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RUIN PROBABILITIES FOR RISK MODELS WITH CONSTANT INTEREST

ІМОВІРНІСТЬ КРАХУ В МОДЕЛЯХ ІЗ РИЗИКОМ І СТАЛИМ ПРИБУТКОМ

We consider continuous-time risk models with m-dependent claim sizes and constant interest rate. Under some special conditions, we obtain the upper bound for the infinite-time ruin probability. Our approach is based on the martingale methods.

Розглянуто моделі ризику з неперервним часом, m-залежними розмірами вимог і сталим прибутком. За деяких спеціальних умов отримано верхню межу для ймовірності краху через нескінченний проміжок часу. Запропонований підхід базується на мартингальних методах.

1. Introduction. Consider the classical risk model with continuous time. The inter-claim times $\{t_i, i = 1, 2, \ldots\}$ are a sequence of independent identically distributed (i.i.d.) nonnegative random variables and the claim sizes $\{X_n\}$ are a sequence of i.i.d. nonnegative random variables independent of $\{t_n\}$. Let $T_n = \sum_{k=1}^{n} t_k$ be the time of the nth claim, $T_0 = 0$, then $N_t = \sup\{n | T_n \leq t\}$ is the number of claims up to time t and the aggregate claim amount up to time t is $S(t) = \sum_{n=1}^{N(t)} X_n$. If the insurer's initial surplus is $u \geq 0$, the risk model is given by

$$U(t) = u + ct - S(t), \tag{1}$$

where $c > 0$ is the rate of premium income and $U(0) = u$. The ruin probability up to a finite time T is defined by

$$\Psi(u, T) = \mathbb{P}\{U(t) < 0 \quad \text{for some} \quad t \leq T\},$$

and the ultimate ruin probability by

$$\Psi(u) := \Psi(u, \infty) = \lim_{T \to +\infty} \Psi(u, T).$$

According to Lundberg's inequality we obtain the following evaluation of ruin probability:

$$\Psi(u) = \Psi(u, \infty) \leq e^{-Ru},$$

where R is the smallest positive root of the equation

$$\mathbb{E}(e^{-R(X_1 - ct_1)}) = 1.$$

We refer to [7, 9] for reviewing results and developments of Lundberg's inequality for ruin probabilities.

In many studies the claim sizes $\{X_n, n \geq 1\}$ are assumed to be a sequence of i.i.d. nonnegative random variables (see, e.g., [1–3, 5, 8]). Some studies for the model with dependent claims are presented in, e.g., [6, 10], where the authors considered autoregressive model and used martingale method to obtain an estimate of the ruin probabilities. In this paper we assume that the claim sizes are m-dependent random variables and derive an analog of Lundberg's inequality.

2. Preliminaries and main results. To begin with, we give the concept of m-dependent random variables and some examples.

Definition 1. *Let m be an integer. The sequence of random variables $\{X_n, n \geq 1\}$ is called m-dependent, if σ-algebras $\Im_n = \sigma\{X_1, X_2, \ldots, X_n\}$ and $\Im^{n+k} = \sigma\{X_{n+k}, X_{n+k+1}, \ldots\}$ are independent for all $k \geq m+1$ and $n \geq 1$.*

***Example* 1.** Sequence of independent random variables $\{X_n, n \geq 1\}$ is called 0-dependent.

***Example* 2.** Let $\{Z_n, n \geq 1\}$ be a sequence of independent random variables. For each $k \geq 1$, let $\varphi_k : \mathbb{R}^m \to \mathbb{R}$ be measurable functions and denote

$$X_k = \varphi_k(Z_k, Z_{k+1}, \ldots, Z_{k+m-1}).$$

Then $\{X_n, n \geq 1\}$ is a sequence of m-dependent random variables. Moreover, if $\varphi_k = \varphi$ for all k and (Z_k)'s are identical distributed random variables, then (X_n) is a sequence of identically distributed random variables.

***Example* 3.** If $\{X_n, n \geq 1\}$ is a sequence of 1-dependent random variables, then $\{X_1, X_3, X_5, \ldots\}$ and $\{X_2, X_4, X_6, \ldots\}$ are dependent sequences of independent random variables.

Consider model (1), assume that the sequence of claim sizes $\{X_n, n \geq 1\}$ are m-dependent and the model includes interest rate. Furthermore, we assume that the interest $\delta > 0$ of the surplus is a constant continuously compounded. Let $U_\delta(t)$ denote the surplus of the insurance company up to time t. Then

$$U_\delta(t) := ue^{\delta t} + c\int_0^t e^{\delta t}dt - \int_0^t e^{\delta(t-v)}dS(v) = ue^{\delta t} + c(e^{\delta t} - 1)/\delta - \int_0^t e^{\delta(t-v)}dS(v),$$

where $U_\delta(0) = u$. Denote by $\tau_\delta := \inf\{t : U_\delta(t) < 0\}$ the first time the surplus process is negative. Then ruin probability is defined as follows:

$$\Psi_\delta(u) = \mathbb{P}\{\tau_\delta < \infty\} = \mathbb{P}\left\{\bigcup_{t\geq 0}(U_\delta(t) < 0)\right\}.$$

However, since the ruin can occur only at the time of a claim, we get

$$\Psi_\delta(u) = \mathbb{P}\left\{\bigcup_{n=1}^{\infty}(U_\delta(T_n) < 0)\right\} = \mathbb{P}\left\{\bigcup_{n=1}^{\infty}(V_\delta(T_n) < 0)\right\},$$

where $V_\delta(T_n) = U_\delta(T_n)e^{-\delta T_n}$ is the present value at time 0 of $U_\delta(T_n)$. We have (see [2], (1.7))

$$V_\delta(T_{n+1}) = V_\delta(T_n) + c\left(e^{-\delta T_n} - e^{-\delta T_{n+1}}\right)/\delta - X_{n+1}e^{-\delta T_{n+1}} =$$

$$= V_\delta(T_n) + e^{-\delta T_n}\left[c(1 - e^{-\delta t_{n+1}})/\delta - X_{n+1}e^{-\delta t_{n+1}}\right],$$

where $V_\delta(0) = u$. So we obtain

$$\Psi_\delta(u, n) := \mathbb{P}\left\{\bigcup_{k=1}^{n}(U_\delta(T_k) < 0)\right\} = \mathbb{P}\left\{\bigcup_{k=1}^{n}(V_\delta(T_k) < 0)\right\} = \mathbb{P}\left\{\bigcup_{k=1}^{n}(S_k > u)\right\},$$

and, hence,

$$\Psi_\delta(u) = \lim_{n\to\infty} \Psi_\delta(u,n),$$

where

$$S_n = \sum_{k=1}^n X_k e^{-\delta T_k} - c(1 - e^{-\delta \sum_{k=1}^n t_k})/\delta = \sum_{k=1}^n X_k e^{-\delta T_k} - c(1 - e^{-\delta T_n})/\delta.$$

Lemma 1. *Set* $\tau = T_{m+1}$ *and*

$$\phi(R) = \mathbb{E}\left[\exp\left(R\left(X_1 e^{-\delta\tau} - \frac{c(1-e^{-\delta\tau})}{(m+1)\delta}\right)\right)\right].$$

Assume that:

(A_1) *there exists some* $R_0 > 0$ *such that* $\phi(R_0) < \infty$;

(A_2) $\mathbb{E}(X_1) < \dfrac{c}{(m+1)\delta}\dfrac{1-\mathbb{E}(e^{-\delta\tau})}{\mathbb{E}(e^{-\delta\tau})}$;

(A_3) $\mathbb{P}\left(X_1 e^{-\delta\tau} - \dfrac{c(1-e^{-\delta\tau})}{(m+1)\delta} > 0\right) > 0$.

Then there exists a unique positive number R *such that* $\phi(R) = 1$.

Proof. Let $R_1 = \sup\{R > 0 : \phi(R) < \infty\}$. Note that $R_1 \geq R_0 > 0$. It follows from Hölder's inequality that $\phi(R) < \infty$ for any $0 < R < R_1$ and $\phi(R) = \infty$ for any $R > R_1$. For any $R < R_1$, we get

$$\phi'(R) = \mathbb{E}\left[\left(X_1 e^{-\delta\tau} - \frac{c(1-e^{-\delta\tau})}{(m+1)\delta}\right)\exp\left(R\left(X_1 e^{-\delta\tau} - \frac{c(1-e^{-\delta\tau})}{(m+1)\delta}\right)\right)\right],$$

$$\phi''(R) = \mathbb{E}\left[\left(X_1 e^{-\delta\tau} - \frac{c(1-e^{-\delta\tau})}{(m+1)\delta}\right)^2\exp\left(R\left(X_1 e^{-\delta\tau} - \frac{c(1-e^{-\delta\tau})}{(m+1)\delta}\right)\right)\right].$$

It is seen that $\phi''(R) > 0$ for all $R \in (0, R_1)$. Hence, ϕ is a strictly convex function on $(0, R_1)$. It follows from condition (A_3) that $\lim_{R\to\infty}\phi(R) = \infty$. Thanks to the definition of R_1, we also have $\lim_{R\to R_1}\phi(R) = \infty$. It follows from condition (A_2) that $\mathbb{E}\left(X_1 e^{-\delta\tau} - \dfrac{c(1-e^{-\delta\tau})}{(m+1)\delta}\right) < 0$ which implies that $\phi'(0) < 0$. Since $\phi(0) = 1$, the equation $\phi(R) = 1$ has a unique positive solution.

Throughout the rest of this section, we denote by R the unique positive solution to equation $\phi(R) = 1$. We write

$$S_n = \sum_{k=1}^n X_k e^{-\delta T_k} + \frac{c}{\delta}\left(e^{-\delta T_n} - 1\right).$$

For each $j = 1, \ldots, m+1$, we set

$$X_k^{(j)} = X_{j+k(m+1)}, \quad T_k^{(j)} = T_{j+k(m+1)}, \quad k = 0, 1, \ldots,$$

and

$$S_p^{(j)} = \sum_{k=0}^{p} X_k^{(j)} e^{-\delta T_k^{(j)}} + \frac{c(e^{-\delta T_p^{(j)}} - 1)}{(m+1)\delta}, \quad p = 0, 1, \ldots .$$

Note that

$$X_p^{(j)} \overset{\text{df}}{=} X_1 \quad \text{and} \quad T_{k+1}^{(j)} - T_k^{(j)} \overset{\text{df}}{=} \tau. \tag{2}$$

Let

$$Z_p^{(j)} = e^{RS_p^{(j)}}, \qquad 1 \le j \le m+1, \quad p \ge 0,$$

and

$$\mathcal{F}_p^{(j)} = \sigma(X_k^{(j)}, T_k^{(j)} : 0 \le k \le p).$$

Note that

$$T_{p+1}^{(j)} - T_p^{(j)} \text{ and } X_{p+1}^{(j)} \text{ are independent of } \mathcal{F}_p^{(j)}. \tag{3}$$

Lemma 2. *Assume that conditions* (A_1)–(A_3) *hold. For each* $j = 1, \ldots, m+1,$ *the sequence* $(Z_p^{(j)}, \mathcal{F}_p^{(j)})_{p \ge 0}$ *is a supermartingale.*

Proof. Since

$$Z_{p+1}^{(j)} = Z_p^{(j)} \exp\left(Re^{-\delta T_p^{(j)}} \left[X_{p+1}^{(j)} e^{-\delta(T_{p+1}^{(j)} - T_p^{(j)})} - \frac{c\left(1 - e^{-\delta(T_{p+1}^{(j)} - T_p^{(j)})}\right)}{(m+1)\delta} \right] \right),$$

we have

$$\mathbb{E}(Z_{p+1}^{(j)} | \mathcal{F}_p^{(j)}) = Z_p^{(j)} \mathbb{E}\left(\exp\left(Re^{-\delta T_p^{(j)}} \left[X_{p+1}^{(j)} e^{-\delta(T_{p+1}^{(j)} - T_p^{(j)})} - \frac{c\left(1 - e^{-\delta(T_{p+1}^{(j)} - T_p^{(j)})}\right)}{(m+1)\delta} \right] \right) \middle| \mathcal{F}_p^{(j)} \right).$$

It follows from (2) and (3) that

$$\mathbb{E}(Z_{p+1}^{(j)} | \mathcal{F}_p^{(j)}) = Z_p^{(j)}\, \mathbb{E}\left(\exp\left(Re^{-\delta t} \left[X_1 e^{-\delta\tau} - \frac{c(1 - e^{-\delta\tau})}{(m+1)\delta} \right] \right) \right) \Bigg|_{t = T_p^{(j)}}.$$

By applying Hölder's inequality, we get

$$\mathbb{E}(Z_{p+1}^{(j)} | \mathcal{F}_p^{(j)}) \le Z_p^{(j)} \left\{ \mathbb{E}\left(\exp\left(R\left[X_1 e^{-\delta\tau} - \frac{c(1 - e^{-\delta\tau})}{(m+1)\delta} \right] \right) \right) \right\}^{e^{-\delta t}} \Bigg|_{t = T_p^{(j)}}.$$

It follows from Lemma 1 that

$$\mathbb{E}(Z_{p+1}^{(j)} | \mathcal{F}_p^{(j)}) \le Z_p^{(j)},$$

which implies the desired result.

Theorem 1. *Suppose that conditions* (A_1)–(A_3) *hold. Then*

$$\Psi_\delta(u) \le e^{-\frac{Ru}{m+1}} \sum_{j=1}^{m+1} \mathbb{E}\left[\exp\left(R\left(X_1 e^{-\delta T_j} - \frac{c(1 - e^{-\delta T_j})}{(m+1)\delta} \right) \right) \right].$$

Proof. We have

$$S_n \le \sum_{j=1}^{m+1} S^{(j)}_{[(n-j)/(m+1)]},$$

where $[(n-j)/(m+1)]$ is the integer part of $(n-j)/(m+1)$. This implies

$$\Psi_\delta(u) = \mathbb{P}\left(\bigcup_{n=1}^{\infty}(S_n > u)\right) \le \mathbb{P}\left(\bigcup_{j=1}^{m+1}\bigcup_{p=0}^{\infty}\left(S_p^{(j)} > \frac{u}{m+1}\right)\right) \le$$

$$\le \sum_{j=1}^{m+1}\mathbb{P}\left(\bigcup_{p=0}^{\infty}\left(S_p^{(j)} > \frac{u}{m+1}\right)\right) =$$

$$= \sum_{j=1}^{m+1}\mathbb{P}\left(\bigcup_{p=0}^{\infty}\left(Z_p^{(j)} > e^{\frac{Ru}{m+1}}\right)\right).$$

Since $(Z_p^{(j)})_{p\ge 0}$ is a nonnegative supermartingale, it follows from Doob's maximal inequality that

$$\mathbb{P}\left(\bigcup_{p=0}^{\infty}\left(Z_p^{(j)} > e^{\frac{Ru}{m+1}}\right)\right) \le e^{-\frac{Ru}{m+1}}\mathbb{E}(Z_0^{(j)}) =$$

$$= e^{-\frac{Ru}{m+1}}\mathbb{E}\left[\exp\left(R\left(X_1 e^{-\delta T_j} - \frac{c(1-e^{-\delta T_j})}{(m+1)\delta}\right)\right)\right].$$

Therefore,

$$\Psi_\delta(u) \le e^{-\frac{Ru}{m+1}}\sum_{j=1}^{m+1}\mathbb{E}\left[\exp\left(R\left(X_1 e^{-\delta T_j} - \frac{c(1-e^{-\delta T_j})}{(m+1)\delta}\right)\right)\right].$$

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