UDC 517.54

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A CLASS OF MEROMORPHIC BAZILEVIČ-TYPE FUNCTIONS DEFINED BY A DIFFERENTIAL OPERATOR

КЛАС МЕРОМОРФНИХ ФУНКЦІЙ ТИПУ БАЗІЛЕВИЧА, ЩО ВИЗНАЧЕНІ ДИФЕРЕНЦІАЛЬНИМ ОПЕРАТОРОМ

We define a new subclass of meromorphic Bazilevič-type functions by using a differential operator. We study some interesting properties, such as the arc length, the growth of coefficients, and the integral representation of functions from this class.

Визначено новий підклас мероморфних функцій типу Базілевича, що визначені за допомогою диференціального оператора. Вивчаються деякі цікаві властивості, такі як довжина дуги, зростання коефіцієнтів та інтегральні зображення функцій із цього класу.

1. Introduction and definitions. Let H denote the class of functions p, given by

$$p(z) = 1 + p_1 z + p_2 z^2 + \ldots,$$

that are analytic in the unit disk $E = \{z : |z| < 1\}$. A function $f \in H$ is said to be subordinate to a function g written as

$$f(z) \prec g(z),$$

if there exists a Schwarz function w with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1$$

such that

$$f(z) = g(w(z)).$$

In particular, if g is univalent in E, then

$$f(0) = g(0) \quad \text{and} \quad f(E) \subset g(E).$$

A function p analytic in E belongs to the class $P(\alpha, A, B)$, if

$$p(z) \prec (1-\alpha)\frac{1+Az}{1+Bz} + \alpha,$$

where $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$. By using Herglotz representation, a function $p \in$ $\in P(\alpha, A, B)$ if and only if

$$p(z) = \alpha + (1-\alpha) \int_0^{2\pi} \frac{1 + Aze^{-it}}{1 + Bze^{-it}} d\mu(t), \qquad z \in E,$$

where μ is a non decreasing function in $[0, 2\pi]$ such that $\int_0^{2\pi} d\mu(t) = 1$. For some details about Herglotz theorem, see [6, 8].

A function p analytic in E belongs to the class $P_m(\alpha, A, B)$, $m \geq 2$, $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, if and only if

$$p(z) = \alpha + \frac{1-\alpha}{2} \int_0^{2\pi} \frac{1 + Aze^{-it}}{1 + Bze^{-it}} d\mu(t), \qquad z \in E, \tag{1.1}$$

where μ is a real valued function of bounded variation on $[0, 2\pi]$ satisfying

$$\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq m. \tag{1.2}$$

The function $p \in P_m(\alpha, A, B)$ can also be written as

$$p(z) = \left(\frac{m}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) p_2(z), \qquad z \in E,$$

where $p_1, p_2 \in P(\alpha, A, B)$.

Let M denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n,$$

which are analytic in

$$E^* = \{z : 0 < |z| < 1\}.$$

A function f analytic in E^* belongs to the class $MR_m(\alpha, A, B)$, $m \geq 2$, $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, if and only if

$$-\frac{zf'(z)}{f(z)} \in P_m(\alpha, A, B), \qquad z \in E.$$

By using (1.1), one can easily prove the following representation formula for the class $MR_m(\alpha, A, B)$:

$$f(z)=\begin{cases}\dfrac{1}{z}\exp\dfrac{(1-\alpha)(B-A)}{2B}\displaystyle\int_0^{2\pi}\log\left(1+Bze^{-it}\right)d\mu(t), & B\neq 0,\\ \dfrac{1}{z}\exp(\alpha-1)\dfrac{A}{2}\displaystyle\int_0^{2\pi}ze^{-it}d\mu(t), & B=0,\end{cases}\qquad z\in E^*, \tag{1.3}$$

where μ is a real valued function of bounded variation on $[0,2\pi]$ satisfying the conditions in (1.2). For $m=2$ and $\alpha=0$, we have the class $MR_2(A,B)=MR(A,B)$ studied by Ali et al. [1]. For $A=1$ and $B=-1$, we get the class $MR_m(\alpha)$ studied by Dziok [4], and, for $A=1,\ B=-1$ and $\alpha=0$, we obtain the class MR_m of meromorphic functions of bounded radius rotation defined and studied by Pfaltzgraff et al. [15]. Further a function f analytic in E^* belongs to the class $MV_m(\alpha,A,B)$, $m\geq 2$, $-1\leq B<A\leq 1$ and $0\leq\alpha<1$, if and only if

$$-\frac{(zf'(z))'}{f'(z)}\in P_m(\alpha,A,B),\qquad z\in E.$$

Also for the function f belongs to the class $MV_m(\alpha,A,B)$, one can easily prove the following representation formula:

$$f(z)=\begin{cases}-\dfrac{1}{z^2}\exp\dfrac{(1-\alpha)(B-A)}{2B}\displaystyle\int_0^{2\pi}\log\left(1+Bze^{-it}\right)d\mu(t), & B\neq 0,\\ -\dfrac{1}{z^2}\exp(\alpha-1)\dfrac{A}{2}\displaystyle\int_0^{2\pi}ze^{-it}d\mu(t), & B=0,\end{cases}\qquad z\in E^*, \tag{1.4}$$

where μ is a real valued function of bounded variation on $[0,2\pi]$ satisfying the conditions

$$\int_0^{2\pi}d\mu(t)=2,\qquad \int_0^{2\pi}|d\mu(t)|\leq m,\qquad \int_0^{2\pi}e^{-it}d\mu(t)=0.$$

For $A=1$ and $B=-1$, we get the class $MV_m(\alpha)$ studied by Dziok [4], and, for $A=1,\ B=-1$ and $\alpha=0$, we obtain the class MV_m of meromorphic functions of bounded boundary rotation defined by Pfaltzgraff et al. [15] and studied by Noonan [12]. Next following Pommerenke [16], we denote by $MR^*(A,B)$ the class of all functions h of the form

$$h(\xi)=\xi+c_0+\frac{c_1}{\xi}+\ldots,$$

that are analytic in $1<|\xi|<\infty$ and satisfies

$$\frac{\xi h'(\xi)}{h(\xi)}\in P(A,B),\qquad |\xi|>1.$$

Let $V_m(A,B)$, $-1\leq B<A\leq 1$, denote the class of functions g, given by

$$g(z) = z + b_2 z^2 + \ldots,$$

that are analytic in E, satisfy the condition $g'(z) \neq 0$ in E, and map E onto a domain with boundary rotation at most $m\pi$. It can be shown that $g \in V_m(A, B)$ if and only if

$$f'(z) = \begin{cases} \exp \dfrac{A-B}{2B} \displaystyle\int_0^{2\pi} \log\left(1 + Bze^{-it}\right) d\mu(t), & B \neq 0, \\ \exp \dfrac{A}{2} \displaystyle\int_0^{2\pi} ze^{-it} d\mu(t), & B = 0, \end{cases} \qquad z \in E. \tag{1.5}$$

For some details about $V_m(A, B)$, we refer [14]. Moreover, a function $h = z + h_2 z^2 + \ldots$ belongs to the class $T_m(A, B)$ if and only if there exists a function $g \in V_m(A, B)$ such that

$$\frac{h'(z)}{g'(z)} \in P_m(A, B),$$

where $m \geq 2$ and $-1 \leq B < A \leq 1$.

The class of Bazilevič functions in the open unit disc was introduced by Bazilevič [3]. He defined Bazilevič function by the relation

$$f(z) = \frac{\eta}{1+\beta^2} \int_0^z (p(t) - i\beta)\, t^{-\frac{\eta\beta i}{2} - 1} g^{\frac{\eta}{1+\beta^2}}(t) dt, \qquad z \in E,$$

where $p \in P$, $g \in S^*$(the class of analytic starlike functions), β is real and $\eta > 0$. Many authors, by using different techniques, studied Bazilevič functions and related concepts. For some details see [18–20].

The classes of meromorphic Bazilevič functions were studied by many authors, for instance, Thomas [21], introduced and studied the class B_α of all meromorphic Bazilevič functions of order α and in [7] the estimates for the initial coefficients of the meromorphic Bazilevič functions were obtained.

For any two meromorphic functions f and g with

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n z^n, \quad z \in E^*,$$

the convolution is given by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n b_n z^n, \qquad z \in E^*,$$

where $(*)$ stands for convolution. We now consider the operator defined by Al-Oboudi and Al-Zakeri [2]. For λ real and $n \in N_0 = N \cup \{0\}$, we define the linear operator $D_\lambda^n : M \to M$ by

$$D_\lambda^n f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} [1 + \lambda (k+1)]^n a_k z^k, \qquad z \in E^*.$$

It is noted that

$$\lambda z \left(D_\lambda^n f(z)\right)' = D_\lambda^{n+1} f(z) - (\lambda + 1) D_\lambda^n f(z), \qquad z \in E^*. \tag{1.6}$$

By using the class $MR_m(\alpha, A, B)$ of functions, we define the following class of meromorphic functions.

Definition 1. *A function $f \in M$ is said to belong to the class $MB_{m,\eta}(\alpha, n, \lambda, A, B)$ if and only if there exists a function $D_\lambda^n g \in MR_m(\alpha, A, B)$ such that*

$$\left| \arg \frac{\left[D_\lambda^{n+1} f(z)\right]^\eta \left[D_\lambda^n f(z)\right]^{1-\eta}}{D_\lambda^n g(z)} \right| \leq \frac{\beta\pi}{2}, \quad z \in E^*, \tag{1.7}$$

where $n \in N \cup \{0\}$, λ is real, $\eta \geq 0,\ m \geq 2,\ -1 \leq B < A \leq 1$, and $0 \leq \alpha < 1$.

Special cases:

i) For $n = 0,\ \lambda = -1,\ \alpha = 0,\ A = 1,\ B = -1,\ m = 2$ and $\delta = \dfrac{1}{\eta}$, $\eta \neq 0$, we have the class of functions introduced and studied by Thomas [21].

ii) For $n = 0,\ \lambda = -1,\ \alpha = 0,\ A = 1,\ B = -1$, and $\eta = 1$, we obtain the meromorphic analogue of the class defined and studied by Noor [13].

iii) For $n = 0,\ \lambda = -1,\ \alpha = 0,\ A = 1,\ B = -1$, and $m = 2$, we have the class of meromorphic close to convex functions introduced and studied by Libera et al. [11].

2. A set of lemmas. Each of the following lemmas will be needed in our present investigation.

Lemma 1. *Let $f \in MR_m(\alpha, A, B)$. Then, for $m \geq 2,\ -1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$,*

$$f(z) = \frac{1}{z} \left(z f_1(z)\right)^{1-\alpha}, \qquad z \in E^*,$$

where $f_1 \in MR_m(A, B)$.

Proof. The proof is an immediate consequence of the representation formula given in (1.3).

Lemma 2. *Let $p \in P(A, B)$. Then, for $|z| = r < 1$,*

$$\frac{1 - Ar}{1 - Br} \leq \Re(p(z)) \leq |p(z)| \leq \frac{1 + Ar}{1 + Br}.$$

These bounds are sharp.

***Remark* 1.** Lemma 2 was independently proved in [10].

The following result is an extension of the result due to Pommerenke [16].

Lemma 3. *Let $f \in MR(A, B)$, $-1 \leq B < A \leq 1$ with $B \neq 0$. Let f_1 be defined by $f_1(\zeta) = f\left(\dfrac{1}{\zeta}\right)$, $|\zeta| > 1$. Then*

$$\left(1 + Br^{-1}\right)^{\frac{B-A}{B}} \leq \left|\frac{f_1(\zeta)}{\zeta}\right| \leq \left(1 - Br^{-1}\right)^{\frac{B-A}{B}}. \tag{2.1}$$

Proof. Let $f_1(\zeta) = f\left(\dfrac{1}{\zeta}\right) \in MR^*(A,B)$, $-1 \leq B < A \leq 1$ with $B \neq 0$. Then, for $|\zeta| > 1$, f_1 can be written as

$$f_1(\zeta) = \zeta \exp \frac{B-A}{B} \int_0^{2\pi} \log\left(1 + B\zeta^{-1}e^{it}\right) d\mu(t), \qquad B \neq 0, \tag{2.2}$$

where μ is a non decreasing function in $[0, 2\pi]$ such that $\int_0^{2\pi} d\mu(t) = 1$. From the above equation and using the fact that geometric mean is not greater than the arithmetic mean, we see that

$$\left|\zeta^{-1} f_1(\zeta)\right|^{\frac{2B}{B-A}} \leq \int_0^{2\pi} \left|1 + B\zeta^{-1}e^{it}\right|^2 d\mu(t) =$$

$$= \int_0^{2\pi} \left(1 + B^2 r^{-2}\right) d\mu(t) + 2\ \mathrm{Re}\left(\zeta^{-1} \int_0^{2\pi} e^{it} d\mu(t)\right). \tag{2.3}$$

Also from (2.2)

$$a_0 = (B-A) \int_0^{2\pi} e^{it} d\mu(t).$$

Hence,

$$|a_0| \leq A - B.$$

Now (2.3) can be written as

$$\left|\zeta^{-1} f_1(\zeta)\right|^{\frac{2B}{B-A}} \leq 1 + B^2 r^{-2} + \frac{2|a_0| B r^{-1}}{B-A} \leq \left(1 - B r^{-1}\right)^2,$$

from which we obtain the inequality (2.1). We shall now prove the second part. Since

$$\frac{\zeta f_1'(\zeta)}{f_1(\zeta)} = 1 - a_0 \zeta^{-1} + \ldots \quad \text{and} \quad \frac{\zeta f_1'(\zeta)}{f_1(\zeta)} \prec \frac{1 + A\zeta}{1 + B\zeta},$$

therefore, the function

$$\phi(\zeta) = \xi \frac{\dfrac{\zeta f_1'(\zeta)}{f_1(\zeta)} - 1}{-B \dfrac{z f_1'(\zeta)}{f_1(\zeta)} + A} = -\frac{a_0}{A-B} + \ldots$$

is analytic in $|\zeta| > 1$ and satisfies $|\phi(\zeta)| < 1$. Then

$$\frac{\partial}{\partial r} \log\left|\zeta^{-1} f_1(\zeta)\right| = -\frac{1}{r} + \Re\left[\frac{\zeta^{-1} f_1'(\zeta)}{r f_1(\zeta)}\right] \leq \frac{a_0}{A-B} \left|\frac{\zeta^{-1}\phi(\zeta)}{r\left(1 + B\zeta^{-1}\phi(\zeta)\right)}\right|. \tag{2.4}$$

If $l = |\phi(\infty)| = \dfrac{|a_0|}{A-B}$, then, from [5, p. 287], we have

$$|\phi(\zeta)| \leq \frac{lr+1}{l+r},$$

hence, by (2.4)

$$\frac{\partial}{\partial r}\log\left|\zeta^{-1}f_1(\zeta)\right| \leq (A-B)\frac{lr+1}{r\left(r^2+l(1+B)r+B\right)}.$$

Integration over $[r,+\infty]$ gives

$$\log\left|\zeta^{-1}f_1(\zeta)\right| \geq$$

$$\geq \frac{B-A}{2B}\left(\frac{l(1-B)}{2\sqrt{\left(\frac{l(1+B)}{2}\right)^2-B}}+1\right)\times$$

$$\times\log\left(1+\left(\frac{l(1+B)}{2}-\sqrt{\left(\frac{l(1+B)}{2}\right)^2-B}\right)r^{-1}\right)+$$

$$+\frac{B-A}{2B}\left(\frac{l(1-B)}{2\sqrt{\left(\frac{l(1+B)}{2}\right)^2-B}}-1\right)\times$$

$$\times\log\left(1+\left(\frac{l(1+B)}{2}+\sqrt{\left(\frac{l(1+B)}{2}\right)^2-B}\right)r^{-1}\right).$$

Since $l=\dfrac{|a_0|}{A-B}=|\phi(\infty)|\leq 1$, therefore, we have

$$\left|\zeta^{-1}f_1(\zeta)\right| \geq \left(1+Br^{-1}\right)^{\frac{B-A}{B}}.$$

Lemma 3 is proved.

Lemma 4. *Let* $f\in MR_m(A,B)$, $m\geq 2$ *and* $-1\leq B<A\leq 1$. *Then there exist functions* $f_1,\ f_2\in MR(A,B)$ *such that, for all* $z\in E^*$,

$$f(z)=\frac{1}{z}\frac{(zf_1(z))^{\frac{m+2}{4}}}{(zf_2(z))^{\frac{m-2}{4}}}.$$

Proof. Since $f\in MR_m(A,B)$, therefore, we have

$$-\frac{zf'(z)}{f(z)}=\frac{m+2}{4}\frac{zf_1'(z)}{f_1(z)}-\frac{m-2}{4}\frac{zf_2'(z)}{f_2(z)}, \qquad z\in E^*,$$

where f_1 and f_2 belong to $MR(A,B)$. Now

$$\frac{f'(z)}{f(z)}+\frac{1}{z}=\frac{m+2}{4}\left(\frac{f_1'(z)}{f_1(z)}+\frac{1}{z}\right)-\frac{m-2}{4}\left(\frac{f_2'(z)}{f_2(z)}+\frac{1}{z}\right), \qquad z\in E^*,$$

which on integration gives the required result.

Lemma 4 is proved.

3. Main results. In this section, we will prove our main results.

Theorem 1. *Let* $f\in MR_m(\alpha,A,B)$, $m\geq 2, -1\leq B<A\leq 1$ *and* $0\leq\alpha<1$. *Then, with* $z=re^{i\theta}$ *and* $0\leq\theta_1<\theta_2\leq 2\pi$,

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{zf'(z)}{f(z)}\right\}d\theta<\frac{A-B}{1-B}(1-\alpha)\left(\frac{m}{2}-1\right)\pi.$$

Proof. Since $P(A,B)\subset P(\rho)$, $\rho=\dfrac{1-A}{1-B}$, therefore, from Lemma 1, one can write

$$f(z)=\frac{1}{z}(zs(z))^{(1-\alpha)\frac{A-B}{1-B}}, \qquad z\in E^*,$$

where $f\in MR_m(\alpha,A,B)$ and $s\in MR_m$. Taking argument on both sides and differentiating with respect to θ from θ_1 to θ_2 with $0\leq\theta_1<\theta_2\leq 2\pi$, we have

$$\Im\left[\frac{\partial}{\partial\theta}\log zf(z)\right]=\Im\left[\frac{A-B}{1-B}(1-\alpha)\frac{\partial}{\partial\theta}\log zs(z)\right].$$

Some simple calculations yield us

$$\Re\left(1+\frac{zf'(z)}{f(z)}\right)=\frac{A-B}{1-B}(1-\alpha)\Re\left(1+\frac{zs'(z)}{s(z)}\right).$$

Upon integrating from θ_1 to θ_2 with $\theta_1<\theta_2$ and taking argument to be continuous for $|z|<1$, we obtain

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{zf'(z)}{f(z)}\right\}d\theta=\frac{A-B}{1-B}(1-\alpha)\int_{\theta_1}^{\theta_2}\Re\left\{\frac{zs'(z)}{s(z)}\right\}d\theta. \qquad (3.1)$$

But Noonan [12], proved that

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{(zs_1'(z))'}{s_1'(z)}\right\}d\theta<\left(\frac{m}{2}-1\right)\pi,$$

where $s_1\in MV_m$. By using Alexander-type relation between MV_m and MR_m, we have

$$\int_{\theta_1}^{\theta_2}\Re\left\{\frac{zs'(z)}{s(z)}\right\}d\theta<\left(\frac{m}{2}-1\right)\pi. \qquad (3.2)$$

Thus, from (3.1) and (3.2) we get

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{zf'(z)}{f(z)}\right\} d\theta < \frac{A-B}{1-B}(1-\alpha)\left(\frac{m}{2}-1\right)\pi.$$

Theorem 1 is proved.

Theorem 2. *A function* $f \in MB_{m,\eta}(\alpha, n, \lambda, A, B)$ *if and only if*

$$\int_{\theta_1}^{\theta_2} \Re\left\{J(\alpha,\eta,\lambda,n,\beta)\right\} d\theta < \left[\beta + \frac{A-B}{1-B}(1-\alpha)\left(\frac{m}{2}-1\right)\right]\pi,$$

where $m \geq 2,\ 0 \leq \theta_1 < \theta_2 \leq 2\pi,\ 0 \leq \alpha < 1,\ n \in N_0,\ z = re^{i\theta,}\ 0 < \beta \leq 1,\ -1 \leq B < A \leq 1,$ $\eta \geq 0,\ \lambda$ *is real and*

$$J(\alpha,\eta,\lambda,n,\beta) = \Re\left\{\frac{\eta}{\lambda}\left[\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)} - (1+\lambda)\right] + \frac{1-\eta}{\lambda}\left[\frac{D_\lambda^{n+1}f(z)}{D_\lambda^{n}f(z)} - (1+\lambda)\right]\right\}. \tag{3.3}$$

Proof. For $z = re^{i\theta}$, $r \in (0\ 1)$ and θ real, we define the following classes of functions:

$$F(r,\theta) = \arg\left\{\left[(D_\lambda^{n+1}f(z)\right]^{\eta}\left[D_\lambda^{n}f(z)\right]^{1-\eta}\right\} \tag{3.4}$$

and

$$G(r,\theta) = \arg\left\{D_\lambda^{n}g(z)\right\}. \tag{3.5}$$

Since $f \in MB_{m,\eta}(\alpha, n, \lambda, A, B)$, therefore, from (1.7) it follows

$$|F(r,\theta) - G(r,\theta)| \leq \beta\frac{\pi}{2}, \qquad \beta \in (0\ \ 1].$$

Also we have $D_\lambda^n g \in MR_m(\alpha, A, B)$, then, by using Theorem 1, we get

$$\int_{\theta_1}^{\theta_2} \Re\left\{\frac{z(D_\lambda^n g)'(z)}{D_\lambda^n g(z)}\right\} d\theta < (1-\alpha)\frac{A-B}{1-B}\left(\frac{m}{2}-1\right)\pi. \tag{3.6}$$

Now from (3.4), (3.5), and (3.6) we obtain

$$|F(r,\theta_1) - F(r,\theta_2)| = |F(r,\theta_2) - G(r,\theta_2)| - |F(r,\theta_1) - G(r,\theta_1)| + |G(r,\theta_2) - G(r,\theta_1)| \leq$$

$$\leq \left[\beta + (1-\alpha)\frac{A-B}{1-B}\left(\frac{m}{2}-1\right)\right]\pi.$$

Moreover, from (3.4) we have

$$\frac{d}{d\theta}F(r,\theta) = \Re\left\{\frac{\eta}{\lambda}\left[\frac{D_\lambda^{n+2}f(z)}{D_\lambda^{n+1}f(z)} - (1+\lambda)\right] + \frac{1-\eta}{\lambda}\left[\frac{D_\lambda^{n+1}f(z)}{D_\lambda^{n}f(z)} - (1+\lambda)\right]\right\}.$$

Thus, we obtain

$$\int_{\theta_1}^{\theta_2} \Re J\left\{(\alpha,\eta,\lambda,n,\beta)\right\} d\theta < \left[\beta + \left(\frac{m}{2}-1\right)\frac{A-B}{1-B}\right]\pi.$$

Theorem 2 is proved.

***Corollary* 1.** *For* $\alpha = 0,\ A = 1,\ B = -1,\ \lambda = -1,\ n = 0$ *and* $\beta = \eta = 1,$ *we have that* $f \in MT_m$ *is the class of meromorphic close-to-convex functions of bounded boundary rotation and*

$$\int\limits_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} d\theta < \frac{m}{2}\pi.$$

***Corollary* 2** [11]**.** *For* $\alpha = 0,\ A = 1,\ B = -1,\ \lambda = -1,\ n = 0,\ m = 2$ *and* $\beta = \eta = 1,$ *we have that* $f \in MT$ *is the class of meromorphic close-to-convex functions and*

$$\int\limits_{\theta_1}^{\theta_2} \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} d\theta < \pi.$$

Theorem 3. *Let* $F = D^n f$*. Then* $F \in MB_{m,1}(0, n, -1, A, B),\ m \geq 2,\ -1 \leq B < A \leq 1$ *and* $n \in N_0,$ *if and only if*

$$F'(z) = -\frac{1}{z^2}\left[\frac{(u_1)^{\frac{m+2}{4}}}{(u_2)^{\frac{m-2}{4}}}\right], \qquad z \in E^*,$$

where u_1 *and* u_2 *are suitable meromorphic functions.*

Proof. Let $f \in MB_{m,1}(0, n, -1, A, B)$, then from definition (1.7) we can write

$$D^{n+1} f(z) = G(z)\, p(z),$$

where $G = D^n g \in MR_m(A, B)$ and $p \in P(A, B)$. By using (1.6) with $\lambda = -1,$ we have

$$-z\,(D^n f(z))' = G(z)\, p(z). \tag{3.7}$$

By using Lemma 4, we get

$$F'(z) = (D^n f(z))' = -\frac{1}{z^2}\frac{(zh_1(z))^{\frac{m+2}{4}}}{(zh_2(z))^{\frac{m-2}{4}}}p(z), \tag{3.8}$$

where $h_i \in MR(A, B)$. We can write the above equation as

$$F'(z) = -\frac{1}{z^2}\frac{[zh_1(z)\,p]^{\frac{m+2}{4}}}{[zh_2(z)\,p]^{\frac{m-2}{4}}} = -\frac{1}{z^2}\left[\frac{(u_1)^{\frac{m+2}{4}}}{(u_2)^{\frac{m-2}{4}}}\right], \qquad z \in E^*.$$

Theorem 3 is proved.

Theorem 4. *A function* $f \in MB_{m,1}(0, 0, -1, A, B),\ m \geq 2$ *and* $-1 \leq B < A \leq 1,$ *if and only if there exists a function* $h \in T_m(A, B)$ *such that*

$$-\frac{1}{z^2 F'(z)} = h'(z), \qquad z \in E^*.$$

Proof. From (3.7) we can write

$$F'(z) = G_1'(z)\, p(z), \qquad z \in E, \tag{3.9}$$

where $G_1 \in MV_m(A,B)$ and $p \in P(A,B)$. Now using the representation formulas (1.4) and (1.5) for the classes $MV_m(A,B)$ and $V_m(A,B)$ with the fact that

$$2b_2 = \frac{(1-\alpha)(A-B)}{2} \int\limits_0^{2\pi} e^{-it} d\mu(t),$$

we obtain

$$\frac{-1}{z^2 G'(z)} = \phi'(z), \qquad z \in E^*, \tag{3.10}$$

where $\phi(z) = z + \sum_{n=2}^{\infty} b_n z^n \in V_m(A,B)$ with $b_2 = 0$. From (3.9) and (3.10) we have

$$-\frac{1}{z^2 F'(z)} = p_1(z)\,\phi'(z) = h'(z) \qquad \left(p_1(z) = \frac{1}{p(z)}\right).$$

Theorem 4 is proved.

Theorem 5. *Let* $F = D^n f \in MB_{m,1}(0,n,-1,A,B)$, $m \geq 2, -1 \leq B < A \leq 1$ *and* $n \in N_0$, *then, for* $z = re^{i\theta}$, $0 < r < 1$,

$$(1-Ar)\frac{(1+Br)^{\frac{m+2}{4}\frac{B-A}{B}}}{(1-Br)^{\frac{m-2}{4}\frac{B-A}{B}+1}}. \leq \left|z^2 F'(z)\right| \leq (1+Ar)\frac{(1-Br)^{\frac{m+2}{4}\frac{B-A}{B}}}{(1+Br)^{\frac{m-2}{4}\frac{B-A}{B}+1}}.$$

Proof. From (3.8) we obtain

$$F'(z) = -\frac{1}{z^2}\frac{(zh_1(z))^{\frac{m+2}{4}}}{(zh_2(z))^{\frac{m-2}{4}}} p(z).$$

By using Lemmas 2 and 3, we get

$$\left|z^2 F'(z)\right| \leq \frac{(1-Br)^{\frac{m+2}{4}\frac{B-A}{B}}}{(1+Br)^{\frac{m-2}{4}\frac{B-A}{B}}}\frac{1+Ar}{1+Br},$$

or, equivalently,

$$\left|z^2 F'(z)\right| \leq (1+Ar)\frac{(1-Br)^{\frac{m+2}{4}\frac{B-A}{B}}}{(1+Br)^{\frac{m-2}{4}\frac{B-A}{B}+1}}.$$

Similarly we can prove the other case.

Theorem 5 is proved.

Theorem 6. *A function* $f \in MB_{m,\eta}(\alpha,n,\lambda,A,B)$, $m \geq 2$, $-1 \leq B < A \leq 1$, $n \in N_0$, $\lambda > 0$ $\eta > 0$ *and* $0 \leq \alpha < 1$, *if and only if*

$$D_\lambda^n f(z) = \left\{ \frac{1}{\lambda\eta z^{\frac{1+\lambda}{\lambda\eta}}} \int\limits_0^z t^{\frac{1}{\lambda\eta}-1} \left[th(t)\right]^{\frac{1-\alpha}{\eta}} p^{\frac{\beta}{\eta}}(t)dt \right\}^\eta, \qquad z \in E^*,$$

where $h \in MR_m(A,B)$ *and* $p \in P(A,B)$.

Proof. From (1.7), we can write

$$\left[D_\lambda^{n+1} f(z)\right]^\eta \left[D_\lambda^n f(z)\right]^{1-\eta} = h_1(z)\, p^\beta(z),$$

where $h_1 = D_\lambda^n g \in MR_m(\alpha, A, B)$ and $p \in P(A, B)$. By using (1.6), we have

$$\frac{1}{\eta} z (D_\lambda^n f(z))' \left[D_\lambda^n f(z)\right]^{\frac{1}{\eta}-1} + \frac{1+\lambda}{\lambda\eta}\left[D_\lambda^n f(z)\right]^{\frac{1}{\eta}} = \frac{1}{\eta\lambda} h_1^{\frac{1}{\eta}}(z)\, p^{\frac{\beta}{\eta}}(z).$$

Multiplying with $z^{\frac{1+\lambda}{\lambda\eta}-1}$ and using Lemma 1, we obtain

$$\left[z^{\frac{1+\lambda}{\lambda\eta}} \left[D_\lambda^n f(z)\right]^{\frac{1}{\eta}}\right]' = \frac{1}{\lambda\eta} z^{\frac{1}{\lambda\eta}-1} (zh(z))^{\frac{1-\alpha}{\eta}} p^{\frac{\beta}{\eta}}(z).$$

Integrating from 0 to z, we obtain the required result.

Theorem 6 is proved.

Theorem 7. *Let $f \in MB_{m,\eta}(\alpha, n, \lambda, A, B)$, $m \geq 2$, $B \in [-1, 0)$, $A \in [0, 1]$, $n \in N_0$, $\lambda > 0$, $\eta > 0$, $0 \leq \alpha < 1$, and $M(r) = \max_{|z|=r} |D_\lambda^n f|$ with $0 \leq \theta \leq 2\pi$. Then*

$$M^{\frac{1}{\eta}}(r) \leq \frac{1}{r^{\frac{1}{\eta}}} (1-B)^{\frac{m+2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B}} (1+A)^{\frac{\beta}{\eta}} (1+Br)^{-\frac{m-2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B}-\frac{\beta}{\eta}} \times$$

$$\times {}_2F_1\left(1, \frac{m-2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B} + \frac{\beta}{\eta}, \frac{1}{\lambda\eta}+1, \frac{Br}{1+Br}\right), \tag{3.11}$$

where ${}_2F_1$ is the hypergeometric function.

Proof. From Theorem 6, we have

$$\left[D_\lambda^n f(z)\right]^{\frac{1}{\eta}} = \frac{1}{\lambda\eta z^{\frac{1+\lambda}{\lambda\eta}}} \int_0^z t^{\frac{1}{\lambda\eta}-1} \left[th(t)\right]^{\frac{1-\alpha}{\eta}} p^{\frac{\beta}{\eta}}(t)\,dt, \qquad z \in E^*,$$

where $h \in MR_m(A, B)$ and $p \in P(A, B)$. By using Lemma 4, we get

$$\left[D_\lambda^n f(z)\right]^{\frac{1}{\eta}} = \frac{1}{\lambda\eta z^{\frac{1+\lambda}{\lambda\eta}}} \int_0^z t^{\frac{1}{\lambda\eta}-1} \left[\frac{\left[tf_1(t)\right]^{\frac{m}{4}+\frac{1}{2}}}{\left[tf_2(t)\right]^{\frac{m}{4}-\frac{1}{2}}}\right]^{\frac{1-\alpha}{\eta}} \left[p(t)\right]^{\frac{\beta}{\eta}} dt, \qquad z \in E^*.$$

Since $f_i \in MR(A, B)$ for $i = 1, 2$ and $p \in P(A, B)$, therefore, by using Lemmas 2 and 3 with suitable simplifications, we obtain

$$M^{\frac{1}{\eta}}(r) \leq \frac{1}{\eta\lambda r^{\frac{1}{\eta}}} (1-B)^{\frac{m+2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B}} (1+A)^{\frac{\beta}{\eta}} \times$$

$$\times r^{-\frac{1}{\lambda\eta}} \int_0^z r^{\frac{1}{\lambda\eta}-1} (1+Br)^{-\frac{m-2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B}-\frac{\beta}{\eta}} dt =$$

$$= \frac{1}{r^{\frac{1}{\eta}}} (1-B)^{\frac{m+2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B}} (1+A)^{\frac{\beta}{\eta}} (1+Br)^{-\frac{m-2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B}-\frac{\beta}{\eta}} \times$$

$$\times {}_2F_1\left(1,\ \frac{m-2}{4}\frac{1-\alpha}{\eta}\frac{B-A}{B}+\frac{\beta}{\eta},\ \frac{1}{\lambda\eta}+1,\ \frac{Br}{1+Br}\right),$$

where ${}_2F_1$ is the hypergeometric function.

Theorem 7 is proved.

Theorem 8. *Let* $f \in MB_m(\alpha,\eta,\rho,\beta)$, $0<\beta\leq 1$, $-1\leq B<A\leq 1$, $0\leq\alpha<1$, $\lambda\neq 0$ *is real,* $m>\left[2+\dfrac{\eta\,(1-B)}{(1-\alpha)\,(A-B)}\right]$, *and* $0<\eta<2\beta$. *Then, for* $F=D_\lambda^n f$ *and* $M(r)=$ $=\max_{|z|=r}|D_\lambda^n f|$,

$$L_rF(z)\leq 2\pi r\left|1+\frac{1}{\lambda}\right|M\,(r)+C\,(x)\,M\,(r)^{1-\frac{1}{\eta}}\left(\frac{1}{1-r}\right)^{\frac{m-2}{2}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}+\frac{\beta}{\eta}-1}\qquad (r\longrightarrow 1),$$

where $C\,(x)$ *is a constant depending upon* m, A, B, α, β, *and* η.

Proof. It is known that

$$L_rF\,(z)=\int\limits_0^{2\pi}\left|z^2\left[D_\lambda^n f(z)\right]'\right|d\theta=$$

$$=\int\limits_0^{2\pi}|z|\left|\frac{1}{\lambda}z^{\frac{-1}{\eta}}\,(zh\,(z))^{\frac{1-\alpha}{\eta}}\,(D_\lambda^n f\,(z))^{1-\frac{1}{\eta}}\,p^{\frac{\beta}{\eta}}\,(z)-\left(1+\frac{1}{\lambda}\right)D_\lambda^n f\,(z)\right|d\theta,$$

$$z=re^{i\theta},\quad 0<r<1,\quad 0\leq\theta\leq 2\pi,$$

where we have used (1.6) and (1.7). Some manipulations gives us

$$L_rF(z)\leq 2\pi r\left|1+\frac{1}{\lambda}\right|M\,(r)+\frac{1}{\lambda r^{\frac{1}{\eta}-1}}M^{1-\frac{1}{\eta}}\,(r)\times$$

$$\times\int\limits_0^{2\pi}|zh\,(z)|^{\frac{1-\alpha}{\eta}}\,|p\,(z)|^{\frac{\beta}{\eta}}.$$

By using Lemma 4, we get

$$L_rF(z)\leq 2\pi r\left|1+\frac{1}{\lambda}\right|M\,(r)+\frac{1}{\lambda r^{\frac{1}{\eta}-1}}M^{1-\frac{1}{\eta}}\,(r)\times$$

$$\times\int\limits_0^{2\pi}\frac{|zf_1\,(z)|^{\frac{m+2}{4}\frac{1-\alpha}{\eta}}}{|zf_2\,(z)|^{\frac{m-2}{4}\frac{1-\alpha}{\eta}}}\,|p\,(z)|^{\frac{\beta}{\eta}}\,d\theta.$$

Since f_i for $i=1,2\in MR\,(A,B)\subset MR\,(\rho)$ with $\rho=\dfrac{1-A}{1-B}$, therefore, by using Lemma 1 in a modified form, we obtain

$$L_rF(z) \le 2\pi r\left|1+\frac{1}{\lambda}\right| M(r) + \frac{1}{\lambda r^{\frac{1}{\eta}-1}} M^{1-\frac{1}{\eta}}(r) \times$$

$$\times \int_0^{2\pi} \frac{|zs_1(z)|^{\frac{m+2}{4}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}}}{|zs_2(z)|^{\frac{m-2}{4}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}}} |p(z)|^{\frac{\beta}{\eta}} d\theta,$$

where $s_i \in MR$ for $i=1,2$. By using Schwarz's inequality and the fact that the functions $\frac{1}{s_i}$, $i=1,2$, belong to the class S^* of analytic starlike functions, we get $|zs_1(z)|<4$ and

$$[zs_2(z)]^{-1} \prec (1-z)^{-2}.$$

Thus,

$$L_rF(z) \le 2\pi r\left|1+\frac{1}{\lambda}\right| M(r) + \frac{2^{\frac{m+2}{2}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}}}{\lambda r^{\frac{1}{\eta}-1}} \times$$

$$\times M^{1-\frac{1}{\eta}}(r)\left(\int_0^{2\pi} \frac{1}{|1-z|^{(m-2)\frac{1-\alpha}{\eta}\frac{A-B}{1-B}}} d\theta\right)^{\frac{1}{2}} \times$$

$$\times \left(\int_0^{2\pi} |p(z)|^{\frac{2\beta}{\eta}} d\theta\right)^{\frac{1}{2}}.$$

Since $p \in P(A,B) \subset P$, therefore, by using a result due to Hayman [9] that for $p \in P$ and $z = re^{i\theta}$,

$$\int_0^{2\pi} |p_1(z)|^{\lambda} d\theta \le c(\lambda)\frac{1}{(1-r)^{\lambda-1}},$$

where $c(\lambda)$ is a constant depending upon λ. Thus, we have

$$L_rF(z) \le 2\pi r\left|1+\frac{1}{\lambda}\right| M(r) + \frac{2^{\frac{m+2}{2}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}+\frac{\beta}{\eta}}}{\lambda r^{\frac{1}{\eta}-1}} C(\beta,\eta)\left(\frac{1}{1-r}\right)^{\frac{\beta}{\eta}-\frac{1}{2}} \times$$

$$\times M^{1-\frac{1}{\eta}}(r)\left(\int_0^{2\pi} \frac{1}{|1-z|^{(m-2)\frac{1-\alpha}{\eta}\frac{A-B}{1-B}}} d\theta\right)^{\frac{1}{2}}.$$

But Pommerenke [17] has shown that

$$\int_0^{2\pi} \frac{1}{(1-z)^{\gamma}} d\theta \sim c(\gamma)\frac{1}{(1-r)^{\gamma-1}} \qquad (r \to 1),$$

whenever $\gamma > 1$. Since $(m-2)\dfrac{1-\alpha}{\eta}\dfrac{A-B}{1-B} > 1$, therefore, we obtain

$$L_r F(z) \leq 2\pi r \left|1+\frac{1}{\lambda}\right| M(r) + C(x)\, M^{1-\frac{1}{\eta}}(r) \left(\frac{1}{1-r}\right)^{\frac{m-2}{2}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}+\frac{\beta}{\eta}-1},$$

where $C(x)$ is a constant depending upon $m, A, B, \alpha, \beta,$ and η.

Theorem 8 is proved.

Theorem 9. *Let* $f \in MB_m(\alpha, \eta, \rho, \beta)$, $0 < \beta \leq 1$, $-1 \leq B < A \leq 1, 0 \leq \alpha < 1, \lambda \neq 0$ *is real,* $m > \left[2 + \dfrac{\eta(1-B)}{(1-\alpha)(A-B)}\right]$, *and* $0 < \eta < 2\beta$. *Then*

$$|a_k| \leq O(1)\, k^{\frac{m-2}{2}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}+\frac{\beta}{\eta}-1}.$$

Proof. For $F(z) = \dfrac{1}{z} + \sum_{k=0}^{\infty} A_k z^k$ with $z = re^{i\theta}$, it is known that

$$kA_k = \frac{1}{2\pi r^{k+1}} L_r F(z).$$

By using Theorem 8, we have

$$k|A_k| \leq \frac{1}{2\pi r^{k+1}} \left[2\pi r \left|1+\frac{1}{\lambda}\right| M(r) + C(x)\left(\frac{1}{1-r}\right)^{\frac{m-2}{2}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}+\frac{\beta}{\eta}-1}\right].$$

Take $r = 1 - \dfrac{1}{n}$ and $A_k = [1+\lambda(k+1)]^n a_k$, to have

$$|a_k| \leq O(1) k^{\frac{m-2}{2}\frac{1-\alpha}{\eta}\frac{A-B}{1-B}+\frac{\beta}{\eta}-\left(2-\frac{1}{\lambda}\right)}.$$

Theorem 9 is proved.

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Received 25.08.16