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GENERALIZATIONS OF SHERMAN'S INEQUALITY VIA FINK'S IDENTITY AND GREEN'S FUNCTION

УЗАГАЛЬНЕННЯ НЕРІВНОСТІ ШЕРМАНА ЗА ДОПОМОГОЮ ТОТОЖНОСТІ ФІНКА ТА ФУНКЦІЇ ГРІНА

New generalizations of Sherman's inequality for n-convex functions are obtained by using Fink's identity and Green's function. By using inequalities for the Chebyshev functional, we establish some new Ostrowski- and Grüss-type inequalities related to these generalizations.

Отримано нові узагальнення нерівності Шермана для n-опуклих функцій за допомогою тотожності Фінка та функції Гріна. За допомогою нерівностей для функціонала Чебишова встановлено деякі нові нерівності типу Островського та Грюсса, пов'язані з цими узагальненнями.

1. Introduction. S. Sherman [9] obtained generalization of the well known majorization theorem, proved by G. H. Hardy et al. [4], which can be stated as follows: For every convex function $\phi\colon [\alpha,\beta]\to\mathbb{R}$, the inequality

$$\sum_{i=1}^{m} b_i\phi(y_i) \le \sum_{j=1}^{l} a_j\phi(x_j) \tag{1.1}$$

holds, where $\mathbf{x}\in[\alpha,\beta]^l$, $\mathbf{y}\in[\alpha,\beta]^m$, $\mathbf{a}\in[0,\infty)^l$, $\mathbf{b}\in[0,\infty)^m$ and

$$\mathbf{y}=\mathbf{x}\mathbf{A}^T \qquad \text{and} \qquad \mathbf{a}=\mathbf{b}\mathbf{A} \tag{1.2}$$

is satisfied for some row stochastic matrix $\mathbf{A}\in\mathcal{M}_{ml}(\mathbb{R})$, i.e., matrix with

$$a_{ij}\ge 0 \quad \text{for all} \quad i=1,\ldots,m, \quad j=1,\ldots,l,$$

$$\sum_{j=1}^{l} a_{ij}=1 \quad \text{for all} \quad i=1,\ldots,m,$$

while $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$. If ϕ is concave, then the reverse inequality in (1.1) holds. Some related results can be found in [1, 6, 7].

Sherman's result holds for convex functions under assumption of non negativity of entries of vectors $\mathbf{a}$, $\mathbf{b}$ and matrix $\mathbf{A}$. The main purpose of this paper is to present generalizations of Sherman's theorem for convex function of higher order (n-convex functions) which are in a special case convex in the usual sense. Moreover, obtained generalizations hold for real choice, not necessary nonnegative, of vectors $\mathbf{a}$, $\mathbf{b}$ and matrix $\mathbf{A}$. For more details about n-convexity see [8].

The techniques that we use are based on the classical real analysis and an application of Fink's identity and Green's function which we introduce in the sequel.

Theorem 1.1 [3]. *Let* $n \geq 1$ *and* $\phi\colon [\alpha, \beta] \to \mathbb{R}$ *be such that* $\phi^{(n-1)}$ *is absolutely continuous on* $[\alpha, \beta]$. *Then*

$$\phi(x) = \frac{n}{\beta - \alpha} \int\limits_{\alpha}^{\beta} \phi(t)\, dt - \sum_{w=1}^{n-1} \frac{n - w}{w!} \frac{\phi^{(w-1)}(\alpha)(x - \alpha)^w - \phi^{(w-1)}(\beta)(x - \beta)^w}{\beta - \alpha} +$$

$$+ \frac{1}{(n-1)!(\beta - \alpha)} \int\limits_{\alpha}^{\beta} (x - t)^{n-1} k(t, x) \phi^{(n)}(t)\, dt, \tag{1.3}$$

where

$$k(t, x) = \begin{cases} t - \alpha, & \alpha \leq t \leq x \leq \beta, \\ t - \beta, & \alpha \leq x < t \leq \beta. \end{cases} \tag{1.4}$$

The sum in (1.3) *is zero when* $n = 1$.

Green's function $G\colon [\alpha, \beta] \times [\alpha, \beta] \to \mathbb{R}$ is defined by

$$G(t, s) = \begin{cases} \dfrac{(t - \beta)(s - \alpha)}{\beta - \alpha}, & \alpha \leq s \leq t, \\ \dfrac{(s - \beta)(t - \alpha)}{\beta - \alpha}, & t \leq s \leq \beta. \end{cases} \tag{1.5}$$

This function is convex and continuous with respect to both variables s and t. Furthermore, for any function $\phi \in C^2([\alpha, \beta])$, it can be easily shown integration by parts that the next identity is valid

$$\phi(x) = \frac{\beta - x}{\beta - \alpha} \phi(\alpha) + \frac{x - \alpha}{\beta - \alpha} \phi(\beta) + \int\limits_{\alpha}^{\beta} G(x, s) \phi''(s)\, ds. \tag{1.6}$$

For more details see [10].

To establish some new Ostrowski- and Grüss-type inequalities related to obtained generalizations, we use recent results for the Chebyshev functional, which for two Lebesgue integrable functions $f, g\colon [\alpha, \beta] \to \mathbb{R}$ is defined by

$$T(f, g) := \frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} f(t) g(t)\, dt - \frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} f(t)\, dt \frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} g(t)\, dt.$$

Theorem 1.2 ([2], Theorem 1). *Let* $f\colon [\alpha, \beta] \to \mathbb{R}$ *be Lebesgue integrable and* $g\colon [\alpha, \beta] \to \mathbb{R}$ *be absolutely continuous with* $(\cdot - \alpha)(\beta - \cdot)(g')^2 \in L_1[\alpha, \beta]$. *Then*

$$|T(f, g)| \leq \frac{1}{\sqrt{2}} [T(f, f)]^{1/2} \frac{1}{\sqrt{\beta - \alpha}} \left(\int\limits_{\alpha}^{\beta} (x - \alpha)(\beta - x)[g'(x)]^2 dx \right)^{1/2}. \tag{1.7}$$

The constant $\dfrac{1}{\sqrt{2}}$ *in* (1.7) *is the best possible.*

Theorem 1.3 ([2], Theorem 2). *Let $g\colon [\alpha,\beta] \to \mathbb{R}$ be monotonic nondecreasing and $f\colon [\alpha,\beta] \to \mathbb{R}$ be absolutely continuous with $f' \in L_\infty[\alpha,\beta]$. Then*

$$|T(f,g)| \leq \frac{1}{2(\beta-\alpha)}\|f'\|_\infty \int\limits_\alpha^\beta (x-\alpha)(\beta-x)\,dg(x). \tag{1.8}$$

The constant $\frac{1}{2}$ in (1.8) *is the best possible.*

With $\|\cdot\|_p$, $1 \leq p \leq \infty$, we denote the usual Lebesgue norms on space $L_p[\alpha,\beta]$.

Through the paper, we consider simultaneously two aspect, i.e., represent two types of results, in first case results obtained by using only Fink's identity and in another case results obtained by using Fink's identity with Green's function.

2. Main results. We start with two identities which are very useful for us to obtain generalizations.

Theorem 2.1. *Let $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ and $\mathbf{b} \in \mathbb{R}^m$ be such that* (1.2) *holds for some matrix $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ whose entries satisfy the condition $\sum_{j=1}^{l} a_{ij} = 1$ for $i = 1,\ldots,m$. Let $k(\cdot,\cdot)$ and $G(\cdot,\cdot)$ be defined as in* (1.4) *and* (1.5), *respectively. Let $\phi\colon [\alpha,\beta] \to \mathbb{R}$ be such that $\phi^{(n-1)}$ is absolutely continuous on $[\alpha,\beta]$.*

(i) *For $n \geq 1$, the identity*

$$\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) =$$

$$= \frac{1}{\beta-\alpha}\sum_{w=2}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\beta)\left(\sum_{j=1}^{l} a_j(x_j-\beta)^w - \sum_{i=1}^{m} b_i(y_i-\beta)^w\right) -$$

$$-\frac{1}{\beta-\alpha}\sum_{w=2}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\alpha)\left(\sum_{j=1}^{l} a_j(x_j-\alpha)^w - \sum_{i=1}^{m} b_i(y_i-\alpha)^w\right) +$$

$$+\frac{1}{(n-1)!(\beta-\alpha)}\int\limits_\alpha^\beta \left[\sum_{j=1}^{l} a_j(x_j-t)^{n-1}k(t,x_j) - \sum_{i=1}^{m} b_i(y_i-t)^{n-1}k(t,y_i)\right]\phi^{(n)}(t)\,dt \tag{2.1}$$

holds.

(ii) *For $n \geq 3$, the identity*

$$\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) = \sum_{w=0}^{n-3}\frac{n-w-2}{(\beta-\alpha)w!}\times$$

$$\times\int\limits_\alpha^\beta\left(\sum_{j=1}^{l} a_jG(x_j,s) - \sum_{i=1}^{m} b_iG(y_i,s)\right)\left(\phi^{(w+1)}(\beta)(s-\beta)^w - \phi^{(w+1)}(\alpha)(s-\alpha)^w\right)ds+$$

$$+\frac{1}{(n-3)!(\beta-\alpha)}\int_{\alpha}^{\beta}\phi^{(n)}(t)\left(\int_{\alpha}^{\beta}\left(\sum_{j=1}^{l}a_jG(x_j,s)-\sum_{i=1}^{m}b_iG(y_i,s)\right)(s-t)^{n-3}k(t,s)ds\right)dt \tag{2.2}$$

holds.

Proof. (i) By using (1.3) in the difference $\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)$, we get

$$\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)=$$

$$=\frac{1}{\beta-\alpha}\sum_{w=1}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\beta)\left(\sum_{j=1}^{l}a_j(x_j-\beta)^w-\sum_{i=1}^{m}b_i(y_i-\beta)^w\right)-$$

$$-\frac{1}{\beta-\alpha}\sum_{w=1}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\alpha)\left(\sum_{j=1}^{l}a_j(x_j-\alpha)^w-\sum_{i=1}^{m}b_i(y_i-\alpha)^w\right)+$$

$$+\frac{1}{(n-1)!(\beta-\alpha)}\int_{\alpha}^{\beta}\left[\sum_{j=1}^{l}a_j(x_j-t)^{n-1}k(t,x_j)-\sum_{i=1}^{m}b_i(y_i-t)^{n-1}k(t,y_i)\right]\phi^{(n)}(t)\,dt.$$

Since under assumption (1.2) we have

$$\sum_{j=1}^{l}a_j(x_j-\alpha)-\sum_{i=1}^{m}b_i(y_i-\alpha)=\sum_{j=1}^{l}a_j(x_j-\beta)-\sum_{i=1}^{m}b_i(y_i-\beta)=0,$$

the identity (2.1) immediately follows.

(ii) By using (1.2), and (1.6) in the difference $\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)$, we obtain

$$\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)=\int_{\alpha}^{\beta}\left(\sum_{j=1}^{l}a_jG(x_j,s)-\sum_{i=1}^{m}b_iG(y_i,s)\right)\phi''(s)\,ds. \tag{2.3}$$

Applying Fink's identity (1.3) to ϕ'' we get

$$\phi''(s)=\sum_{w=0}^{n-3}\frac{n-w-2}{w!}\frac{\phi^{(w+1)}(\beta)(s-\beta)^w-\phi^{(w+1)}(\alpha)(s-\alpha)^w}{\beta-\alpha}+$$

$$+\frac{1}{(n-3)!(\beta-\alpha)}\int_{\alpha}^{\beta}(s-t)^{n-3}k(t,s)\phi^{(n)}(t)\,dt. \tag{2.4}$$

By an easy calculation, using (2.3) and (2.4) we have

$$\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)=\int_{\alpha}^{\beta}\left[\left(\sum_{j=1}^{l}a_jG(x_j,s)-\sum_{i=1}^{m}b_iG(y_i,s)\right)\times\right.$$

$$\times\left(\sum_{w=0}^{n-3}\frac{n-w-2}{w!}\cdot\frac{\phi^{(w+1)}(\beta)(s-\beta)^w-\phi^{(w+1)}(\alpha)(s-\alpha)^w}{\beta-\alpha}+\right.$$

$$\left.\left.+\frac{1}{(n-3)!(\beta-\alpha)}\int\limits_\alpha^\beta(s-t)^{n-3}k(t,s)\phi^{(n)}(t)\,dt\right)\right]ds.$$

After interchanging the order of summation and integration and applying Fubini's theorem we get (2.2).

The following generalizations of Sherman's theorem for n-convex functions hold.

Theorem 2.2. *Let all the assumptions of Theorem* 2.1 *be satisfied. Additionally, let* ϕ *be* n-*convex on* $[\alpha,\beta]$.

(i) *If* $n\geq 1$ *and*

$$\sum_{j=1}^{l}a_j(x_j-t)^{n-1}k(t,x_j)-\sum_{i=1}^{m}b_i(y_i-t)^{n-1}k(t,y_i)\geq 0,\quad \alpha\leq t\leq\beta, \tag{2.5}$$

then

$$\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)\geq$$

$$\geq\frac{1}{\beta-\alpha}\sum_{w=2}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\beta)\left(\sum_{j=1}^{l}a_j(x_j-\beta)^w-\sum_{i=1}^{m}b_i(y_i-\beta)^w\right)-$$

$$-\frac{1}{\beta-\alpha}\sum_{w=2}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\alpha)\left(\sum_{j=1}^{l}a_j(x_j-\alpha)^w-\sum_{i=1}^{m}b_i(y_i-\alpha)^w\right). \tag{2.6}$$

If the reverse inequality in (2.5) *holds, then the reverse inequality in* (2.6) *holds.*

(ii) *If* $n\geq 3$ *and*

$$\int\limits_\alpha^\beta\left(\sum_{j=1}^{l}a_jG(x_j,s)-\sum_{i=1}^{m}b_iG(y_i,s)\right)(s-t)^{n-3}k(t,s)\,ds\geq 0,\quad \alpha\leq s,t\leq\beta, \tag{2.7}$$

then

$$\sum_{j=1}^{l}a_j\phi(x_j)-\sum_{i=1}^{m}b_i\phi(y_i)\geq$$

$$\geq\sum_{w=0}^{n-3}\frac{n-w-2}{(\beta-\alpha)w!}\int\limits_\alpha^\beta\left(\sum_{j=1}^{l}a_jG(x_j,s)-\sum_{i=1}^{m}b_iG(y_i,s)\right)\times$$

$$\times\left(\phi^{(w+1)}(\beta)(s-\beta)^w-\phi^{(w+1)}(\alpha)(s-\alpha)^w\right)ds. \tag{2.8}$$

If the reverse inequality in (2.7) *holds, then the reverse inequality in* (2.8) *holds.*

Proof. (i) Since ϕ is n-convex on $[\alpha,\beta]$, we may assume without loss of generality that ϕ is n-times differentiable and $\phi^{(n)}(t)\geq 0,\ t\in[\alpha,\beta]$ (see [8, p. 16]).

By using this fact and the assumption (2.5), applying Theorem 2.1, we obtain (2.6).

(ii) Analogous to the part (i).

The following generalizations under Sherman's assumption of non negativity are also valid.

Theorem 2.3. *Let all the assumptions of Theorem* 2.1 *be satisfied. Additionally, let* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{A}$ *be nonnegative and* ϕ *be* n*-convex on* $[\alpha,\beta]$.

(i) *If* n *is even and* $n\geq 2$, *then* (2.6) *holds. Moreover, if the function*

$$\bar{F}(\cdot)=\frac{1}{\beta-\alpha}\sum_{w=2}^{n-1}\frac{n-w}{w!}\left[\phi^{(w-1)}(\beta)(\cdot-\beta)^{w}-\phi^{(w-1)}(\alpha)(\cdot-\alpha)^{w}\right] \tag{2.9}$$

is convex on $[\alpha,\beta]$, *then* (1.1) *holds.*

(ii) *If* n *is even and* $n\geq 4$, *then* (2.8) *holds. Moreover, if the function*

$$\tilde{F}(\cdot)=\sum_{w=0}^{n-3}\frac{n-w-2}{(\beta-\alpha)w!}\int_{\alpha}^{\beta}G(\cdot,s)\Big[\phi^{(w+1)}(\beta)(s-\beta)^{w}-\phi^{(w+1)}(\alpha)(s-\alpha)^{w}\Big]ds, \tag{2.10}$$

where $s\in[\alpha,\beta]$, *is convex on* $[\alpha,\beta]$, *then* (1.1) *holds.*

Proof. (i) Consider the function $s\colon[\alpha,\beta]\to\mathbb{R}$ defined by

$$s(x)=(x-t)^{n-1}k(t,x)=\begin{cases}(x-t)^{n-1}(t-\alpha), & \alpha\leq t\leq x\leq\beta,\\ (x-t)^{n-1}(t-\beta), & \alpha\leq x<t\leq\beta.\end{cases}$$

Since

$$s''(x)=\begin{cases}(n-1)(n-2)(x-t)^{n-3}(t-\alpha), & \alpha\leq t\leq x\leq\beta,\\ (n-1)(n-2)(x-t)^{n-3}(t-\beta), & \alpha\leq x<t\leq\beta,\end{cases}$$

it follows that for even $n\geq 2$, s is convex on $[\alpha,\beta]$. Then by Sherman's theorem, the inequality (2.5) holds. Therefore, by Theorem 2.2, the inequality (2.6) holds. Changing the order of summation, the right-hand side of (2.6) can be written in the form

$$\sum_{j=1}^{l}a_j\bar{F}(x_j)-\sum_{i=1}^{m}b_i\bar{F}(y_i),$$

where $\bar{F}$ is defined as in (2.9). If $\bar{F}$ is convex, then by Sherman's theorem we have

$$\sum_{j=1}^{l}a_j\bar{F}(x_j)-\sum_{i=1}^{m}b_i\bar{F}(y_i)\geq 0,$$

i.e., the right-hand side of (2.6) is nonnegative, so the inequality (1.1) immediately follows.

(ii) Further, the function $G(\cdot,s)$, $s\in[\alpha,\beta]$, is convex on $[\alpha,\beta]$ and by Sherman's theorem we obtain

$$\sum_{j=1}^{l}a_jG(x_j,s)-\sum_{i=1}^{m}b_iG(y_i,s)\geq 0.$$

It is easy to see that for even $n > 3$, the inequality

$$\int\limits_{\alpha}^{\beta} \left(\sum_{j=1}^{l} a_j G(x_j, s) - \sum_{i=1}^{m} b_i G(y_i, s) \right) (s - t)^{n-3} k(t, s)\, ds \geq 0, \quad \alpha \leq s \leq t, \tag{2.11}$$

holds, while for every $n \geq 3$, we get

$$\int\limits_{\alpha}^{\beta} \left(\sum_{j=1}^{l} a_j G(x_j, s) - \sum_{i=1}^{m} b_i G(y_i, s) \right) (s - t)^{n-3} k(t, s)\, ds \geq 0, \quad t \leq s \leq \beta.$$

Now applying Theorem 2.3, we have the inequality (2.8).

The rest of proof is analog to the part (ii), whereby instead of $\bar{F}$ we consider the function $\tilde{F}$ defined by (2.10).

***Remark* 2.1.** Let all the assumptions of the previous theorem be satisfied.

(i) For even $n \geq 2$, the inequality (2.6) holds. Further, the function $s \mapsto (s - \alpha)^w$ is convex on $[\alpha, \beta]$ for every w, while $s \mapsto (s - \beta)^w$ is convex on $[\alpha, \beta]$ for even w and concave for odd w.

If for even w, $\phi^{(w-1)}(\alpha) \leq 0$ and $\phi^{(w-1)}(\beta) \geq 0$ and for odd w, $\phi^{(w-1)}(\alpha) \leq 0$ and $\phi^{(w-1)}(\beta) \leq 0$, then the right-hand side of (2.6) is nonnegative. Therefore, (1.1) immediately follows.

(ii) For even $n \geq 4$, the inequality (2.8) holds. Further, when $\alpha \leq s \leq \beta$, we have $(s - \alpha)^w \geq 0$ for every w while $(s - \beta)^w \geq 0$ for even w and $(s - \beta)^w \leq 0$ for odd w.

If for even w, $\phi^{(w+1)}(\alpha) \leq 0$ and $\phi^{(w+1)}(\beta) \geq 0$ and for odd w, $\phi^{(w+1)}(\alpha) \leq 0$ and $\phi^{(w+1)}(\beta) \leq 0$, then the right-hand side of (2.8) is nonnegative and the inequality (1.1) immediately follows.

3. The Ostrowski- and Grüss-type inequalities. To avoid many notations, we define the functions $\mathcal{B}, \tilde{\mathcal{B}} \colon [\alpha, \beta] \to \mathbb{R}$ by

$$\mathcal{B}(t) = \sum_{j=1}^{l} a_j (x_j - t)^{n-1} k(t, x_j) - \sum_{i=1}^{m} b_i (y_i - t)^{n-1} k(t, y_i), \quad n \geq 1, \tag{3.1}$$

$$\tilde{\mathcal{B}}(t) = \int\limits_{\alpha}^{\beta} \left(\sum_{j=1}^{l} a_j G(x_j, s) - \sum_{i=1}^{m} b_i G(y_i, s) \right) (s - t)^{n-3} k(t, s) ds, \qquad \alpha \leq s \leq \beta, \quad n \geq 3,$$

where $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ and $\mathbf{b} \in \mathbb{R}^m$ are such that (1.2) holds for some matrix $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ whose entries satisfy the condition $\sum_{j=1}^{l} a_{ij} = 1$ for $i = 1, \ldots, m$, and $G(\cdot, \cdot)$ and $k(\cdot, \cdot)$ are defined by (1.5) and (1.4), respectively.

We also consider the Chebyshev functionals defined by

$$T(\mathcal{B}, \mathcal{B}) = \frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} \mathcal{B}^2(t)\, dt - \left(\frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} \mathcal{B}(t)\, dt \right)^2,$$

$$T(\tilde{\mathcal{B}}, \tilde{\mathcal{B}}) = \frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} \tilde{\mathcal{B}}^2(t)\, dt - \left(\frac{1}{\beta - \alpha} \int\limits_{\alpha}^{\beta} \tilde{\mathcal{B}}(t)\, dt \right)^2.$$

Theorem 3.1. *Let* $\mathbf{x} \in [\alpha, \beta]^l$, $\mathbf{y} \in [\alpha, \beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ *and* $\mathbf{b} \in \mathbb{R}^m$ *be such that* (1.2) *holds for some matrix* $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ *whose entries satisfy the condition* $\sum_{j=1}^{l} a_{ij} = 1$ *for* $i = 1, \ldots, m$. *Let* $\mathcal{B}$, $\tilde{\mathcal{B}}$ *be defined as in* (3.1). *Let* $\phi \colon [\alpha, \beta] \to \mathbb{R}$ *be such that* $\phi^{(n)}$ *is absolutely continuous with* $(\cdot - \alpha)(\beta - \cdot)(\phi^{(n+1)})^2 \in L_1[\alpha, \beta]$.

(i) *For* $n \geq 1$, *we have*

$$\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) =$$

$$= \frac{1}{\beta - \alpha} \sum_{w=2}^{n-1} \frac{n-w}{w!} \phi^{(w-1)}(\beta) \left(\sum_{j=1}^{l} a_j (x_j - \beta)^w - \sum_{i=1}^{m} b_i (y_i - \beta)^w \right) -$$

$$- \frac{1}{\beta - \alpha} \sum_{w=2}^{n-1} \frac{n-w}{w!} \phi^{(w-1)}(\alpha) \left(\sum_{j=1}^{l} a_j (x_j - \alpha)^w - \sum_{i=1}^{m} b_i (y_i - \alpha)^w \right) +$$

$$+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(\beta - \alpha)^2 (n-1)!} \int_{\alpha}^{\beta} \mathcal{B}(t)\, dt + R_n(\phi; \alpha, \beta), \tag{3.2}$$

where the remainder satisfies

$$\left| R_n(\phi; \alpha, \beta) \right| \leq \frac{1}{\sqrt{2(\beta - \alpha)}(n-1)!} \left[T(\mathcal{B}, \mathcal{B}) \right]^{1/2} \left(\int_{\alpha}^{\beta} (t - \alpha)(\beta - t)[\phi^{(n+1)}(t)]^2\, dt \right)^{1/2}. \tag{3.3}$$

(ii) *For* $n \geq 3$, *we get*

$$\sum_{j=1}^{l} a_j \phi(x_j) - \sum_{i=1}^{m} b_i \phi(y_i) =$$

$$= \sum_{w=0}^{n-3} \frac{n-w-2}{(\beta - \alpha) w!} \int_{\alpha}^{\beta} \mathcal{G}(s) \left(\phi^{(w+1)}(\beta)(s - \beta)^w - \phi^{(w+1)}(\alpha)(s - \alpha)^w \right) ds +$$

$$+ \frac{\phi^{(n-1)}(\beta) - \phi^{(n-1)}(\alpha)}{(n-3)!(\beta - \alpha)^2} \int_{\alpha}^{\beta} \tilde{\mathcal{B}}(t)\, dt + \tilde{R}_n(\phi; \alpha, \beta), \tag{3.4}$$

where $\mathcal{G}(s) = \sum_{j=1}^{l} a_j G(x_j, s) - \sum_{i=1}^{m} b_i G(y_i, s)$ *for* $G(\cdot, \cdot)$ *defined by* (1.5), *and the remainder satisfies*

$$\left| \tilde{R}_n(\phi; \alpha, \beta) \right| \leq \frac{1}{\sqrt{2(\beta - \alpha)}(n-3)!} \left[T(\tilde{\mathcal{B}}, \tilde{\mathcal{B}}) \right]^{1/2} \left(\int_{\alpha}^{\beta} (t - \alpha)(\beta - t)[\phi^{(n+1)}(t)]^2 dt \right)^{1/2}. \tag{3.5}$$

Proof. Our proof proceeds similarly to the proof of Theorem 9 in [1].

By using Theorem 1.3, we obtain the Grüss-type inequality.

Theorem 3.2. *Let* $\mathbf{x} \in [\alpha,\beta]^l$, $\mathbf{y} \in [\alpha,\beta]^m$, $\mathbf{a} \in \mathbb{R}^l$ *and* $\mathbf{b} \in \mathbb{R}^m$ *be such that* (1.2) *holds for some matrix* $\mathbf{A} \in \mathcal{M}_{ml}(\mathbb{R})$ *whose entries satisfy the condition* $\sum_{j=1}^{l} a_{ij} = 1$ *for* $i = 1,\ldots,m$. *Let* $\mathcal{B}$, $\tilde{\mathcal{B}}$ *be defined as in* (3.1). *Let* $\phi\colon [\alpha,\beta] \to \mathbb{R}$ *be such that* $\phi^{(n)}$ *is absolutely continuous and* $\phi^{(n+1)} \geq 0$ *on* $[\alpha,\beta]$.

(i) *For* $n \geq 1$, *the representation* (3.2) *holds and the remainder* $R_n(\phi;\alpha,\beta)$ *satisfies*

$$|R_n(\phi;\alpha,\beta)| \leq \frac{1}{(n-1)!}\left\|\mathcal{B}'\right\|_\infty \left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{3.6}$$

(ii) *For* $n \geq 3$, *the representation* (3.4) *holds and the remainder* $\tilde{R}_n(\phi;\alpha,\beta)$ *satisfies*

$$\left|\tilde{R}_n(\phi;\alpha,\beta)\right| \leq \frac{1}{(n-3)!}\left\|\tilde{\mathcal{B}}'\right\|_\infty \left[\frac{\phi^{(n-1)}(\beta)+\phi^{(n-1)}(\alpha)}{2} - \frac{\phi^{(n-2)}(\beta)-\phi^{(n-2)}(\alpha)}{\beta-\alpha}\right]. \tag{3.7}$$

Proof. Our proof proceeds similarly to the proof of Theorem 10 in [1].

We present the Ostrowski-type inequality related to the identity (2.1).

Theorem 3.3. *Suppose that all assumptions of Theorem* 2.1 *hold. Furthermore, let* $\mathcal{B}$, $\tilde{\mathcal{B}}$ *be defined as in* (3.1). *Let* (p,q) *be a pair of conjugate exponents, that is* $1 \leq p,q \leq \infty$, $1/p+1/q=1$ *and* $\phi^{(n)} \in L_p[\alpha,\beta]$.

(i) *For* $n \geq 1$, *we have*

$$\left|\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) - \right.$$

$$-\frac{1}{\beta-\alpha}\sum_{w=2}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\beta)\left(\sum_{j=1}^{l} a_j(x_j-\beta)^w - \sum_{i=1}^{m} b_i(y_i-\beta)^w\right) +$$

$$\left. +\frac{1}{\beta-\alpha}\sum_{w=2}^{n-1}\frac{n-w}{w!}\phi^{(w-1)}(\alpha)\left(\sum_{j=1}^{l} a_j(x_j-\alpha)^w - \sum_{i=1}^{m} b_i(y_i-\alpha)^w\right)\right| \leq$$

$$\leq \frac{1}{(n-1)!(\beta-\alpha)}\left(\int_\alpha^\beta |\mathcal{B}(t)|^q\,dt\right)^{1/q}\left\|\phi^{(n)}\right\|_p. \tag{3.8}$$

The constant on the right-hand side of (3.8) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

(ii) *For* $n \geq 3$, *we get*

$$\left|\sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) - \right.$$

$$\left. -\sum_{w=0}^{n-3} \frac{n-w-2}{(\beta-\alpha)w!} \int_{\alpha}^{\beta} \mathcal{G}(s) \left(\phi^{(w+1)}(\beta)(s-\beta)^w - \phi^{(w+1)}(\alpha)(s-\alpha)^w\right) ds \right| \leq$$

$$\leq \frac{1}{(n-3)!(\beta-\alpha)} \left(\int_{\alpha}^{\beta} \left|\tilde{\mathcal{B}}(t)\right|^q dt \right)^{1/q} \left\|\phi^{(n)}\right\|_p, \tag{3.9}$$

where $\mathcal{G}(s) = \sum_{j=1}^{l} a_j G(x_j, s) - \sum_{i=1}^{m} b_i G(y_i, s)$ *for* $G(\cdot,\cdot)$ *defined by* (1.5). *The constant on the right-hand side of* (3.9) *is sharp for* $1 < p \leq \infty$ *and the best possible for* $p = 1$.

Proof. Our proof proceeds similarly to the proof of Theorem 11 in [1].

4. Some applications. Under the assumptions of Theorem 2.2, using the inequality (2.6) and (2.8), we can define two linear functionals

$$A_1(\phi) = \sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) -$$

$$-\frac{1}{\beta-\alpha} \sum_{w=2}^{n-1} \frac{n-w}{w!} \phi^{(w-1)}(\beta) \left(\sum_{j=1}^{l} a_j(x_j-\beta)^w - \sum_{i=1}^{m} b_i(y_i-\beta)^w \right) +$$

$$+\frac{1}{\beta-\alpha} \sum_{w=2}^{n-1} \frac{n-w}{w!} \phi^{(w-1)}(\alpha) \left(\sum_{j=1}^{l} a_j(x_j-\alpha)^w - \sum_{i=1}^{m} b_i(y_i-\alpha)^w \right)$$

and

$$A_2(\phi) = \sum_{j=1}^{l} a_j\phi(x_j) - \sum_{i=1}^{m} b_i\phi(y_i) -$$

$$-\sum_{w=0}^{n-3} \frac{n-w-2}{(\beta-\alpha)w!} \int_{\alpha}^{\beta} \mathcal{G}(s) \left(\phi^{(w+1)}(\beta)(s-\beta)^w - \phi^{(w+1)}(\alpha)(s-\alpha)^w\right) ds,$$

where $\mathcal{G}(s) = \sum_{j=1}^{l} a_j G(x_j, s) - \sum_{i=1}^{m} b_i G(y_i, s)$ for $G(\cdot,\cdot)$ defined by (1.5).

For any n-convex function $\phi\colon [\alpha,\beta] \to \mathbb{R}$ we have

$$A_p(\phi) \geq 0, \quad p = 1, 2.$$

By using the linearity and positivity of defined functionals, we can apply Exponentially convex method, established in [5], in order to interpret our results in the form of exponentially or in the special case logarithmically convex functions. As outcome we can get some new classes of two-parameter Cauchy-type means. For such constructions we can use the same ideas as in paper [1].

References

1. *Agarwal R. P., Ivelić Bradanović S., Pečarić J.* Generalizations of Sherman's inequality by Lidstone's interpolating polynomial // J. Inequal Appl. – 2016. – **6**.
2. *Cerone P., Dragomir S. S.* Some new Ostrowski-type bounds for the Čebyšev functional and applications // Math J. Inequal. – 2014. – **8**, № 1. – P. 159 – 170.
3. *Fink A. M.* Bounds of the deviation of a function from its averages // Czechoslovak Math. J. – 1992. – **42**, № 117. – P. 289 – 310.
4. *Hardy G. H., Littlewood J. E., Pólya G.* Inequalities. – 2nd ed. – Cambridge: Cambridge Univ. Press, 1952.
5. *Jakšetić J., Pečarić J.* Exponential convexity method // J. Convex Anal. – 2013. – **20**, № 1. – P. 181 – 197.
6. *Niezgoda M.* Remarks on Sherman like inequalities for (α, β)-convex functions // Math. Ineqal. and Appl. – 2014. – **17**, № 4. – P. 1579 – 1590.
7. *Niezgoda M.* Vector joint majorization and generalization of Csiszár – Körner's inequality for f-divergence // Discrete and Appl. Math. – 2016. – **198**. – P. 195 – 205.
8. *Pečarić J., Proschan F., Tong Y. L.* Convex functions, partial orderings and statistical applications. – New York: Acad. Press, 1992.
9. *Sherman S.* On a theorem of Hardy, Littlewood, Pólya and Blackwell // Proc. Nat. Acad. Sci. USA. – 1957. – **37**, № 1. – P. 826 – 831.
10. *Widder D. V.* Completely convex function and Lidstone series // Trans. Amer. Math. Soc. – 1942. – **51**. – P. 387 – 398.

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