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SPACE-LIKE SURFACES IN MINKOWSKI SPACE \mathbb{E}_1^4 WITH POINTWISE 1-TYPE GAUSS MAP*

ПРОСТОРОВО-ПОДІБНІ ПОВЕРХНІ У ПРОСТОРІ МІНКОВСЬКОГО \mathbb{E}_1^4 З ПОТОЧКОВИМ ГАУССОВИМ ВІДОБРАЖЕННЯМ ПЕРШОГО ТИПУ

We first classify space-like surfaces in the Minkowski space \mathbb{E}_1^4 , de Sitter space \mathbb{S}_1^3 , and hyperbolic space \mathbb{H}^3 with harmonic Gauss map. Then we give a characterization and classification of space-like surfaces with pointwise 1-type Gauss map of the first kind. We also present some explicit examples.

Насамперед наведено класифікацію просторово-подібних поверхонь у просторі Мінковського \mathbb{E}_1^4 , просторі де Сіттера \mathbb{S}_1^3 і гіперболічному просторі \mathbb{H}^3 з гармонічним гауссовим відображенням. Після цього охарактеризовано і наведено класифікацію просторово-подібних поверхонь першого типу з поточковим гауссовим відображенням першого типу. Також наведено деякі конкретні приклади.

1. Introduction. In late 1970's B. Y. Chen introduced the notion of finite type submanifolds of Euclidean space [6]. Since then many works have been done to characterize or classify submanifolds of Euclidean space or pseudo-Euclidean space in terms of finite type. Also, B. Y. Chen and P. Piccinni extended the notion of finite type to differentiable maps, in particular, to Gauss map of submanifolds in [12]. A smooth map ϕ on a submanifold M of a Euclidean space or a pseudo-Euclidean space is said to be of *finite type* if ϕ can be expressed as a finite sum of eigenfunctions of the Laplacian Δ of M , that is, $\phi = \phi_0 + \sum_{i=1}^k \phi_i$, where ϕ_0 is a constant map, ϕ_1, \dots, ϕ_k are non-constant maps such that $\Delta\phi_i = \lambda_i\phi_i$, $\lambda_i \in \mathbb{R}$, $i = 1, \dots, k$.

If a submanifold M of a Euclidean space or a pseudo-Euclidean space has 1-type Gauss map ν , then ν satisfies $\Delta\nu = \lambda(\nu + C)$ for some $\lambda \in \mathbb{R}$ and some constant vector C . In [12], B. Y. Chen and P. Piccinni studied compact submanifolds of Euclidean spaces with finite type Gauss map. Several articles also appeared on submanifolds with finite type Gauss map (cf. [2–5, 24, 25]).

However, the Laplacian of the Gauss map of several surfaces and hypersurfaces such as helicoids of the 1st, 2nd, and 3rd kind, conjugate Enneper's surface of the second kind and B-scrolls in a 3-dimensional Minkowski space \mathbb{E}_1^3 , generalized catenoids, spherical n -cones, hyperbolic n -cones and Enneper's hypersurfaces in \mathbb{E}_1^{n+1} take the form

$$\Delta\nu = f(\nu + C) \quad (1.1)$$

for some smooth function f on M and some constant vector C [17, 21]. A submanifold of a pseudo-Euclidean space is said to have *pointwise 1-type Gauss map* if its Gauss map satisfies (1.1) for some smooth function f on M and some constant vector C . In particular, if C is zero, it is said to be of *the first kind*. Otherwise, it is said to be of *the second kind* (cf. [1, 10, 15, 16, 18, 20, 22]).

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Remark 1.1. The Gauss map ν of a totally geodesic submanifold M in \mathbb{E}_1^m is a constant vector and $\Delta\nu = 0$, i.e., it is harmonic. For $f = 0$ if we write $\Delta\nu = 0 \cdot \nu$, then M has pointwise 1-type Gauss map of the first kind. If we choose $C = -\nu$, then (1.1) holds for any non-zero smooth function f . In this case M has pointwise 1-type Gauss map of the second kind. Therefore, a totally geodesic submanifold in \mathbb{E}_1^m is a trivial submanifold with pointwise 1-type Gauss map of both the first kind and the second kind.

The complete classification of ruled surfaces in \mathbb{E}_1^3 with pointwise 1-type Gauss map of the first kind was obtained in [21]. Also, a complete classification of rational surfaces of revolution in \mathbb{E}_1^3 satisfying (1.1) was recently given in [20], and it was proved that a right circular cone and a hyperbolic cone in \mathbb{E}_1^3 are the only rational surfaces of revolution in \mathbb{E}_1^3 with pointwise 1-type Gauss map of the second kind. The first author studied rotational hypersurfaces in Lorentz–Minkowski space with pointwise 1-type Gauss map [17], Moreover, in [23] a complete classification of cylindrical and non-cylindrical surfaces in \mathbb{E}_1^m with pointwise 1-type Gauss map of the first kind was obtained.

In this article, we study space-like surfaces in \mathbb{E}_1^4 with pointwise 1-type Gauss map of the first kind. Surfaces with harmonic Gauss map in \mathbb{E}_1^4 are of global 1-type Gauss map of the first kind. We first give a characterization and classification of maximal surfaces and non-maximal space-like surfaces in \mathbb{E}_1^4 with harmonic Gauss map. We also prove that oriented maximal surfaces and surfaces with light-like mean curvature vector in \mathbb{E}_1^4 with harmonic Gauss map are the only surfaces in \mathbb{E}_1^4 with (global) 1-type Gauss map of the first kind.

Then we obtain the necessary and sufficient conditions on non-maximal space-like surfaces in \mathbb{E}_1^4 with pointwise 1-type Gauss map of the first kind, and we give a classification of such surfaces. Further, we prove that an oriented non-maximal space-like surface in \mathbb{E}_1^4 has (global) 1-type Gauss map of the first kind if and only if the surface has constant Gaussian curvature and parallel mean curvature.

2. Prelimineries. Let \mathbb{E}_t^m denote the pseudo-Euclidean m -space with the canonical pseudo-Euclidean metric tensor of index t given by

$$g = - \sum_{i=1}^t dx_i^2 + \sum_{j=t+1}^m dx_j^2,$$

where (x_1, x_2, \dots, x_m) is a rectangular coordinate system in \mathbb{E}_t^m . We put

$$\begin{aligned} \mathbb{S}_t^{m-1}(r^2) &= \{x \in \mathbb{E}_t^m : \langle x, x \rangle = r^2\}, \\ \mathbb{H}_{t-1}^{m-1}(-r^2) &= \{x \in \mathbb{E}_t^m : \langle x, x \rangle = -r^2\}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the indefinite inner product of \mathbb{E}_t^m . Then $\mathbb{S}_t^{m-1}(r^2)$ and $\mathbb{H}_{t-1}^{m-1}(-r^2)$, $m \geq 3$, are complete pseudo-Riemannian manifolds of constant curvature r^2 and $-r^2$, respectively. The Lorentzian manifolds \mathbb{E}_1^m and $\mathbb{S}_1^{m-1}(r^2)$ are known as the Minkowski and de Sitter spaces, respectively. For $t = 1$

$$\mathbb{H}^{m-1}(-r^2) = \left\{ x = (x_1, \dots, x_m) \in \mathbb{E}_1^m : \langle x, x \rangle = -r^2 \text{ and } x_1 > 0 \right\}$$

is the hyperbolic space in \mathbb{E}_1^m .

The light cone \mathcal{LC}^{n-1} with vertex at the origin in \mathbb{E}_t^m is defined to be

$$\mathcal{LC}^{n-1} = \{x \in \mathbb{E}_t^m : \langle x, x \rangle = 0\}.$$

A vector v in \mathbb{E}_t^m is called space-like (resp., time-like) if $\langle v, v \rangle > 0$ (resp., $\langle v, v \rangle < 0$). A vector v is called light-like if it is nonzero and it satisfies $\langle v, v \rangle = 0$.

Let M be an n -dimensional pseudo-Riemannian submanifold of the pseudo-Euclidean space \mathbb{E}_t^m . We denote Levi-Civita connections of \mathbb{E}_t^m and M by $\tilde{\nabla}$ and ∇ , respectively. In this section, we shall use letters X, Y, Z, W (resp., ξ, η) to denote vectors fields tangent (resp., normal) to M . The Gauss and Weingarten formulas are given, respectively, by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.1)$$

$$\tilde{\nabla}_X \xi = -A_\xi(X) + D_X \xi, \quad (2.2)$$

where h , D and A are the second fundamental form, the normal connection and the shape operator of M , respectively.

For each $\xi \in T_p^\perp M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space $T_p M$ at $p \in M$. The shape operator and the second fundamental form are related by $\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle$.

The Gauss, Codazzi and Ricci equations are given, respectively, by

$$\langle R(X, Y, Z, W) \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \quad (2.3)$$

$$(\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z), \quad (2.4)$$

$$\langle R^D(X, Y)\xi, \eta \rangle = \langle [A_\xi, A_\eta]X, Y \rangle, \quad (2.5)$$

where R, R^D are the curvature tensors associated with connections ∇ and D , respectively, and $\tilde{\nabla}h$ is defined by

$$(\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

A submanifold M is said to have flat normal bundle if $R^D = 0$ identically, and the second fundamental form h of M in \mathbb{E}_t^m is called parallel if $\tilde{\nabla}h = 0$. A submanifold with parallel second fundamental form is also known as a parallel submanifold.

Let $\{e_1, e_2, \dots, e_m\}$ be a local orthonormal frame on M with $\varepsilon_A = \langle e_A, e_A \rangle = \pm 1$ such that e_1, e_2, \dots, e_n are tangent to M and $e_{n+1}, e_{n+2}, \dots, e_m$ are normal to M . We use the following convention on the range of indices: $1 \leq A, B, C, \dots \leq m$, $1 \leq i, j, k, \dots \leq n$, $n+1 \leq \beta, \gamma, \dots \leq m$.

Let $\{\omega_{AB}\}$ with $\omega_{AB} + \omega_{BA} = 0$ be the connection 1-forms associated to $\{e_1, \dots, e_m\}$. Then we have

$$\tilde{\nabla}_{e_k} e_i = \sum_{j=1}^n \varepsilon_j \omega_{ij}(e_k) e_j + \sum_{\beta=n+1}^m \varepsilon_\beta h_{i\beta}^\beta e_\beta$$

and

$$\tilde{\nabla}_{e_k} e_\beta = - \sum_{j=1}^n \varepsilon_j h_{kj}^\beta e_j + \sum_{\nu=n+1}^m \varepsilon_\nu \omega_{\beta\nu}(e_k) e_\nu,$$

where h_{ij}^β 's are the coefficients of the second fundamental form h .

The mean curvature vector H , the scalar curvature S and the squared length $\|h\|^2$ of the second fundamental form h are defined by

$$H = \frac{1}{n} \sum_{\beta=n+1}^m \varepsilon_\beta \operatorname{tr} A_\beta e_\beta, \quad (2.6)$$

$$\|h\|^2 = \sum_{i,j,\beta} \varepsilon_i \varepsilon_j \varepsilon_\beta h_{ij}^\beta h_{ji}^\beta, \quad (2.7)$$

$$S = n^2 \langle H, H \rangle - \|h\|^2, \quad (2.8)$$

where $\operatorname{tr} A_\beta$ denotes the trace of shape operator A_β , i.e., $\operatorname{tr} A_\beta = \sum_{i=1}^n \varepsilon_i h_{ii}^\beta$.

The mean curvature vector H of a submanifold of M in \mathbb{E}_t^m is called parallel if $DH = 0$ identically.

The gradient of a smooth function f defined on M into \mathbb{R} is defined by $\nabla f = \sum_{i=1}^n \varepsilon_i e_i(f) e_i$ and the Laplace operator acting on M is $\Delta = \sum_{i=1}^n \varepsilon_i (\nabla_{e_i} e_i - e_i e_i)$. If the position vector x of M in E_s^m satisfies $\Delta x \neq 0$ and $\Delta^2 x = 0$, then M is called biharmonic.

A surface M in \mathbb{E}_1^4 is called space-like if every non-zero tangent vector on M is space-like. Let $\{e_1, e_2, e_3, e_4\}$ be a local orthonormal frame on a space-like surface M such that e_1, e_2 are tangent to M and e_3, e_4 are normal to M with $\varepsilon_\beta = \langle e_\beta, e_\beta \rangle$, $\beta = 3, 4$.

The Gaussian curvature K is defined by $K = R(e_1, e_2; e_2, e_1)$. Note that scalar curvature S and Gaussian curvature of M satisfies $S = 2K$. Thus, (2.8) implies

$$K = 2 \langle H, H \rangle - \|h\|^2 / 2. \quad (2.9)$$

From Gauss equation (2.3) we have $K = \varepsilon_3 (\det A_3 - \det A_4)$. If K vanishes identically, M is said to be flat. On the other hand, M is called maximal if $H = 0$. A surface M is called pseudo-umbilical if its second fundamental form h and the mean curvature vector H satisfies $\langle h(X, Y), H \rangle = \rho \langle X, Y \rangle$ for a smooth function ρ . Moreover, if the equation $h(X, Y) = \langle X, Y \rangle H$ is satisfied, then M is said to be totally umbilical.

If we put $h_{ij,k} = (\nabla_{e_k} h)(e_i, e_j)$, then for a space-like surface M in \mathbb{E}_1^4 the Codazzi equation given by (2.4) becomes

$$\begin{aligned} h_{ij,k}^\beta &= h_{jk,i}^\beta, \quad i, j, k = 1, 2, \quad \beta = 3, 4, \\ h_{jk,i}^\beta &= e_i(h_{jk}^\beta) + \sum_{\gamma=3}^4 \varepsilon_\gamma h_{jk}^\gamma \omega_{\gamma\beta}(e_i) - \sum_{\ell=1}^2 \left(\omega_{j\ell}(e_i) h_{\ell k}^\beta + \omega_{k\ell}(e_i) h_{j\ell}^\beta \right). \end{aligned} \quad (2.10)$$

Let $G(m-n, m)$ be the Grassmannian manifold consisting of all oriented $(m-n)$ -planes through the origin of \mathbb{E}_t^m and $\bigwedge^{m-n} \mathbb{E}_t^m$ the vector space obtained by the exterior product of $m-n$ vectors in \mathbb{E}_t^m . Let $f_{i_1} \wedge \dots \wedge f_{i_{m-n}}$ and $g_{i_1} \wedge \dots \wedge g_{i_{m-n}}$ be two vectors in $\bigwedge^{m-n} \mathbb{E}_t^m$, where $\{f_1, f_2, \dots, f_m\}$ and $\{g_1, g_2, \dots, g_m\}$ are two orthonormal bases of \mathbb{E}_t^m . Define an indefinite inner product \langle, \rangle on $\bigwedge^{m-n} \mathbb{E}_t^m$ by

$$\langle f_{i_1} \wedge \dots \wedge f_{i_{m-n}}, g_{i_1} \wedge \dots \wedge g_{i_{m-n}} \rangle = \det(\langle f_{i_\ell}, g_{j_k} \rangle). \quad (2.11)$$

Therefore, for some positive integer s , we may identify $\bigwedge^{m-n} \mathbb{E}_t^m$ with some pseudo-Euclidean space \mathbb{E}_s^N , where $N = \binom{m}{m-n}$. Let $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ be an oriented local orthonormal frame on an n -dimensional pseudo-Riemannian submanifold M in \mathbb{E}_t^m with $\varepsilon_B = \langle e_B, e_B \rangle = \pm 1$ such that e_1, \dots, e_n are tangent to M and e_{n+1}, \dots, e_m are normal to M . The map $\nu: M \rightarrow G(m-n, m) \subset \mathbb{E}_s^N$ from an oriented pseudo-Riemannian submanifold M into $G(m-n, m)$ defined by

$$\nu(p) = (e_{n+1} \wedge e_{n+2} \wedge \dots \wedge e_m)(p) \quad (2.12)$$

is called the *Gauss map* of M that is a smooth map which assigns to a point p in M the oriented $(m - n)$ -plane through the origin of \mathbb{E}_t^m and parallel to the normal space of M at p [22]. We put $\varepsilon = \langle \nu, \nu \rangle = \varepsilon_{n+1}\varepsilon_{n+2}\dots\varepsilon_m = \pm 1$ and

$$\widetilde{M}_s^{N-1}(\varepsilon) = \begin{cases} \mathbb{S}_s^{N-1}(1) & \text{in } \mathbb{E}_s^N, \text{ if } \varepsilon = 1, \\ \mathbb{H}_{s-1}^{N-1}(-1) & \text{in } \mathbb{E}_s^N, \text{ if } \varepsilon = -1. \end{cases}$$

Then the Gauss image $\nu(M)$ can be viewed as $\nu(M) \subset \widetilde{M}_s^{N-1}(\varepsilon)$.

3. Space-like surfaces in \mathbb{E}_1^4 with harmonic Gauss map. The Laplacian of the Gauss map of an n -dimensional oriented submanifold M of a Euclidean space \mathbb{E}^{n+2} was obtained in [19]. By a similar calculation, for the Laplacian of the Gauss map ν given by (2.12) of an n -dimensional oriented submanifold M of a pseudo-Euclidean space \mathbb{E}_t^{n+2} we have the following lemma.

Lemma 3.1. *Let M be an n -dimensional oriented submanifold of a pseudo-Euclidean space \mathbb{E}_t^{n+2} . Then the Laplacian of Gauss map $\nu = e_{n+1} \wedge e_{n+2}$ is given by*

$$\begin{aligned} \Delta\nu = & \|h\|^2\nu + 2 \sum_{1 \leq j < k \leq n} \varepsilon_j \varepsilon_k R^D(e_j, e_k; e_{n+1}, e_{n+2}) e_j \wedge e_k + \\ & + \nabla(\text{tr}A_{n+1}) \wedge e_{n+2} + e_{n+1} \wedge \nabla(\text{tr}A_{n+2}) + \\ & + n \sum_{j=1}^n \varepsilon_j \omega_{(n+1)(n+2)}(e_j) H \wedge e_j, \end{aligned} \quad (3.1)$$

where $\|h\|^2$ is the squared length of the second fundamental form, R^D is the normal curvature tensor and $\nabla \text{tr}A_r$ is the gradient of $\text{tr}A_r$.

Remark 3.1. From (3.1) we see that if an n -dimensional submanifold M of \mathbb{E}_t^{n+2} has pointwise 1-type Gauss map of the first kind, then equation (1.1) is satisfied for $f = \|h\|^2$ and $C = 0$.

The Gauss map of a surface M in \mathbb{E}_1^4 is said to be harmonic if $\Delta\nu = 0$. Clearly, a harmonic Gauss map is of (global) 1-type of the first kind. In the Euclidean space \mathbb{E}^4 , a plane is the only surface with harmonic Gauss map. However, in the Minkowski space \mathbb{E}_1^4 there are non-planar surfaces with harmonic Gauss map.

Lemma 3.2 [8]. *Let M be a space-like surface with parallel mean curvature vector H in \mathbb{E}_1^4 . Then we have:*

- (a) $\langle H, H \rangle$ is constant,
- (b) $[A_H, A_\xi] = 0$ for any normal vector field ξ .

By combining the part (b) of Lemma 3.2 and the Ricci equation (2.5), we state the following lemma for later use.

Lemma 3.3. *Let M be a non-maximal space-like surface in the Minkowski space \mathbb{E}_1^4 . If the mean curvature vector H of M is parallel, then the normal bundle of M is flat, i.e., $R^D \equiv 0$.*

Proposition 3.1. *Let M be an oriented maximal surface in the Minkowski space \mathbb{E}_1^4 . Then the Gauss map ν of M is harmonic if and only if M is a flat surface in \mathbb{E}_1^4 with flat normal bundle.*

Proof. Let M be a maximal surface in \mathbb{E}_1^4 , i.e., $H \equiv 0$. Then, from (2.9) we have $\|h\|^2 = -2K$. Thus, (3.1) implies

$$\Delta\nu = -2K\nu + 2R^D(e_1, e_2; e_3, e_4)e_1 \wedge e_2. \quad (3.2)$$

Therefore, ν is harmonic if and only if $K = 0$ and $R^D = 0$.

Proposition 3.1 is proved.

Next, we obtain a non-planar maximal surface in \mathbb{E}_1^4 with harmonic Gauss map.

Example 3.1 [11]. Let Ω be an open, connected set in \mathbb{R}^2 and $\phi: \Omega \rightarrow \mathbb{R}$ a smooth function. We consider the surface M in the Minkowski space \mathbb{E}_1^4 given by

$$x(u, v) = (\phi(u, v), u, v, \phi(u, v)). \quad (3.3)$$

This surface lies in the degenerate hyperplane $\mathcal{H}_0 = \{(x_1, x_2, x_3, x_4) \in \mathbb{E}_1^4 \mid x_1 = x_4\}$. By a direct calculation, we see that M is a flat surface with flat normal bundle and the mean curvature vector H of M in \mathbb{E}_1^4 is given by

$$H = (\Delta\phi, 0, 0, \Delta\phi). \quad (3.4)$$

Therefore, M is maximal if and only if ϕ is harmonic.

Hence, Proposition 3.1 implies that if ϕ is a harmonic function, then the surface given by (3.3) has harmonic Gauss map.

Proposition 3.2. *A non-planar flat maximal surface in the Minkowski space \mathbb{E}_1^4 with flat normal bundle is congruent to the surface given by (3.3) for a smooth harmonic function $\phi: \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, where Ω is an open set in \mathbb{R}^2 .*

Proof. Let M be a non-planar flat maximal surface in \mathbb{E}_1^4 with flat normal bundle.

Since M is flat, there exist local coordinates (u, v) on M such that $e_1 = \partial/\partial u$, $e_2 = \partial/\partial v$. Then the induced metric tensor is given by $g = du^2 + dv^2$ and also $\omega_{12} \equiv 0$. Let $\{e_3, e_4\}$ be a local orthonormal normal frame on M with $\varepsilon_3 = -\varepsilon_4 = 1$, where $\varepsilon_\beta = \langle e_\beta, e_\beta \rangle$. As M is maximal, we have $A_\beta = (h_{ij}^\beta)$ with $h_{11}^\beta + h_{22}^\beta = 0$, $\beta = 3, 4$, that is, $\text{tr}A_3 = \text{tr}A_4 = 0$. Moreover, since M is flat, from the Gauss equation (2.3) we get $\det A_3 = \det A_4$. Therefore, the eigenvalues of A_3 and A_4 are equal which imply that $A_3 = \mp A_4$ as $R^D = 0$. Without loss of generality, we may take $A_3 = A_4$.

Let Ω be an open set in \mathbb{R}^2 and $x: \Omega \rightarrow M \subset \mathbb{E}_1^4$ be an isometric immersion. From the Gauss formula we obtain

$$x_{uu} = h_{11}^3(e_3 - e_4), \quad x_{uv} = h_{12}^3(e_3 - e_4), \quad x_{vv} = -h_{11}^3(e_3 - e_4) \quad (3.5)$$

as $\omega_{12} \equiv 0$. Also, the first and second equations in (3.5) imply that

$$x_{uu} + x_{vv} = 0. \quad (3.6)$$

Moreover, x_{uu} , x_{uv} and x_{vv} are pairwise linearly dependent light-like vector fields.

On the other hand, by a direct calculation, using the Weingarten formula and (3.5), we get

$$x_{uuu} = (\partial_u (h_{11}^3) + \omega_{34}(\partial_u)h_{11}^3)(e_3 - e_4), \quad (3.7)$$

$$x_{uvv} = (\partial_v (h_{11}^3) + \omega_{34}(\partial_v)h_{11}^3)(e_3 - e_4). \quad (3.8)$$

Now we define a vector valued function $y = (y^1, y^2, y^3, y^4): \Omega \rightarrow \mathbb{E}_1^4$ as $y = x_{uu}$. From equations (3.5), (3.7) and (3.8) we have $y_u = \gamma_1 y$ and $y_v = \gamma_2 y$ for some smooth functions γ_1 and

γ_2 . Thus, the coordinate functions of y satisfy

$$y_u^i = \gamma_1 y^i, \quad y_v^i = \gamma_2 y^i, \quad i = 1, 2, 3, 4. \quad (3.9)$$

By solving these equations, we get $y^j = c_j y^1$, $i = 2, 3, 4$, for some constants $c_j \in \mathbb{R}$. Thus, we obtain

$$x_{uu} = y^1 \eta_0, \quad (3.10)$$

where $\eta_0 = (1, c_2, c_3, c_4)$ is a constant light-like vector. In a similar way, we get

$$x_{uv} = \phi_2 \eta_0 \quad \text{and} \quad x_{vv} = \phi_3 \eta_0, \quad (3.11)$$

where $\phi_2, \phi_3 : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ are some smooth functions. By integrating (3.10) and (3.11), we obtain

$$x(u, v) = \phi(u, v) \eta_0 + u \eta_1 + v \eta_2,$$

where $\phi : \Omega \rightarrow \mathbb{R}$ is a smooth function and η_1, η_2 are constant vectors such that $\langle \eta_0, \eta_i \rangle = 0$, $\langle \eta_i, \eta_j \rangle = \delta_{ij}$, $i, j = 1, 2$. Equation (3.6) implies that ϕ is harmonic. By choosing $\eta_0 = (1, 0, 0, 1)$, $\eta_1 = (0, 1, 0, 0)$ and $\eta_2 = (0, 0, 1, 0)$, the proof is completed.

By combining Propositions 3.1 and 3.2, we state the following classification theorem for maximal surfaces in \mathbb{E}_1^4 with harmonic Gauss map.

Theorem 3.1. *An oriented maximal surface with harmonic Gauss map in the Minkowski space \mathbb{E}_1^4 is either an open part of a space-like plane or congruent to a surface given by (3.3) for a smooth harmonic function $\phi : \Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$, where Ω is an open set in \mathbb{R}^2 .*

Now we investigate non-maximal space-like surfaces in \mathbb{E}_1^4 with harmonic Gauss map.

Theorem 3.2. *Let M be an oriented non-maximal space-like surface in the Minkowski space \mathbb{E}_1^4 . Then the Gauss map ν of M is harmonic if and only if M is flat in \mathbb{E}_1^4 with light-like and parallel mean curvature vector.*

Proof. Let M be an oriented non-maximal space-like surface in \mathbb{E}_1^4 with harmonic Gauss map ν . Then we have $\Delta \nu = 0$. From (3.1) we obtain $\|h\|^2 = 0$ and $R^D = 0$. That is, the normal bundle is flat. So we can choose a local parallel orthonormal normal frame $\{e_3, e_4\}$ on M . Thus we have $\omega_{34} = 0$, and from (3.1)

$$\nabla(\text{tr}A_3) \wedge e_4 + e_3 \wedge \nabla(\text{tr}A_4) = 0 \quad (3.12)$$

which implies that $\text{tr}A_3$ and $\text{tr}A_4$ are constants. Therefore, $DH = 0$, that is, H is parallel, and $\langle H, H \rangle$ is constant because of part (a) of Lemma 3.2.

Now we will show that H is light-like. Suppose that H is not light-like, that is, $\langle H, H \rangle \neq 0$. As $\|h\|^2 = 0$ we have $K = 2\langle H, H \rangle \neq 0$ from (2.9). Thus, M is not flat.

On the other hand, since the normal bundle is flat and $\langle H, H \rangle \neq 0$, we can choose a local orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on M such that $e_3 = H/\alpha$ and e_4 are parallel, the shape operators are diagonalized, and e_1, e_2 are eigenvectors of A_3 , where $\alpha = \sqrt{|\langle H, H \rangle|}$. So we have $A_3 = \text{diag}(h_{11}^3, h_{22}^3)$, $A_4 = \text{diag}(h_{11}^4, -h_{11}^4)$, $h_{11}^3 + h_{22}^3 = 2\alpha$ and $\omega_{34} = 0$. Considering these, it follows from Codazzi equation (2.10) that

$$e_1(h_{22}^3) = -e_1(h_{11}^3) = \omega_{12}(e_2)(h_{11}^3 - h_{22}^3), \quad (3.13)$$

$$e_1(h_{22}^4) = -e_1(h_{11}^4) = 2\omega_{12}(e_2)h_{11}^4, \quad (3.14)$$

$$e_2(h_{11}^3) = -e_2(h_{22}^3) = \omega_{12}(e_1)(h_{11}^3 - h_{22}^3), \quad (3.15)$$

$$e_2(h_{11}^4) = -e_2(h_{22}^4) = 2\omega_{12}(e_1)h_{11}^4. \quad (3.16)$$

As $\|h\|^2 = 0$, we have

$$(h_{11}^3)^2 + (h_{22}^3)^2 = 2(h_{11}^4)^2, \quad (3.17)$$

from which we obtain

$$h_{11}^3 e_1(h_{11}^3) + h_{22}^3 e_1(h_{22}^3) = 2h_{11}^4 e_1(h_{11}^4),$$

$$h_{11}^3 e_2(h_{11}^3) + h_{22}^3 e_2(h_{22}^3) = 2h_{11}^4 e_2(h_{11}^4).$$

Using (3.13)–(3.16), the above equations become

$$-\omega_{12}(e_2)((h_{11}^3 - h_{22}^3)^2 - 4(h_{11}^4)^2) = 0, \quad (3.18)$$

$$\omega_{12}(e_1)((h_{11}^3 - h_{22}^3)^2 - 4(h_{11}^4)^2) = 0. \quad (3.19)$$

Since M is not flat, at least one of $\omega_{12}(e_1)$ and $\omega_{12}(e_2)$ is not zero. Therefore, (3.18) and (3.19) imply that

$$(h_{11}^3 - h_{22}^3)^2 = 4(h_{11}^4)^2.$$

Considering this and (3.17) we obtain $h_{11}^3 h_{22}^3 + (h_{11}^4)^2 = 0$. Therefore, the Gauss curvature $K = \varepsilon_3(h_{11}^3 h_{22}^3 + (h_{11}^4)^2) = 0$ and hence $2\langle H, H \rangle = K = 0$ which is a contradiction. As a result, H is light-like. Since $\langle H, H \rangle = 0$ and $\|h\|^2 = 0$, (2.9) implies $K = 0$, i.e., M is flat.

Conversely, we assume that M is a flat surface in \mathbb{E}_1^4 with parallel and light-like mean curvature vector H , that is, $K = \langle H, H \rangle = 0$. So we have $\|h\|^2 = 0$ from (2.9). On the other hand, Lemma 3.3 implies that M has flat normal bundle, i.e., $R^D = 0$. Therefore, there exists a local parallel orthonormal frame $\{e_3, e_4\}$ of normal bundle of M with $\varepsilon_3 = -\varepsilon_4 = 1$ and the shape operators A_3 and A_4 can be diagonalized simultaneously by choosing a proper frame $\{e_1, e_2\}$ of tangent bundle of M , namely, we have

$$A_\beta = \text{diag}(h_{11}^\beta, h_{22}^\beta), \quad \beta = 3, 4,$$

also, $\omega_{34} \equiv 0$. Moreover, since H is light-like, we get

$$\text{tr}A_3 = \text{tr}A_4 = \mu \neq 0 \quad \text{and} \quad H = \frac{\mu}{2}(e_3 - e_4).$$

In addition, since H is parallel and $\omega_{34} = 0$, μ is a constant. Thus, we obtain $\nabla(\text{tr}A_3) = \nabla(\text{tr}A_4) = 0$. Therefore, equation (3.1) gives $\Delta\nu = 0$.

Theorem 3.2 is proved.

A space-like surface in the Minkowski space \mathbb{E}_1^4 is called *marginally trapped* (or quasi-minimal) if its mean curvature vector is light-like at each point on the surface. We will use the following classification theorem of marginally trapped surfaces with parallel mean curvature vector in the Minkowski space \mathbb{E}_1^4 obtained in [14].

Theorem 3.3 [14]. *Let M be a marginally trapped surface with parallel mean curvature vector in the Minkowski space-time \mathbb{E}_1^4 . Then, with respect to suitable Minkowskian coordinates (t, x_2, x_3, x_4) on \mathbb{E}_1^4 , M is an open part of one of the following six types of surfaces:*

(i) *a flat parallel biharmonic surface given by*

$$x(u, v) = \left(\frac{1-b}{2}u^2 + \frac{1+b}{2}v^2, u, v, \frac{1-b}{2}u^2 + \frac{1+b}{2}v^2 \right), \quad b \in \mathbb{R};$$

(ii) *a flat parallel surface given by*

$$x(u, v) = a(\cosh u, \sinh u, \cos u, \sin u), \quad a > 0; \quad (3.20)$$

(iii) *a non-parallel flat biharmonic surface with constant light-like mean curvature vector, lying in the hyperplane $\mathcal{H}_0 = \{(t, x_2, x_3, t)\}$, but not in the light cone \mathcal{LC} ;*

(iv) *a non-parallel flat surface lying in the light cone \mathcal{LC} ;*

(v) *a non-parallel surface lying in the de Sitter space-time $S_1^3(r^2)$ for some $r > 0$ such that the mean curvature vector H' of M in $S_1^3(r^2)$ satisfies $\langle H', H' \rangle = -r^2$;*

(vi) *a non-parallel surface lying in the hyperbolic space $H^3(-r^2)$ for some $r > 0$ such that the mean curvature vector H' of M in $H^3(-r^2)$ satisfies $\langle H', H' \rangle = r^2$.*

Conversely, all surfaces of type (i)–(vi) above give rise to marginally trapped surfaces with parallel mean curvature vector in \mathbb{E}_1^4 .

Remark 3.2 [9]. We can combine cases (i) and (iii) of Theorem 3.3 into a single case, namely, flat surfaces defined by (3.3) such that ϕ is a function satisfying $\Delta\phi = c$ for some real number $c \neq 0$.

The surfaces type (i) and (ii) in Theorem 3.3 are two explicit examples for Theorem 3.2. In the next theorem we determine flat surfaces in $S_1^3(r^2) \subset \mathbb{E}_1^4$ with parallel and light-like mean curvature vector in \mathbb{E}_1^4 .

Theorem 3.4. *Let M be a space-like surface in the de Sitter space $S_1^3(r^2) \subset \mathbb{E}_1^4$ for some $r > 0$. If M is a flat surface with parallel and light-like mean curvature vector in \mathbb{E}_1^4 , then M is congruent to the surface given by*

$$x(u, v) = \left(\frac{r}{2}(u^2 + v^2), u, v, \frac{r}{2}(u^2 + v^2) - \frac{1}{r} \right). \quad (3.21)$$

Proof. Suppose that M is a flat space-like surface in $S_1^3(r^2) \subset \mathbb{E}_1^4$ with parallel and light-like mean curvature vector H in \mathbb{E}_1^4 . Since M is flat, there exist local coordinates u and v on M such that the induced metric tensor is $g = du^2 + dv^2$. Let $x: \Omega \rightarrow M \subset S_1^3(r^2) \subset \mathbb{E}_1^4$ be an isometric immersion, where Ω is an open set in \mathbb{R}^2 . Then, we have $\langle x, x \rangle = r^{-2}$. Thus, a local frame field $\{e_1, e_2, e_3, e_4\}$ on M can be chosen as $e_1 = \partial_u$, $e_2 = \partial_v$, $e_3 = rx$, and e_4 is a unit normal vector field orthogonal to e_3 such that $H = -r(e_3 - e_4)$ as H is light-like (see [8], Lemma 2.2).

From the Weingarten formula (2.2), we have $\tilde{\nabla}_{\partial_u} e_3 = r\partial_u$ and $\tilde{\nabla}_{\partial_v} e_3 = r\partial_v$ which imply $A_3 = -rI$, where I is identity operator acting on tangent bundle of M . Moreover, since M is flat and $H = -r(e_3 - e_4)$, we have $\det A_3 = \det A_4$, $\text{tr} A_3 = \text{tr} A_4$ from which and $A_3 = -rI$ we obtain $A_3 = A_4$. Thus, M is pseudo-umbilical. Theorem 4 [7] implies that M is biharmonic.

As M is a biharmonic surface with light-like mean curvature vector, from the proof of [11] (Theorem 6.1), one can see that x is of the form

$$x(u, v) = (\phi(u, v), u, v, \phi(u, v) - \phi_0) \quad (3.22)$$

for a smooth function ϕ and constant $\phi_0 \neq 0$. As $\langle x, x \rangle = r^{-2}$, from (3.22) we obtain

$$-2\phi_0\phi + \phi_0^2 + u^2 + v^2 = r^{-2}.$$

By considering this equation and a linear isometry of \mathbb{E}_1^4 , we may assume that

$$\phi(u, v) = \frac{r}{2}(u^2 + v^2) \quad \text{and} \quad \phi_0 = \frac{1}{r} \quad (3.23)$$

from which and (3.22) we have (3.21).

Theorem 3.4 is proved.

Similarly, we state that the following theorem holds true.

Theorem 3.5. *Let M be a space-like surface in the hyperbolic space $\mathbb{H}^3(-r^2) \subset \mathbb{E}_1^4$ for some $r > 0$. If M is a flat surface with parallel and light-like mean curvature vector in \mathbb{E}_1^4 , then M is congruent to the surface given by*

$$x(u, v) = \left(\frac{1}{r} + \frac{r}{2}(u^2 + v^2), u, v, \frac{r}{2}(u^2 + v^2) \right). \quad (3.24)$$

The proof of this theorem is similar to the proof of Theorem 3.4.

Corollary 3.1. *Up to linear isometries in \mathbb{E}_1^4 , the surface given by (3.21) (resp., (3.24)) is the only surface in $\mathbb{S}_1^3(r^2) \subset \mathbb{E}_1^4$ (resp., $\mathbb{H}^3(r^2) \subset \mathbb{E}_1^4$) with harmonic Gauss map.*

By combining the results given in this section, we state that the following theorem holds true.

Theorem 3.6. *Let M be an oriented space-like surface in the Minkowski space \mathbb{E}_1^4 . Then the Gauss map ν of M is harmonic if and only if M is congruent to one of the following six types of surfaces:*

- (i) *an open part of a space-like plane;*
- (ii) *the flat surface given by (3.3) for a smooth function $\phi: \Omega \rightarrow \mathbb{R}$ satisfying $\Delta\phi = c$, where Ω is an open set in \mathbb{R}^2 and $c \in \mathbb{R}$;*
- (iii) *the flat surface given by (3.20);*
- (iv) *a non-parallel flat surface lying in the light cone \mathcal{LC} ;*
- (v) *the flat surface given by (3.21) lying in the de Sitter space-time $\mathbb{S}_1^3(r^2)$;*
- (vi) *the flat surface given by (3.24) lying in the hyperbolic space $\mathbb{H}^3(-r^2)$.*

4. Space-like surfaces in \mathbb{E}_1^4 with pointwise 1-type Gauss map of the first kind. Let M be an oriented space-like surface in the Minkowski space \mathbb{E}_1^4 with harmonic Gauss map ν . Then ν satisfies (1.1) for $f = 0$ and $C = 0$. Thus, a harmonic Gauss map ν is of pointwise 1-type of the first kind. In this section, we obtain a characterization of surfaces in \mathbb{E}_1^4 with pointwise 1-type Gauss map of the first kind.

Theorem 4.1. *Let M be an oriented maximal surface in the Minkowski space \mathbb{E}_1^4 . Then M has pointwise 1-type Gauss map of the first kind if and only if M has flat normal bundle. Moreover, the Gauss map ν satisfies (1.1) for $f = \|h\|^2$ and $C = 0$.*

Proof. If M is maximal, then the Gauss map ν satisfies (3.2). Hence, ν is of pointwise 1-type of the first kind if and only if $R^D = 0$.

We now give the following lemma.

Lemma 4.1. *Let M be an oriented maximal surface in the Minkowski space \mathbb{E}_1^4 . If M has pointwise 1-type Gauss map of the first kind, then the function $f = \|h\|^2$ satisfies*

$$e_1(f) = -4\varepsilon\omega_{12}(e_2)f, \quad (4.1)$$

$$e_2(f) = 4\varepsilon\omega_{12}(e_1)f, \quad (4.2)$$

where $\{e_1, e_2\}$ is a local orthonormal frame for tangent bundle of M and $\varepsilon \in \{-1, 1\}$.

Proof. Let M be a maximal surface in \mathbb{E}_1^4 with pointwise 1-type Gauss map of the first kind. Then Theorem 4.1 implies that M has flat normal bundle. Thus, the shape operators can be diagonalized simultaneously, i.e., there exists an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on M such that $A_\beta = \text{diag}(h_{11}^\beta, -h_{11}^\beta)$, $\beta = 3, 4$, as $H = 0$. Therefore, we have from (2.7)

$$f = \|h\|^2 = 2(\varepsilon_3(h_{11}^3)^2 + \varepsilon_4(h_{11}^4)^2). \quad (4.3)$$

and Codazzi equation (2.10) yields

$$e_1(h_{11}^3) - \varepsilon_4 h_{11}^4 \omega_{34}(e_1) = -2\omega_{12}(e_2)h_{11}^3, \quad (4.4)$$

$$e_1(h_{11}^4) + \varepsilon_3 h_{11}^3 \omega_{34}(e_1) = -2\omega_{12}(e_2)h_{11}^4, \quad (4.5)$$

$$e_2(h_{11}^3) - \varepsilon_4 h_{11}^4 \omega_{34}(e_2) = 2\omega_{12}(e_1)h_{11}^3, \quad (4.6)$$

$$e_2(h_{11}^4) + \varepsilon_3 h_{11}^3 \omega_{34}(e_2) = 2\omega_{12}(e_1)h_{11}^4. \quad (4.7)$$

By multiplying (4.4) and (4.5), respectively, $\varepsilon_3 h_{11}^3$ and $\varepsilon_4 h_{11}^4$ and adding them, we have

$$\varepsilon_3 h_{11}^3 e_1(h_{11}^3) + \varepsilon_4 h_{11}^4 e_1(h_{11}^4) = -2\omega_{12}(e_2) \left(\varepsilon_3 (h_{11}^3)^2 + \varepsilon_4 (h_{11}^4)^2 \right).$$

By using (4.3) again in this equation, we obtain (4.1). In a similar way, we see that (4.6) and (4.7) give (4.2).

Lemma 4.1 is proved.

Proposition 4.1. *Let M be an oriented maximal surface in the Minkowski space \mathbb{E}_1^4 . Then M has (global) 1-type Gauss map of the first kind if and only if the Gauss map ν of M is harmonic.*

Proof. We assume that M has (global) 1-type Gauss map ν of the first kind. Then Theorem 4.1 implies that M has flat normal bundle. On the other hand, since ν is (global) 1-type of the first kind, (1.1) is satisfied for $f = f_0$, where f_0 is a constant. Moreover, Lemma 4.1 implies that f satisfies (4.1) and (4.2) from which we obtain $\omega_{12}(e_1)f_0 = \omega_{12}(e_2)f_0 = 0$ that imply $f_0 = 0$ or $\omega_{12} = 0$. In the case $f_0 = 0$, we have $\Delta\nu = f_0\nu = 0$, i.e., ν is harmonic. Otherwise M is flat, and it follows from Proposition 3.1 that ν is harmonic.

The converse is obvious.

Proposition 4.1 is proved.

Now we study non-maximal space-like surfaces in \mathbb{E}_1^4 with pointwise 1-type Gauss map of the first kind.

Theorem 4.2. *Let M be an oriented non-maximal space-like surface in \mathbb{E}_1^4 . Then M has pointwise 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector.*

Proof. Let M be an oriented non-maximal space-like surface in \mathbb{E}_1^4 . Suppose that M has pointwise 1-type Gauss map of the first kind. Then (1.1) is satisfied for $f = \|h\|^2$ and $C = 0$. From (1.1) and (3.1) we obtain that $R^D = 0$ and

$$\nabla(\operatorname{tr}A_3) \wedge e_4 + e_3 \wedge \nabla(\operatorname{tr}A_4) + 2 \sum_{j=1}^2 \omega_{34}(e_j)H \wedge e_j = 0. \quad (4.8)$$

Since $R^D = 0$, there exists a local orthonormal frame $\{e_3, e_4\}$ of normal bundle of M such that $\omega_{34} = 0$. So, it follows from (4.8) that $\nabla \operatorname{tr}A_3 = \nabla \operatorname{tr}A_4 = 0$, that is, $\operatorname{tr}A_\beta = \text{constant}$, $\beta = 3, 4$, from which and $\omega_{34} = 0$ we have $DH = 0$.

Conversely, let H be parallel. From Lemma 3.3 we have $R^D = 0$. Thus, there exists a local, orthonormal frame $\{e_3, e_4\}$ of normal bundle of M such that $\omega_{34} \equiv 0$. So, it follows from $DH = 0$ that $\operatorname{tr}A_3$ and $\operatorname{tr}A_4$ are constants. Therefore, equation (3.1) implies that $\Delta\nu = \|h\|^2\nu$, that is, M has pointwise 1-type Gauss map of the first kind.

Theorem 4.2 is proved.

Example 4.1. Let M be a surface in \mathbb{E}_1^4 given by

$$x(u, v) = \frac{1}{\sqrt{2}}(u \cosh \sqrt{2}v, u \sinh \sqrt{2}v, \sqrt{2} \sin \sqrt{2}u - u \cos \sqrt{2}u, \sqrt{2} \cos \sqrt{2}u + u \sin \sqrt{2}u).$$

Then the mean curvature vector H of M is parallel and light-like [14]. Moreover, the Gaussian curvature of M is $K = u^{-4}$ which implies $\|h\|^2 = -2u^{-4}$ from (2.9). Therefore, M has proper 1-type Gauss map of the first kind because of Theorem 4.2, that is, (1.1) is satisfied for $C = 0$ and $f = \|h\|^2 = -2u^{-4}$.

In [13], a complete classification of space-like surfaces with parallel mean curvature vector was given. By combining Theorem 3.1 [13] and Theorem 4.2, we have the following theorem.

Theorem 4.3. *Let M be an oriented non-maximal space-like surface in \mathbb{E}_1^4 with space-like or time-like mean curvature vector. Then M has pointwise 1-type Gauss map of the first kind if and only if M is a CMC surface lying in the light cone $\mathcal{LC} \subset \mathbb{E}_1^4$, a Euclidean hyperplane $\mathbb{E}^3 \subset \mathbb{E}_1^4$, a Lorentzian hyperplane $\mathbb{E}_1^3 \subset \mathbb{E}_1^4$, the de Sitter space-time $\mathbb{S}_1^3(c^2) \subset \mathbb{E}_1^4$, or the hyperbolic space $\mathbb{H}^3(-c^2) \subset \mathbb{E}_1^4$.*

In the next proposition we obtain characterization of non-maximal surfaces with (global) 1-type Gauss map of the first kind.

Proposition 4.2. *Let M be an oriented space-like surface in the Minkowski space \mathbb{E}_1^4 with light-like mean curvature vector. Then M has (global) 1-type Gauss map of the first kind if and only if the Gauss map ν of M is harmonic.*

Proof. Suppose that M has non-harmonic, (global) 1-type Gauss map of the first kind with light-like mean curvature vector. Then (1.1) is satisfied for $f = f_0$ and $C = 0$, where $f_0 \neq 0$ is a constant. Moreover, Theorem 4.2 implies that the mean curvature vector H of M is parallel. Since ν is non-harmonic, it follows from Theorems 3.2 and 3.3 that M is congruent to a non-flat surface lying in either $\mathbb{S}_1^3(r^2)$ or $\mathbb{H}^3(-r^2)$, and if M is lying in $\mathbb{S}_1^3(r^2)$ (resp., in $\mathbb{H}^3(-r^2)$), then its mean curvature vector H' in $\mathbb{S}_1^3(c^2)$ (resp., in $\mathbb{H}^3(-r^2)$) satisfies $\langle H', H' \rangle = -r^2$ (resp., $\langle H', H' \rangle = r^2$). Also, from Remark 3.1 we have $\|h^2\| = f_0$.

Let x be the position vector of M in \mathbb{E}_1^4 and $\langle x, x \rangle = \varepsilon_3 r^{-2}$, where $\varepsilon_3 = \pm 1$. We choose a local orthonormal frame $\{e_3, e_4\}$ of the normal bundle of M such that $e_3 = rx$ and $H = \varepsilon_3 r^2(e_3 - e_4)$. Since $e_3 = rx$ is parallel, we have $\omega_{34} = 0$. Moreover, $R^D = 0$, i.e., M has flat

normal bundle. Thus, the shape operators of M are simultaneously diagonalizable. So, there exists a local orthonormal frame $\{e_1, e_2\}$ of tangent bundle of M such that $A_\beta = \text{diag}(h_{11}^\beta, h_{22}^\beta)$, $\beta = 3, 4$, and

$$h_{11}^3 + h_{22}^3 = h_{11}^4 + h_{22}^4 = 2r^2. \quad (4.9)$$

On the other hand, since $\|h\|^2$ and $\langle H, H \rangle$ are constants, (2.9) implies the Gaussian curvature K of M is constant, i.e., we have $K_0 = \varepsilon_3(h_{11}^3 h_{22}^3 - h_{11}^4 h_{22}^4)$, where $K_0 \neq 0$ is a constant, from which and (4.9) we obtain

$$e_i(h_{11}^3)h_{22}^3 + h_{11}^3 e_i(h_{22}^3) = e_i(h_{11}^4)h_{22}^4 + h_{11}^4 e_i(h_{22}^4)$$

and

$$e_i(h_{11}^\beta) = -e_i(h_{22}^\beta), \quad i = 1, 2, \quad \beta = 3, 4. \quad (4.10)$$

Using these equations we get

$$e_i(h_{11}^3)(h_{11}^3 - h_{22}^3) = e_i(h_{11}^4)(h_{11}^4 - h_{22}^4). \quad (4.11)$$

In addition, considering (4.10), the Codazzi equation (2.4) yields

$$e_1(h_{11}^\beta) = -\omega_{12}(e_2)(h_{11}^\beta - h_{22}^\beta), \quad (4.12)$$

$$e_2(h_{11}^\beta) = \omega_{12}(e_1)(h_{11}^\beta - h_{22}^\beta), \quad \beta = 3, 4. \quad (4.13)$$

So, it follows from these equations and (4.11) that

$$\omega_{12}(e_i) \left((h_{11}^3 - h_{22}^3)^2 - (h_{11}^4 - h_{22}^4)^2 \right) = 0, \quad i = 1, 2. \quad (4.14)$$

As M is not flat, we have $\omega_{12} \neq 0$. Thus, (4.14) implies $(h_{11}^3 - h_{22}^3)^2 = (h_{11}^4 - h_{22}^4)^2$ from which and (2.9) we get $f_0 = \|h\|^2 = 0$ which is a contradiction. Therefore, the Gauss map ν is harmonic.

The converse is obvious.

Proposition 4.2 is proved.

Next we give a characterization for non-maximal space-like surfaces in the Minkowski space \mathbb{E}_1^4 with (global) 1-type Gauss map of the first kind.

Theorem 4.4. *Let M be an oriented non-maximal surface in the Minkowski space \mathbb{E}_1^4 . Then M has (global) 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector and constant Gaussian curvature.*

Proof. Let M be an oriented non-maximal surface in Minkowski space \mathbb{E}_1^4 . First we assume that M has (global) 1-type Gauss map of the first kind. Then it follows from (1.1) and (3.1) that $\|h\|^2 = f_0$ for some constant f_0 . Also, Theorem 4.2 implies that M has parallel mean curvature vector which implies $\langle H, H \rangle$ is constant. Therefore, (2.9) implies that the Gaussian curvature K of M is constant.

Conversely, let M has parallel mean curvature vector and constant Gaussian curvature. By Theorem 4.2 we have $\Delta\nu = \|h\|^2\nu$. Also, equation (2.9) implies that $\|h\|^2$ is constant. Therefore, the Gauss map of M is of 1-type of the first kind.

Theorem 4.4 is proved.

Next we give an example of a surface with non-harmonic (global) 1-type Gauss map of the first kind.

Example 4.2. Let M be a surface in \mathbb{E}_1^4 given by

$$x(u, v) = (a \cosh u, a \sinh u, b \cos v, b \sin v), \quad b^2 - a^2 \neq 0, \quad ab \neq 0.$$

Let $c = \sqrt{|b^2 - a^2|}$. Then we have $M = H^1(-a^{-1}) \times S^1(b^{-1}) \subset S_1^3(c^{-2}) \subset \mathbb{E}_1^4$ if $b^2 - a^2 > 0$, and $M = H^1(-a^{-1}) \times S^1(b^{-1}) \subset H^3(-c^{-2}) \subset \mathbb{E}_1^4$ if $b^2 - a^2 < 0$. By a direct calculation, it can be seen that M has parallel mean curvature vector and constant Gaussian curvature. Hence, Theorem 4.4 implies M has (global) 1-type Gauss map of the first kind.

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