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GREEN'S FUNCTIONAL FOR HIGHER-ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH GENERAL NONLOCAL CONDITIONS AND VARIABLE PRINCIPAL COEFFICIENT

ФУНКЦІОНАЛ ГРІНА ДЛЯ ЗВИЧАЙНИХ ДИФЕРЕНЦІАЛЬНИХ РІВНЯНЬ ВИЩОГО ПОРЯДКУ З НЕЛОКАЛЬНИМИ УМОВАМИ ЗАГАЛЬНОГО ВИГЛЯДУ ТА ЗМІННИМ ГОЛОВНИМ КОЕФІЦІЄНТОМ

The method of Green's functional is a little-known technique for the construction of fundamental solutions to linear ordinary differential equations (ODE) with nonlocal conditions. We apply this technique to a higher order linear ODE involving general nonlocal conditions. A novel kind of adjoint problem and Green's functional are constructed for the completely inhomogeneous problem. Several illustrative applications of the theoretical results are provided.

Метод функціонала Гріна є маловідомою технікою побудови фундаментальних розв'язків лінійних звичайних диференціальних рівнянь (ЗДР) з нелокальними умовами. В роботі цю техніку застосовано до лінійних ЗДР вищого порядку з нелокальними умовами загального вигляду. Спряжену проблему нового типу та функціонал Гріна побудовано для повністю неоднорідної задачі. Наведено також кілька ілюстративних застосувань теоретичних результатів.

1. Introduction. As is known, a fundamental function is a distributional formulation and plays a crucial role in mathematical analysis of differential equations. Once the fundamental function is found, the desired solution of the original equation can be easily obtained by superposition principle. Green's function is a kind of fundamental function which is principally based on the theory of linear operators and the theory of generalized functions in mathematical analysis [7 – 9, 11, 12, 18, 20, 32, 34 – 36], and its construction by classical methods [35] such as Green's function method and the method of variation of parameters for a class of nonclassical problems such as differential problems with nonlocal condition(s) may not be possible or may be troublesome. In order to overcome, in the literature, a small number of studies exist on the investigations of this class of nonclassical problems by Green's functional method, the origin of which dates back to S. S. Akhiev [1 – 5, 21 – 25, 27 – 29, 33]. The governing equation of the considered problems in most of these studies is in the standard form with principal coefficient equal to one. The case with variable principal coefficient has been considered only in [3]. Moreover, the case of a third order ODE with variable principal coefficient has been studied in [26]. To the best of our knowledge, any generalization on the case of the mth order ODE with variable principal coefficient does not exist for an integer m greater than or equal to four. In order to contribute to the enrichment of the literature in this context, we aim to extend the method for the second and third order linear ODEs in [3, 26] to the higher order linear ODEs with nonlocal conditions.

The structure of our work is organized as follows. In Section 2, the problem considered throughout the work is stated in detail. In Section 3, the solution space and its adjoint space are introduced. In Section 4, the adjoint operator, adjoint system and solvability conditions for the completely nonhomogeneous problem are presented. In Section 5, Green's functional and the special adjoint system are defined. In Section 6, several applications are provided. In Section 7, the conclusions are emphasized.

2. Statement of the problem. Let R be the set of real numbers, $X = (x_0, x_1)$ be a bounded open interval in R, and L_p with $1 \leq p < \infty$ be the space of the p-integrable functions on X, let L_∞ be the space of the measurable and essentially bounded functions on X. The aim in this work is to investigate the solvability conditions and Green's function of the mth order ordinary differential equation

$$(V_m u)(x) \equiv \big(g_{m-1}(x)(g_{m-2}(x)\ldots(g_2(x)(g_1(x)u'(x))')'\ldots)'\big)' + A_0(x)u(x) = z_m(x), \quad x \in X, \tag{1}$$

subject to the nonlocal conditions

$$V_i u \equiv a_i^0 u(x_0) + a_i^1 u_1(x_0) + a_i^2 u_2(x_0) + \ldots + a_i^{m-1} u_{m-1}(x_0) +$$

$$+ \int\limits_{x_0}^{x_1} B_i(\xi) u'_{m-1}(\xi)\, d\xi = z_i, \quad i = 0,1,2,\ldots,m-1, \tag{2}$$

where $m \geq 3$ is an integer, $z_m \in L_p$, $z_i \in R$ for $i = 0,1,2,\ldots,m-1$, $u_1(x) = g_1(x)u'(x)$, $u_2(x) = g_2(x)u'_1(x)$, $u_3(x) = g_3(x)u'_2(x), \ldots, u_{m-1}(x) = g_{m-1}(x)u'_{m-2}(x)$. Here, the following assumptions are hold: $g_1, g_2, \ldots, g_{m-1} \in L_\infty$ are the given functions such that $\dfrac{1}{g_1}, \dfrac{1}{g_2}, \ldots, \dfrac{1}{g_{m-1}} \in$ $\in L_p$ and the following integrals belong to the space L_p as functions of s_{m-1}:

$$\int\limits_{x_0}^{s_{m-1}} \frac{ds_{m-2}}{g_2(s_{m-2})g_1(s_{m-1})} \in L_p,$$

$$\int\limits_{x_0}^{s_{m-1}} \int\limits_{x_0}^{s_{m-2}} \frac{ds_{m-3}\, ds_{m-2}}{g_3(s_{m-3})g_2(s_{m-2})g_1(s_{m-1})} \in L_p,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\int\limits_{x_0}^{s_{m-1}} \int\limits_{x_0}^{s_{m-2}} \ldots \int\limits_{x_0}^{s_2} \frac{ds_1 \ldots ds_{m-3}\, ds_{m-2}}{g_{m-1}(s_1)\ldots g_2(s_{m-2})g_1(s_{m-1})} \in L_p$$

for $1 \leq p \leq \infty$, and $A_0 \in L_p, B_i \in L_q$ for $i = 0,1,2,\ldots,m-1$, where $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $a_i^0, a_i^1, \ldots, a_i^{m-1} \in R$ for $i = 0,1,2,\ldots,m-1$, and integration variables s_i are considered for $i \geq 1$ throught the work.

Equation (1) is one of the generalized forms of the mth order linear ODEs in Sturm – Liouville theory [19]. This equation is assumed to have generally nonsmooth coefficient becoming some general properties such as p-integrability and boundedness, and its principal part $\big(g_{m-1}(g_{m-2}\ldots(g_2(g_1u')')'\ldots)'\big)'$ may have weak singularities at finite number points (in the closure $\overline{X}$ of X) where some or all of functions $g_{m-1}, g_{m-2}, \ldots, g_1$ are continuous and zero.

Form (2) for the conditions can be considered as a generalization of the linearly local and nonlocal conditions for such mth order ODEs. Many conditions such as the initial and classical type boundary conditions and also multipoint [13, 14] and integral type conditions arising in modelling of many physical phenomena by such an equation are specific forms of (2) for a suitable choice of $a_i^0, a_i^1, \ldots, a_i^{m-1}$ and $B_i(\xi)$.

Problem (1), (2) may not have a classical adjoint problem. In such a case, some serious difficulties arise in applying the classical methods for this problem. In order to overcome these difficulties, a novel method based on [1 – 3, 5] is presented. By this method, the isomorphic decompositions of a weighted space of solutions and its adjoint space are used. A novel adjoint problem called as adjoint system for problem (1), (2) is introduced by these decompositions. This adjoint system is constructed by more different and easier calculations than constructing the traditional adjoint problem. This system consists of $m+1$ integro-algebraic equations for an unknown $(m+1)$-tuple $(F_m(\xi), F_{m-1}, F_{m-2}, \ldots, F_0)$ of a function $F_m(\xi)$ and m real numbers $F_{m-1}, F_{m-2}, \ldots, F_0$. One of these equations is an integral equation and the others are algebraic equations. This system has a similar role to that of the adjoint operator in general theory of the linear operators in Banach spaces [10, 15–17]. The solvability conditions of the completely nonhomogeneous problem and corresponding adjoint system are derived. Green's functional concept for the problem is introduced as a solution $\big(F_m(\xi,x), F_{m-1}(x), F_{m-2}(x), \ldots, F_0(x)\big)$ of the adjoint system with a free term depending on $x \in \overline{X}$ as a parameter. This concept is more natural than the classical Green's function concept. $F_m(\xi,x)$ corresponds to Green's function for the problem.

3. The solution space and its adjoint space. Equation (1) can be considered as a system of m first order ODEs with unknowns $u,\ u_1,\ u_2, \ldots, u_{m-1} \in W_p^{(1)}$, where $W_p^{(1)}$ is the space of all functions $u \in L_p$ having derivative $u' \in L_p$. Hence, the solvability of problem (1), (2) can be studied in space W_p with the weights $g_{m-1}, g_{m-2}, \ldots, g_1$ consecutively of all $u \in L_p$ with $u' \in L_p$, $(g_1u')' \in L_p$, $(g_2(g_1u')')' \in L_p, \ldots, \big(g_{m-1}(g_{m-2}\ldots(g_2(g_1u')')'\ldots)'\big)' \in L_p$ and also

$$\|u\|_{W_p} = \|u\|_{L_p} + \|u'\|_{L_p} + \|u_1'\|_{L_p} + \|u_2'\|_{L_p} + \ldots + \|u_{m-1}'\|_{L_p},$$

$$u_1 = g_1u',\ u_2 = g_2u_1',\ \ldots,\ u_{m-1} = g_{m-1}u_{m-2}'.$$

This problem, which is linear completely nonhomogeneous, can be considered as an equation in the form

$$Vu = z \tag{3}$$

with operator $V = (V_m, V_{m-1}, \ldots, V_0)$ and $(m+1)$-tuples $z = (z_m(x), z_{m-1}, \ldots, z_0)$. By the assumptions considered, V is a linear bounded operator from W_p into the Banach space

$$E_p \equiv L_p \times \underbrace{R \times R \times \ldots \times R}_{m \text{ times}}$$

consisting of the $(m+1)$-tuples $z = (z_m(x), z_{m-1}, \ldots, z_0)$ with $\|z\|_{E_p} = \|z_m\|_{L_p} + |z_{m-1}| + \ldots$ $\ldots + |z_0|$.

Some of the principal features concerning with solution space W_p can be given as follows: The trace or value operators

$$D_0u = u(\gamma),\ D_1u = u_1(\gamma),\ \ldots,\ D_{m-1}u = u_{m-1}(\gamma)$$

are surjective and bounded from W_p onto R for a given point γ of $\overline{X}$. The operator

$$Nu = \big(u_{m-1}'(x), u_{m-2}(\gamma), \ldots, u_1(\gamma), u(\gamma)\big)$$

is a linear homeomorphism from W_p onto E_p and has a bounded inverse. On the other hand, any function $u \in W_p$ can be represented as

$$u(x) = u(x_0) + u_1(x_0)\int_{x_0}^{x} \frac{ds_{m-1}}{g_1(s_{m-1})} + u_2(x_0)\int_{x_0}^{x}\int_{x_0}^{s_{m-1}} \frac{ds_{m-2}\,ds_{m-1}}{g_2(s_{m-2})g_1(s_{m-1})} +$$

$$+u_3(x_0)\int_{x_0}^{x}\int_{x_0}^{s_{m-1}}\int_{x_0}^{s_{m-2}}\frac{ds_{m-3}\,ds_{m-2}\,ds_{m-1}}{g_3(s_{m-3})g_2(s_{m-2})g_1(s_{m-1})}+\ldots$$

$$\ldots+u_{m-1}(x_0)\int_{x_0}^{x}\int_{x_0}^{s_{m-1}}\ldots\int_{x_0}^{s_2}\frac{ds_1\ldots ds_{m-2}\,ds_{m-1}}{g_{m-1}(s_1)\ldots g_2(s_{m-2})g_1(s_{m-1})}+$$

$$+\int_{x_0}^{x}u'_{m-1}(\xi)\int_{\xi}^{x}\int_{\xi}^{s_{m-1}}\ldots\int_{\xi}^{s_2}\frac{ds_1\ldots ds_{m-2}\,ds_{m-1}}{g_{m-1}(s_1)\ldots g_2(s_{m-2})g_1(s_{m-1})}\,d\xi. \tag{4}$$

The structure of adjoint space W_p^* is determined by the following theorem for the general case of $g_{m-1}, g_{m-2}, \ldots, g_1$ [1–3, 5, 26].

Theorem 1. *If $1 \leq p < \infty$, then any linear bounded functional $F \in W_p^*$ can be represented by*

$$F(u)=\int_{x_0}^{x_1}u'_{m-1}(x)\varphi_m(x)\,dx+u_{m-1}(x_0)\varphi_{m-1}+$$

$$+u_{m-2}(x_0)\varphi_{m-2}+\ldots+u_1(x_0)\varphi_1+u(x_0)\varphi_0,\quad u\in W_p, \tag{5}$$

by means of a unique element $\varphi=\big(\varphi_m(x),\varphi_{m-1},\varphi_{m-2},\ldots,\varphi_1,\varphi_0\big)\in E_q$, where $\dfrac{1}{p}+\dfrac{1}{q}=1$. Any linear bounded functional $F\in W_\infty^*$ can be represented by

$$F(u)=\int_{x_0}^{x_1}u'_{m-1}(x)\,d\varphi_m+u_{m-1}(x_0)\varphi_{m-1}+$$

$$+u_{m-2}(x_0)\varphi_{m-2}+\ldots+u_1(x_0)\varphi_1+u(x_0)\varphi_0,\quad u\in W_\infty, \tag{6}$$

by means of a unique element

$$\varphi=\big(\varphi_m(e),\varphi_{m-1},\varphi_{m-2},\ldots,\varphi_1,\varphi_0\big)\in\widehat{E_1}=\Big(BA\Big(\sum,\mu\Big)\Big)\times\underbrace{R\times R\times\ldots\times R}_{m\ times},$$

where μ is Lebesgue measure on R, $\sum$ is σ-algebra of the μ-measurable subsets $e\subset X$ and $BA\big(\sum,\mu\big)$ is the space of all bounded additive functions $\varphi_m(e)$ defined on $\sum$ such that $\varphi_m(e)=0$ when $\mu(e)=0$ [15]. *The inverse is also true*: *That is, if $\varphi\in E_q$, then* (5) *is bounded on W_p for $1\leq p<\infty$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$. If $\varphi\in\widehat{E_1}$, then* (6) *is bounded on W_∞* [3, 26].

Proof. Theorem 1 can be proved as in [3, 26]. The adjoint operator N^* of N with $\gamma=x_0$ is a linear homeomorphism from E_p^* onto W_p^*. Thus, for any linear bounded functional $F\in W_p^*$ there exists one and only one $\varphi\in E_p^*$ such that $F=N^*\varphi$ or $F(u)=\varphi(Nu)$ for all $u\in W_p$. Any $\varphi\in E_p^*$ is an $(m+1)$-tuple

$$\varphi=(\varphi_m,\varphi_{m-1},\varphi_{m-2},\ldots,\varphi_1,\varphi_0)\in L_p^*\times\underbrace{R^*\times R^*\times\ldots\times R^*}_{m\text{ times}}$$

of the linear bounded functionals $\varphi_m, \varphi_{m-1}, \varphi_{m-2}, \ldots, \varphi_1, \varphi_0$ defined on $L_p,\ R,\ R, \ldots, R,\ R$ respectively. That is

$$F(u) = \varphi_m(u'_{m-1}) + \varphi_{m-1}\big(u_{m-1}(x_0)\big) +$$
$$+\varphi_{m-2}\big(u_{m-2}(x_0)\big) + \ldots + \varphi_1\big(u_1(x_0)\big) + \varphi_0\big(u(x_0)\big), \quad u \in W_p.$$

Furthermore, $R^* = R$, $L_p^* = L_q$ for $1 \le p < \infty$ and $L_\infty^* = BA\big(\sum, \mu\big)$ in the sense of an isomorphism [15]. Therefore, any $F \in W_p^*$ can be represented by (5) for $1 \le p < \infty$ and by (6) for $p = \infty$. The inverse is obtained from (5) and (6).

As can be seen from Theorem 1, undoubtedly, conditions (2) are the most general ones as linear conditions for the continuous differential operator $V_m : W_p \to L_p$ corresponding to equation (1). Namely, each condition $V_i u = z_i$ where V_i, $i = 0, 1, 2, \ldots, m-1$, is a continuous linear functional on W_p can be written in form (2) provided that $p < \infty$.

4. Adjoint operator, adjoint system and solvability conditions. An explicit expression for the adjoint operator V^* of V is investigated in this section. To this end, any $F = (F_m(x), F_{m-1}, \ldots$ $\ldots, F_0) \in E_q$ is taken as a linear bounded functional on E_p, and

$$F(Vu) \equiv \int\limits_{x_0}^{x_1} F_m(x)(V_m u)(x)\,dx + \sum_{i=0}^{m-1} F_i(V_i u), \quad u \in W_p, \tag{7}$$

can be supposed. By substituting expressions (1), (2) and (4) into (7), we have

$$F(Vu) \equiv \int\limits_{x_0}^{x_1} F_m(x) \left[u'_{m-1}(x) + A_0(x) \left\{ u(x_0) + u_1(x_0) \int\limits_{x_0}^{x} \frac{ds_{m-1}}{g_1(s_{m-1})} + \right.\right.$$
$$+u_2(x_0) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \frac{ds_{m-2}\,ds_{m-1}}{g_2(s_{m-2})g_1(s_{m-1})} +$$
$$+u_3(x_0) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \int\limits_{x_0}^{s_{m-2}} \frac{ds_{m-3}\,ds_{m-2}\,ds_{m-1}}{g_3(s_{m-3})g_2(s_{m-2})g_1(s_{m-1})} + \ldots$$
$$\ldots + u_{m-1}(x_0) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \ldots \int\limits_{x_0}^{s_2} \frac{ds_1 \ldots ds_{m-2}\,ds_{m-1}}{g_{m-1}(s_1)\ldots g_2(s_{m-2})g_1(s_{m-1})} +$$
$$\left.\left. + \int\limits_{x_0}^{x} u'_{m-1}(\xi) \int\limits_{\xi}^{x} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} \frac{ds_1 \ldots ds_{m-2}\,ds_{m-1}}{g_{m-1}(s_1)\ldots g_2(s_{m-2})g_1(s_{m-1})}\,d\xi \right\} \right] dx +$$
$$+ \sum_{i=0}^{m-1} F_i \left[a_i^0 u(x_0) + a_i^1 u_1(x_0) + a_i^2 u_2(x_0) + \ldots + a_i^{m-1} u_{m-1}(x_0) + \int\limits_{x_0}^{x_1} B_i(\xi) u'_{m-1}(\xi)\,d\xi \right].$$

Hence, we can obtain the following result by rearrangement:

$$F(Vu) \equiv \int\limits_{x_0}^{x_1} F_m(x)(V_m u)(x)\,dx + \sum_{i=0}^{m-1} F_i(V_i u) =$$

$$= \int\limits_{x_0}^{x_1} (w_m F)(\xi) u'_{m-1}(\xi)\,d\xi +$$

$$+ \sum_{i=1}^{m-1} (w_i F) u_i(x_0) + (w_0 F) u(x_0) \equiv (wF)(u) \quad \forall F \in E_q \quad \forall u \in W_p, \tag{8}$$

where

$$(w_m F)(\xi) = \int\limits_{\xi}^{x_1} F_m(x) A_0(x) \int\limits_{\xi}^{x} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} \frac{ds_1 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-1}(s_1) \ldots g_2(s_{m-2}) g_1(s_{m-1})}\,dx +$$

$$+ F_m(\xi) + \sum_{i=0}^{m-1} F_i B_i(\xi), \quad \xi \in X,$$

$$w_{m-1} F = \int\limits_{x_0}^{x_1} F_m(x) A_0(x) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \ldots \int\limits_{x_0}^{s_2} \frac{ds_1 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-1}(s_1) \ldots g_2(s_{m-2}) g_1(s_{m-1})}\,dx + \sum_{i=0}^{m-1} F_i a_i^{m-1},$$

$$w_{m-2} F = \int\limits_{x_0}^{x_1} F_m(x) A_0(x) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \ldots \int\limits_{x_0}^{s_3} \frac{ds_2 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-2}(s_2) \ldots g_2(s_{m-2}) g_1(s_{m-1})}\,dx + \sum_{i=0}^{m-1} F_i a_i^{m-2},$$

$$w_{m-3} F = \int\limits_{x_0}^{x_1} F_m(x) A_0(x) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \ldots \int\limits_{x_0}^{s_4} \frac{ds_3 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-3}(s_3) \ldots g_2(s_{m-2}) g_1(s_{m-1})}\,dx + \sum_{i=0}^{m-1} F_i a_i^{m-3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \tag{9}$$

$$w_3 F = \int\limits_{x_0}^{x_1} F_m(x) A_0(x) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \int\limits_{x_0}^{s_{m-2}} \frac{ds_{m-3}\, ds_{m-2}\, ds_{m-1}}{g_3(s_{m-3}) g_2(s_{m-2}) g_1(s_{m-1})}\,dx + \sum_{i=0}^{m-1} F_i a_i^3,$$

$$w_2 F = \int\limits_{x_0}^{x_1} F_m(x) A_0(x) \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \frac{ds_{m-2}\, ds_{m-1}}{g_2(s_{m-2}) g_1(s_{m-1})}\,dx + \sum_{i=0}^{m-1} F_i a_i^2,$$

$$w_1 F = \int\limits_{x_0}^{x_1} F_m(x) A_0(x) \int\limits_{x_0}^{x} \frac{ds_{m-1}}{g_1(s_{m-1})}\,dx + \sum_{i=0}^{m-1} F_i a_i^1,$$

$$w_0 F = \int\limits_{x_0}^{x_1} F_m(x) A_0(x)\,dx + \sum_{i=0}^{m-1} F_i a_i^0.$$

The operators $w_m, w_{m-1}, \ldots, w_1$ and w_0 are linear and bounded from the space E_q of the $(m + 1)$-tuples $F = \big(F_m(\xi), F_{m-1}, F_{m-2}, \ldots, F_0\big)$ into the spaces $L_q(X),\ R,\ R, \ldots, R$, respectively. So, the operator $w = (w_m, w_{m-1}, \ldots, w_0)\colon E_q \to E_q$ represented by

$$wF = (w_m F, w_{m-1} F, \ldots, w_0 F)$$

is linear and bounded. By (8) and Theorem 1, V^* becomes w for $1 \leq p < \infty$ and w^* becomes VS for $1 < p \leq \infty$ where S is the inverse of N with $\gamma = x_0$. The operators VS and w can be considered as adjoint operators to each other. Consequently, the equation

$$wF = \varphi, \tag{10}$$

with an unknown $F = \big(F_m(\xi), F_{m-1}, F_{m-2}, \ldots, F_0\big) \in E_q$ and a given

$$\varphi = \big(\varphi_m(\xi), \varphi_{m-1}, \varphi_{m-2}, \ldots, \varphi_0\big) \in E_q$$

can be considered as an adjoint equation of (3) for all $1 \leq p \leq \infty$. The restrictions imposed on the coefficients $A_0, B_0, B_1, \ldots, B_{m-1}$ and the aforementioned assumptions assure that the operator $Q \equiv w - I_q\colon E_q \to E_q$ is completely continuous, where I_q is the identity operator on E_q and $1 < p < \infty$. Thus, (10) is a canonical Fredholm equation and S is a right regularizer of V [3, 5, 6, 16, 17, 26].

(10) can be written in explicit form as the following system of the integro-algebraic equations:

$$\begin{aligned} (w_m F)(\xi) &= \varphi_m(\xi), \quad \xi \in X, \\ w_{m-1} F &= \varphi_{m-1}, \\ w_{m-2} F &= \varphi_{m-2}, \\ &\ldots\ldots\ldots\ldots\ldots \\ w_0 F &= \varphi_0. \end{aligned} \tag{11}$$

As can be seen from (9), the first equation in (11) is generally an integral equation for $F_m(\xi)$ and it may include $F_0,\ F_1, \ldots, F_{m-1}$ as parameters; on the other hand, the other m equations in (11) are algebraic equations for the unknowns $F_0, F_1, \ldots, F_{m-1}$ and they may include some integral functionals on $F_m(\xi)$. In other words, (11) is a system of $m+1$ integro-algebraic equations. This system, called the adjoint system for (3) is constructed by using (8), which is actually a formula of integration by parts in a nonclassical form, and the traditional (classical) form of an adjoint problem is defined by the classical Green's formula of integration by parts [3, 4, 26, 35], therefore, the traditional form only works for some restricted class of problems.

From this point, Fredholm alternative can be stated by the following theorem in the context of solvability of the problem:

Theorem 2. *If $1 < p < \infty$, then $Vu = 0$ has either only the trivial solution or a finite number of linearly independent solutions in W_p:*

(1) *If $Vu = 0$ has only the trivial solution in W_p, then also $wF = 0$ has only the trivial solution in E_q. Then, the operators $V : W_p \to E_p$ and $w\colon E_q \to E_q$ become linear homeomorphisms.*

(2) *If* $Vu = 0$ *has* n *linearly independent solutions* $u_{(1)}, \ldots, u_{(n)}$ *in* W_p, *then* $wF = 0$ *has also* n *linearly independent solutions*

$$F^{\star 1\star} = \big(F_m^{\star 1\star}(x), F_{m-1}^{\star 1\star}, \ldots, F_0^{\star 1\star}\big), \ldots, F^{\star n\star} = \big(F_m^{\star n\star}(x), F_{m-1}^{\star n\star}, \ldots, F_0^{\star n\star}\big)$$

in E_q. *In this case*, (3) *and* (10) *have solutions* $u \in W_p$ *and* $F \in E_q$ *for given* $z \in E_p$ *and* $\varphi \in E_q$ *if and only if the conditions*

$$\int\limits_{x_0}^{x_1} F_m^{\star i\star}(\xi) z_m(\xi)\, d\xi + F_{m-1}^{\star i\star} z_{m-1} + \ldots + F_0^{\star i\star} z_0 = 0, \quad i = 1, \ldots, n, \tag{12}$$

and

$$\int\limits_{x_0}^{x_1} \varphi_m(\xi) u'_{(i),m-1}(\xi)\, d\xi + \varphi_{m-1} u_{(i),m-1}(x_0) + \ldots$$

$$\ldots + \varphi_1 u_{(i),1}(x_0) + \varphi_0 u_{(i)}(x_0) = 0, \quad i = 1, \ldots, n, \tag{13}$$

where

$$u_{(i),1}(x) = g_1(x) u'_{(i)}(x),\ u_{(i),2}(x) = g_2(x) u'_{(i),1}(x),\ \ldots\ ,\ u_{(i),m-1}(x) = g_{m-1}(x) u'_{(i),m-2}(x)$$

are satisfied, respectively [3, 4, 26].

5. Green's functional and a specific adjoint system. Consider the following equation given in the form of a functional identity

$$(wF)(u) = u(x) \quad \forall u \in W_p, \tag{14}$$

where $F = (F_m(\xi), F_{m-1}, \ldots, F_0) \in E_q$ is an unknown $(m+1)$-tuple and $x \in \overline{X}$ is a parameter.

Identity (14) is equivalent to the following system:

$$(w_m F)(\xi) = H(x - \xi) \int\limits_{\xi}^{x} \int\limits_{\xi}^{s_{m-1}} \cdots \int\limits_{\xi}^{s_2} \frac{ds_1 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-1}(s_1) \ldots g_2(s_{m-2}) g_1(s_{m-1})},$$

$$w_{m-1} F = \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \cdots \int\limits_{x_0}^{s_2} \frac{ds_1 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-1}(s_1) \ldots g_2(s_{m-2}) g_1(s_{m-1})},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$w_3 F = \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \int\limits_{x_0}^{s_{m-2}} \frac{ds_{m-3}\, ds_{m-2}\, ds_{m-1}}{g_3(s_{m-3}) g_2(s_{m-2}) g_1(s_{m-1})}, \tag{15}$$

$$w_2 F = \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \frac{ds_{m-2}\, ds_{m-1}}{g_2(s_{m-2}) g_1(s_{m-1})},$$

$$w_1 F = \int\limits_{x_0}^{x} \frac{ds_{m-1}}{g_1(s_{m-1})},$$

$$w_0 F = 1,$$

where $H(x-\xi)$ is a Heaviside function on R, and $\xi \in X$.

Definition 1. *Let* $F(x) = \big(F_m(\xi,x), F_{m-1}(x), F_{m-2}(x), \ldots, F_0(x)\big) \in E_q$ *be an unknown* $(m+1)$*-tuple with a parameter* $x \in \overline{X}$. *If* $F(x)$ *is a solution of* (15) *for a given* $x \in \overline{X}$, *then* $F(x)$ *is called a Green's functional of* V [3, 4, 26, 27, 29].

Theorem 3. *If a Green's functional*

$$F(x) = \big(F_m(\xi,x), F_{m-1}(x), F_{m-2}(x), \ldots, F_0(x)\big)$$

of V *exists*, *then any solution* $u \in W_p$ *of* (3) *can be represented by*

$$u(x) = \int\limits_{x_0}^{x_1} F_m(\xi,x) z_m(\xi)\, d\xi + \sum_{i=0}^{m-1} F_i(x) z_i. \tag{16}$$

Additionally, $\operatorname{Ker} V = \{0\}$ *where* $\operatorname{Ker} V$ *denotes the kernel of* V.

Proof. By (8), we can obtain the following identity:

$$\int\limits_{x_0}^{x_1} F_m(\xi,x) z_m(\xi)\, d\xi + \sum_{i=0}^{m-1} F_i(x) z_i =$$

$$= \int\limits_{x_0}^{x_1} H(x-\xi) \int\limits_{\xi}^{x} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} \frac{ds_1 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-1}(s_1)\ldots g_2(s_{m-2}) g_1(s_{m-1})} u'_{m-1}(\xi)\, d\xi +$$

$$+ \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \ldots \int\limits_{x_0}^{s_2} \frac{ds_1 \ldots ds_{m-2}\, ds_{m-1}}{g_{m-1}(s_1)\ldots g_2(s_{m-2}) g_1(s_{m-1})} u_{m-1}(x_0) + \ldots$$

$$\ldots + \int\limits_{x_0}^{x} \int\limits_{x_0}^{s_{m-1}} \frac{ds_{m-2}\, ds_{m-1}}{g_2(s_{m-2}) g_1(s_{m-1})} u_2(x_0) + \int\limits_{x_0}^{x} \frac{ds_{m-1}}{g_1(s_{m-1})} u_1(x_0) + u(x_0).$$

By (4), the right-hand side of the above identity equals to $u(x)$. Thus, (16) is valid. The verity of Ker $V = \{0\}$ is obtained from (16).

Theorem 3 is related to the necessary condition for the existence a Green's functional of V, and the following theorem is related to the sufficient condition for the existence of a Green's functional.

Theorem 4 [3, 26]. *If* V *has a priori estimate as follows*:

$$\|u\|_{W_p} \le c_0 \|Vu\|_{E_p}, \quad u \in W_p, \tag{17}$$

where c_0 *is a positive constant*, *then a Green's functional of* V *exists*, *where* $1 \le p \le \infty$.

Proof. V has a bounded left inverse by (17). Then the image Im w of w equals to E_q [3, 15]. Hence (15) has a solution

$$F(x) = \big(F_m(\xi, x), F_{m-1}(x), F_{m-2}(x), \ldots, F_0(x)\big) \in E_q$$

for all $x \in \overline{X}$.

***Remark* 1.** If w has a priori estimate as follows:

$$\|F\|_{E_q} \leq c_1 \|wF\|_{E_q}, \quad F \in E_q, \tag{18}$$

where c_1 is a positive constant, then (1), (2) has always a solution $u \in W_p$. If both (17) and (18) are valid, then V and w become homeomorphisms and a unique Green's functional of V exists. Estimates (17) and (18) are valid if $\sum_{i=0}^{m-1} \|B_i\|_{L_q}$ is sufficiently small and

$$\begin{vmatrix} a_0^0 & a_0^1 & \cdots & a_0^{m-1} \\ a_1^0 & a_1^1 & \cdots & a_1^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-1}^0 & a_{m-1}^1 & \cdots & a_{m-1}^{m-1} \end{vmatrix} \neq 0.$$

Theorem 5 [3, 4, 26, 27, 29]**.** *Assume that* $1 < p < \infty$. *If there exists a Green's functional, then it is unique. Additionally, a Green's functional exists if and only if* $Vu = 0$ *has only the trivial solution.*

6. Several applications. In this section, we consider mth order problems involving generally nonlocal condition(s) in order to support the theoretical presentation and to demonstrate the validity, utility and advantages of the proposed approach.

***Example* 1.** Firstly, in order to demonstrate the applicability for a problem with principal coefficient equal to one, we consider the following problem for which Green's function has been presented in [30, 31],

$$(V_m u)(x) \equiv u^{(m)}(x) = f(x), \quad x \in X = (0, 1),$$

$$V_{m-1}u \equiv u(1) - \gamma u(\eta) = 0,$$

$$V_{m-2}u \equiv u^{(m-2)}(0) = 0,$$

$$V_{m-3}u \equiv u^{(m-3)}(0) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$V_1 u \equiv u'(0) = 0,$$

$$V_0 u \equiv u(0) - \beta u(\alpha) = 0,$$

where $f(x) \in L_p$, $\alpha, \eta \in \overline{X}$ and $\beta, \gamma \in R$. Here $g_i(x) = 1$ for $i = 1, 2, \ldots, m-1$, $A_0(x) = 0$, $z_m(x) = f(x)$,

$$a_{m-1}^0 = 1 - \gamma, \qquad z_{m-1} = 0,$$

$$a_{m-1}^1 = \int_0^1 ds_{m-1} - \gamma \int_0^\eta ds_{m-1},$$

$$a_{m-1}^2 = \int_0^1 \int_0^{s_{m-1}} ds_{m-2}\, ds_{m-1} - \gamma \int_0^\eta \int_0^{s_{m-1}} ds_{m-2}\, ds_{m-1},$$

$$a_{m-1}^3 = \int_0^1 \int_0^{s_{m-1}} \int_0^{s_{m-2}} ds_{m-3}\, ds_{m-2}\, ds_{m-1} -$$

$$-\gamma \int_0^\eta \int_0^{s_{m-1}} \int_0^{s_{m-2}} ds_{m-3}\, ds_{m-2}\, ds_{m-1},$$

. . . .

$$a_{m-1}^{m-1} = \int_0^1 \int_0^{s_{m-1}} \ldots \int_0^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1} -$$

$$-\gamma \int_0^\eta \int_0^{s_{m-1}} \ldots \int_0^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1},$$

$$B_{m-1}(\xi) = \int_\xi^1 \int_\xi^{s_{m-1}} \ldots \int_\xi^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1} -$$

$$-\gamma H(\eta - \xi) \int_\xi^\eta \int_\xi^{s_{m-1}} \ldots \int_\xi^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1},$$

$$a_{m-2}^i = \begin{cases} 1 & \text{for} \quad i = m-2, \\ 0 & \text{for} \quad i \neq m-2, \end{cases} \qquad z_{m-2} = 0, \qquad B_{m-2}(\xi) = 0,$$

$$a_{m-3}^i = \begin{cases} 1 & \text{for} \quad i = m-3, \\ 0 & \text{for} \quad i \neq m-3, \end{cases} \qquad z_{m-3} = 0, \qquad B_{m-3}(\xi) = 0,$$

. . . .

$$a_1^i = \begin{cases} 1 & \text{for} \quad i = 1, \\ 0 & \text{for} \quad i \neq 1, \end{cases} \qquad z_1 = 0, \qquad B_1(\xi) = 0,$$

$$a_0^0 = 1-\beta, \qquad z_0 = 0,$$

$$a_0^1 = -\beta \int\limits_0^{\alpha} ds_{m-1},$$

$$a_0^2 = -\beta \int\limits_0^{\alpha} \int\limits_0^{s_{m-1}} ds_{m-2}\, ds_{m-1},$$

$$a_0^3 = -\beta \int\limits_0^{\alpha} \int\limits_0^{s_{m-1}} \int\limits_0^{s_{m-2}} ds_{m-3}\, ds_{m-2}\, ds_{m-1},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_0^{m-1} = -\beta \int\limits_0^{\alpha} \int\limits_0^{s_{m-1}} \ldots \int\limits_0^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1},$$

$$B_0(\xi) = -\beta H(\alpha-\xi) \int\limits_{\xi}^{\alpha} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1},$$

where $H(\alpha-\xi)$ and $H(\eta-\xi)$ are Heaviside functions on R. System (15) corresponding to the problem can be written in the following form:

$$F_m(\xi) + F_0 \left[-\beta H(\alpha-\xi) \int\limits_{\xi}^{\alpha} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1} \right] +$$

$$+F_{m-1} \left[\int\limits_{\xi}^{1} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1} - \right.$$

$$\left. -\gamma H(\eta-\xi) \int\limits_{\xi}^{\eta} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1} \right] =$$

$$= H(x-\xi) \int\limits_{\xi}^{x} \int\limits_{\xi}^{s_{m-1}} \ldots \int\limits_{\xi}^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1}, \quad \xi \in (0,1), \tag{19}$$

$$F_0 \left[-\beta \int\limits_0^{\alpha} \int\limits_0^{s_{m-1}} \ldots \int\limits_0^{s_2} ds_1 \ldots ds_{m-2}\, ds_{m-1} \right] +$$

$$+F_{m-1}\left[\int_0^1\int_0^{s_{m-1}}\ldots\int_0^{s_2}ds_1\ldots ds_{m-2}\,ds_{m-1}-\right.$$

$$\left.-\gamma\int_0^{\eta}\int_0^{s_{m-1}}\ldots\int_0^{s_2}ds_1\ldots ds_{m-2}\,ds_{m-1}\right]=$$

$$=\int_0^{x}\int_0^{s_{m-1}}\ldots\int_0^{s_2}ds_1\ldots ds_{m-2}\,ds_{m-1}, \tag{20}$$

$$F_0\left[-\beta\int_0^{\alpha}\int_0^{s_{m-1}}\ldots\int_0^{s_3}ds_2\ldots ds_{m-2}\,ds_{m-1}\right]+F_{m-2}+$$

$$+F_{m-1}\left[\int_0^1\int_0^{s_{m-1}}\ldots\int_0^{s_3}ds_2\ldots ds_{m-2}\,ds_{m-1}-\right.$$

$$\left.-\gamma\int_0^{\eta}\int_0^{s_{m-1}}\ldots\int_0^{s_3}ds_2\ldots ds_{m-2}\,ds_{m-1}\right]=$$

$$=\int_0^{x}\int_0^{s_{m-1}}\ldots\int_0^{s_3}ds_2\ldots ds_{m-2}\,ds_{m-1}, \tag{21}$$

. . . .

$$F_0\left[-\beta\int_0^{\alpha}\int_0^{s_{m-1}}ds_{m-2}\,ds_{m-1}\right]+F_{m-1}\left[\int_0^1\int_0^{s_{m-1}}ds_{m-2}\,ds_{m-1}-\right.$$

$$\left.-\gamma\int_0^{\eta}\int_0^{s_{m-1}}ds_{m-2}\,ds_{m-1}\right]+F_2=\int_0^{x}\int_0^{s_{m-1}}ds_{m-2}\,ds_{m-1}, \tag{22}$$

$$-F_0\beta\int_0^{\alpha}ds_{m-1}+F_1+F_{m-1}\left[\int_0^1 ds_{m-1}-\gamma\int_0^{\eta}ds_{m-1}\right]=\int_0^{x}ds_{m-1}, \tag{23}$$

$$F_0(1-\beta)+F_{m-1}(1-\gamma)=1. \tag{24}$$

In order to solve (19)–(24), firstly, F_0 and F_{m-1} are uniquely obtained from (20) and (24) under the condition

$$\Delta\equiv(1-\beta)(1-\gamma\eta^{m-1})+(1-\gamma)\beta\alpha^{m-1}\neq 0$$

and then substituting the obtained values of F_0 and F_{m-1} into the other equations in (19)–(24), we have

$$F_0(x) = \frac{1-\gamma\eta^{m-1}-(1-\gamma)x^{m-1}}{\Delta},$$

$$F_i(x) = \frac{x^i}{i!} + \left[\frac{1-\gamma\eta^{m-1}-(1-\gamma)x^{m-1}}{\Delta}\right]\beta\frac{\alpha^{m-2}}{(m-2)!}-$$

$$-\left[\frac{\beta\alpha^{m-1}+(1-\beta)x^{m-1}}{\Delta}\right]\left\{\frac{1-\gamma\eta^i}{i!}\right\}, \quad i=1,2,\ldots,m-2,$$

$$F_{m-1}(x) = \frac{\beta\alpha^{m-1}+(1-\beta)x^{m-1}}{\Delta},$$

$$F_m(\xi,x) = H(x-\xi)\frac{(x-\xi)^{m-1}}{(m-1)!}-$$

$$-\left[\frac{1-\gamma\eta^{m-1}-(1-\gamma)x^{m-1}}{\Delta}\right]H(\alpha-\xi)(-\beta)\frac{(\alpha-\xi)^{m-1}}{(m-1)!}-$$

$$-\left[\frac{\beta\alpha^{m-1}+(1-\beta)x^{m-1}}{\Delta}\right]\times$$

$$\times\left[\frac{(1-\xi)^{m-1}}{(m-1)!}-\gamma H(\eta-\xi)\frac{(\eta-\xi)^{m-1}}{(m-1)!}\right].$$

Hence, under the condition $(1-\beta)(1-\gamma\eta^{m-1})+(1-\gamma)\beta\alpha^{m-1}\neq 0$ we have Green's functional $F(x)=\big(F_m(\xi,x),F_{m-1}(x),F_{m-2}(x),\ldots,F_0(x)\big)$. The first component $F_m(\xi,x)$ of this functional corresponds to Green's function for the problem. After substitution $\xi=s$ into $F_m(\xi,x)$ for notational compatibility, one can see easily that the first component equals to $G(x,s)$ in Example 3 of [31]. Consequently, by Theorem 3, Green's solution can be represented by

$$u(x) = \int\limits_0^1 \left[H(x-\xi)\frac{(x-\xi)^{m-1}}{(m-1)!}-\right.$$

$$-\left[\frac{1-\gamma\eta^{m-1}-(1-\gamma)x^{m-1}}{\Delta}\right]H(\alpha-\xi)(-\beta)\frac{(\alpha-\xi)^{m-1}}{(m-1)!}-$$

$$-\left[\frac{\beta\alpha^{m-1}+(1-\beta)x^{m-1}}{\Delta}\right]\times$$

$$\left.\times\left[\frac{(1-\xi)^{m-1}}{(m-1)!}-\gamma H(\eta-\xi)\frac{(\eta-\xi)^{m-1}}{(m-1)!}\right]\right]f(\xi)\,d\xi$$

since $z_{m-1}=z_{m-2}=\ldots=z_0=0$ for the problem.

***Example* 2.** Now, we consider the following nonlocal problem:

$$(V_m u)(x) \equiv (e^{-x}u^{(m-1)}(x))' = 0, \quad x\in X=(0,1),$$

$$V_i u \equiv u^{(i)}(0) = 1, \quad \text{for} \quad i=m-1,m-2,\ldots,1,$$

$$V_0 u \equiv \int\limits_0^1 u(\tau)\, d\tau = e - 1.$$

Here $g_i(x) = 1$ for $i = 1, 2, \ldots, m-2,\ g_{m-1}(x) = e^{-x},\ A_0(x) = 0,\ z_m(x) = 0,$

$$a^i_{m-1} = \begin{cases} 1 & \text{for} \quad i = m-1, \\ 0 & \text{for} \quad i \neq m-1, \end{cases} \qquad z_{m-1} = 1, \quad B_{m-1}(\xi) = 0,$$

$$a^i_{m-2} = \begin{cases} 1 & \text{for} \quad i = m-2, \\ 0 & \text{for} \quad i \neq m-2, \end{cases} \qquad z_{m-2} = 1, \quad B_{m-2}(\xi) = 0,$$

. . . .

$$a^i_2 = \begin{cases} 1 & \text{for} \quad i = 2, \\ 0 & \text{for} \quad i \neq 2, \end{cases} \qquad z_2 = 1, \quad B_2(\xi) = 0,$$

$$a^i_1 = \begin{cases} 1 & \text{for} \quad i = 1, \\ 0 & \text{for} \quad i \neq 1, \end{cases} \qquad z_1 = 1, \quad B_1(\xi) = 0,$$

$$a^0_0 = \int\limits_0^1 d\tau, \quad z_0 = e - 1,$$

$$a^1_0 = \int\limits_0^1 \int\limits_0^\tau ds_{m-1}\, d\tau,$$

$$a^2_0 = \int\limits_0^1 \int\limits_0^\tau \int\limits_0^{s_{m-1}} ds_{m-2}\, ds_{m-1}\, d\tau,$$

. . . .

$$a^{m-2}_0 = \int\limits_0^1 \int\limits_0^\tau \int\limits_0^{s_{m-1}} \ldots \int\limits_0^{s_3} ds_2 \ldots ds_{m-2}\, ds_{m-1}\, d\tau,$$

$$a^{m-1}_0 = \int\limits_0^1 \int\limits_0^\tau \int\limits_0^{s_{m-1}} \ldots \int\limits_0^{s_2} e^{s_1}\, ds_1 \ldots ds_{m-2}\, ds_{m-1}\, d\tau,$$

$$B_0(\xi) = \int\limits_\xi^1 \int\limits_\xi^\tau \int\limits_\xi^{s_{m-1}} \ldots \int\limits_\xi^{s_2} e^{s_1}\, ds_1 \ldots ds_{m-2}\, ds_{m-1}\, d\tau.$$

System (15) corresponding to the problem can be written in the form

$$F_m(\xi) + F_0\left[\int\limits_\xi^1\int\limits_\xi^\tau\int\limits_\xi^{s_{m-1}}\ldots\int\limits_\xi^{s_2} e^{s_1}\,ds_1\ldots ds_{m-2}\,ds_{m-1}\,d\tau\right] =$$

$$= H(x-\xi)\int\limits_\xi^x\int\limits_\xi^{s_{m-1}}\ldots\int\limits_\xi^{s_2} e^{s_1}\,ds_1\ldots ds_{m-2}\,ds_{m-1}, \quad \xi \in (0,1),$$

$$F_0\left[e - \frac{1}{(m-1)!} - \frac{1}{(m-2)!} - \ldots - 1 - 1\right] + F_{m-1} =$$

$$= e^x - \frac{x^{m-2}}{(m-2)!} - \frac{x^{m-3}}{(m-3)!} - \ldots - x - 1,$$

$$\frac{F_0}{(m-1)!} + F_{m-2} = \frac{x^{m-2}}{(m-2)!}, \tag{25}$$

. . . .

$$\frac{F_0}{3!} + F_2 = \frac{x^2}{2!},$$

$$\frac{F_0}{2!} + F_1 = x,$$

$$F_0 = 1.$$

Substituting the obtained value of F_0 in (25) into the other equations, we have

$$F_0(x) = 1,$$

$$F_i(x) = \frac{x^i}{i!} - \frac{1}{(i+1)!}, \quad i = 1,2,\ldots,m-2,$$

$$F_{m-1}(x) = e^x - e - \sum_{i=0}^{m-2}\frac{x^i}{i!} + \sum_{i=1}^{m}\frac{1}{(i-1)!},$$

$$F_m(\xi,x) = H(x-\xi)\int\limits_\xi^x\int\limits_\xi^{s_{m-1}}\ldots\int\limits_\xi^{s_2} e^{s_1}\,ds_1\ldots ds_{m-2}\,ds_{m-1} -$$

$$-\int\limits_\xi^1\int\limits_\xi^\tau\int\limits_\xi^{s_{m-1}}\ldots\int\limits_\xi^{s_2} e^{s_1}\,ds_1\ldots ds_{m-2}\,ds_{m-1}\,d\tau.$$

Thus, Green's functional $F(x) = \big(F_m(\xi,x), F_{m-1}(x), F_{m-2}(x), \ldots, F_0(x)\big)$ has been determined. The first component $F_m(\xi,x)$ of this functional corresponds to Green's function for the problem. Consequently, by Theorem 3, we have

$$u(x) = e^x$$

since $z_m(\xi) = 0,\ z_{m-1} = z_{m-2} = \ldots = z_1 = 1,\ z_0 = e-1$ for the problem.

7. Conclusion. The introduced method for the second and third order linear ODEs with variable principal coefficient involving nonlocal conditions in [3, 26] is extended to the mth order linear ODEs with variable principal coefficient involving nonlocal conditions. As can be seen from the theory and illustrations, the proposed method principally is different from the known classical construction methods of Green's function. The structural properties of the space of solutions instead of the classical Green's formula of integration by parts are used. The proposed method can successfully be employed also for the problems resulting from the addition of some delayed, loaded (forced) or neutral terms to the operator V_m, provided that the linearity for the operator is conserved. The applicability easily to a very wide class of linear and some linear boundary-value problems involving linear nonlocal conditions is a natural property of the method. Obviously, this method has a significant advantage especially for cases when the adjoint problem can not be constructed in the classical sense. Because of these properties, we believe that it is one of the scarce methods which aim at the derivation of a solution to such problems by reducing to an integral equation in general manner, the results may be useful for the studies which aim at searching for the existence and uniqueness of the solutions to the problems of interest, and the work is significant for its contribution to the feasibility of such studies also with the weak assumptions such as L_p-integrability and boundedness instead of the strong assumptions such as continuities for the coefficient and right-hand side functions of equation.

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