ *and the nonnegative double matrix* A *are given, and the modulus function* f *is bounded, then*

$$S_\theta^2(A,\varphi) \subset N_\theta^2(A,\varphi,f).$$

(iii) *If the* φ*-function* $\varphi(u)$ *and the nonnegative double matrix* A *are given, and the modulus function* f *is bounded, then*

$$S_\theta^2(A,\varphi) = N_\theta^2(A,\varphi,f).$$

Proof. (i) Let f be a modulus function and $\varepsilon > 0$. We can write the inequalities

$$\frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}} f\left(\left|\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|)\right|\right) \geq$$

$$\geq \frac{1}{h_{r,s}}\sum_{(m,n)\in I_{r,s}^1} f\left(\left|\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|)\right|\right) \geq$$

$$\geq \frac{1}{h_{r,s}} f(\varepsilon)\mu(K_\theta(A,\varphi,\varepsilon)),$$

where

$$I_{r,s}^1 = \left\{(m,n)\in I_{r,s} : \left|\sum_{k,l=0}^{\infty} a_{m,n,k,l}\varphi(|x_{k,l}-L|)\right| \geq \varepsilon\right\}.$$

Finally, if $x \in N_\theta^2(A,\varphi,f)$, then $x \in S_\theta^2(A,\varphi)$.

(ii) Let us suppose that $x \in S_\theta^2(A,\varphi)$. If the modulus function f is a bounded function, then there exists an integer M such that $f(x) < M$ for all $x \geq 0$. Let us write

$$I_{r,s}^2 = \left\{ (m,n) \in I_{r,s} : \left| \sum_{k,l=0}^{\infty} a_{m,n,k,l} \varphi(|x_{k,l} - L|) \right| < \varepsilon \right\}.$$

Thus we write

$$\frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}} f\left(\left| \sum_{k,l=0}^{\infty} a_{m,n,k,l} \varphi(|x_{k,l} - L|) \right| \right) \leq$$

$$\leq \frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}^1} f\left(\left| \sum_{k,l=0}^{\infty} a_{m,n,k,l} \varphi(|x_{k,l} - L|) \right| \right) +$$

$$+ \frac{1}{h_{r,s}} \sum_{(m,n)\in I_{r,s}^2} f\left(\left| \sum_{k,l=0}^{\infty} a_{m,n,k,l} \varphi(|x_{k,l} - L|) \right| \right) \leq$$

$$\leq \frac{1}{h_{r,s}} M\mu(K_\theta(A, \varphi, \varepsilon)) + f(\varepsilon).$$

Taking the limit as $\varepsilon \to 0$, we obtain that $x \in N_\theta^2(A, \varphi)$.

The proof of (iii) follows from (i) and (ii).

Theorem 3.2 is proved.

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Received 27.12.16