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ON CÈSARO AND COPSON NORMS OF NONNEGATIVE SEQUENCES

ПРО НОРМИ ЧЕЗАРО І КОПСОНА НЕВІД'ЄМНИХ ПОСЛІДОВНОСТЕЙ

The Cèsaro and Copson norms of a nonnegative sequence are l^p-norms of its arithmetic means and the corresponding conjugate means. It is well known that, for $1 < p < \infty$, these norms are equivalent. In 1996, G. Bennett posed the problem of finding the best constants in the associated inequalities. The solution of this problem requires the evaluation of four constants. Two of them were found by G. Bennett. We find one of the two unknown constants and also prove one optimal weighted-type estimate regarding the remaining constant.

Норми Чезаро і Копсона невід'ємних послідовностей визначаються як l^p-норми їхніх арифметичних середніх і відповідних спряжених середніх. Відомо, що для $1 < p < \infty$ ці норми еквівалентні. У 1996 р. Г. Беннетт поставив задачу про знаходження найкращих сталих у нерівностях, що описують цю еквівалентність. Розв'язок цієї задачі вимагає оцінок чотирьох сталих. Дві з них були знайдені Г. Беннеттом. У цій статті знайдено одну з двох невідомих сталих. Доведено також оптимальну оцінку вагового типу для сталої, що залишилася.

1. Introduction. Let $1 < p < \infty$. Denote by $\mathrm{ces}(p)$ the set of all sequences $\mathbf{x} = \{x_n\}$ such that

$$\|\mathbf{x}\|_{\mathrm{ces}(p)} = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p \right)^{1/p} < \infty.$$

By Hardy's inequality [4] (Ch. 9), $l^p \subset \mathrm{ces}(p),\ 1 < p < \infty$.

We consider also the space $\mathrm{cop}(p)$ which is defined as the set of all sequences $\mathbf{x} = \{x_n\}$ such that

$$\|\mathbf{x}\|_{\mathrm{cop}(p)} = \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{|x_k|}{k} \right)^p \right)^{1/p} < \infty.$$

For any $1 < p < \infty$, $\mathrm{ces}(p) = \mathrm{cop}(p)$ (see [1], § 10). Moreover, G. Bennett [1] proved the following theorem.

Theorem 1.1. *If* $p \geq 2$, *then*

$$\|\mathbf{x}\|_{\mathrm{ces}(p)} \leq \zeta(p)^{1/p} \|\mathbf{x}\|_{\mathrm{cop}(p)}, \tag{1.1}$$

where

$$\zeta(p) = \sum_{n=1}^{\infty} \frac{1}{n^p}, \qquad 1 < p < \infty.$$

If $1 < p \leq 2$, *then*

$$\|\mathbf{x}\|_{\mathrm{cop}(p)} \leq (p-1)^{1/p} \|\mathbf{x}\|_{\mathrm{ces}(p)}. \tag{1.2}$$

The constants are both best possible.

Furthermore, G. Bennett [1] posed the problem: *find the best constants in the inequalities*

$$\|\mathbf{x}\|_{\mathrm{ces}(p)} \le A_p \|\mathbf{x}\|_{\mathrm{cop}(p)} \quad \textit{for} \quad 1 < p < 2 \tag{1.3}$$

and

$$\|\mathbf{x}\|_{\mathrm{cop}(p)} \le B_p \|\mathbf{x}\|_{\mathrm{ces}(p)} \quad \textit{for} \quad p > 2. \tag{1.4}$$

Similar relationships between L^p-norms of the Hardy operator and its dual for functions on $(0, +\infty)$ were studied in the work [5].

Denote by $\mathcal{M}^+(\mathbb{R}_+)$ the class of all nonnegative measurable functions on $\mathbb{R}_+ \equiv (0, +\infty)$. Let $f \in \mathcal{M}^+(\mathbb{R}_+)$. Set

$$Hf(x) = \frac{1}{x}\int_0^x f(t)\,dt \quad \text{and} \quad H^*f(x) = \int_x^\infty \frac{f(t)}{t}\,dt.$$

By Hardy's inequalities [4] (Ch. 9), these operators are bounded in $L^p(\mathbb{R}_+)$ for any $1 < p < \infty$. Furthermore, it is easy to show that for any $1 < p < \infty$ the L^p-norms of Hf and H^*f are equivalent.

The main result in [5] is the following theorem.

Theorem 1.2. *Let* $f \in \mathcal{M}^+(\mathbb{R}_+)$ *and let* $1 < p < \infty$. *Then*

$$(p-1)\|Hf\|_p \le \|H^*f\|_p \le (p-1)^{1/p}\|Hf\|_p \tag{1.5}$$

if $1 < p \le 2$, *and*

$$(p-1)^{1/p}\|Hf\|_p \le \|H^*f\|_p \le (p-1)\|Hf\|_p \tag{1.6}$$

if $2 \le p < \infty$. *All constants in* (1.5) *and* (1.6) *are the best possible.*

As it was observed in [5], the first inequality in (1.6) can be derived from the results obtained in [3] and [6].

We observe also that the first inequality in (1.6) and the second inequality in (1.5) were obtained in [1] (§ 21). We didn't mention this fact in the paper [5] because we learned about the monograph [1] when [5] was already published.

Note that the constant in the first inequality in (1.6) differs from that in (1.1).

One of the main results of this paper is that the best constant in (1.4) is $B_p = p - 1$. Our proof of this result doesn't rely on the second inequality in (1.6) (apparently it cannot be directly derived from the latter inequality).

As for the best constant in the inequality (1.3), this problem remains open. However, we prove the following result: *if* $1 < p \le 2$, *then*

$$\sum_{n=1}^\infty \left(\frac{1}{n}\sum_{k=1}^n x_k\right)^p \le (p-1)^{-(p-1)} \sum_{n=1}^\infty \left(\sum_{k=n}^\infty \frac{x_k}{k} w_p(k)\right)^p, \tag{1.7}$$

where

$$w_p(n) = \left(n^{p-1} \sum_{k=n}^{\infty} \frac{1}{k^p} \right)^{1/p}.$$

The constant in (1.7) *is optimal.*

Since $w_p(n) \le \zeta(p)^{1/p}$ (see Lemma 2.1 below), (1.7) implies that

$$\|\mathbf{x}\|_{\mathrm{ces}(p)} \le \frac{\zeta(p)^{1/p}}{(p-1)^{1/p'}} \|\mathbf{x}\|_{\mathrm{cop}(p)}, \qquad 1 < p \le 2. \tag{1.8}$$

This result agrees with (1.1) for $p = 2$. Besides, the constant in (1.8) has the optimal asymptotic behaviour as $p \to 1+$. However, it is not difficult to show that for $1 < p < 2$ the *value* of the constant in (1.8) is not the best possible.

We observe that the main role in this paper belongs to Lemma 2.3 which gives explicit link between Cèsaro and Copson norms.

2. Lemmas. The following lemma was proved in [1, p. 14].

Lemma 2.1. *Let* $1 < p < \infty$. *Set*

$$\nu_p(n) = n^{p-1} \sum_{k=n}^{\infty} \frac{1}{k^p}, \qquad n \in \mathbb{N}. \tag{2.1}$$

Then $\{\nu_p(n)\}$ *strictly decreases as* n *increases and*

$$\frac{1}{p-1} < \nu_p(n) \le \zeta(p), \qquad n \in \mathbb{N}. \tag{2.2}$$

The constants in (2.2) *are both best possible and there is equality on the right only when* $n = 1$.

Remark **2.1.** We observe that the decrease of the sequence $\{\nu_p(n)\}$ was stated in [1] without proof. However, the proof follows immediately from the well-known representation

$$\sum_{k=n}^{\infty} \frac{1}{k^p} = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{t^{p-1} e^{-nt}}{1 - e^{-t}}\, dt. \tag{2.3}$$

Indeed, from (2.3) we have

$$\nu_p(n) = \frac{1}{\Gamma(p)} \int_0^{\infty} \frac{z^{p-1} e^{-z}}{n(1 - e^{-z/n})}\, dz.$$

It remains to note that the function $\varphi(x) = x/(1 - e^{-x})$, $x > 0$, increases on $(0, \infty)$.

Observe also that

$$\sum_{k=n+1}^{\infty} \frac{1}{k^p} \le \int_n^{\infty} \frac{dx}{x^p} = \frac{1}{(p-1)n^{p-1}}. \tag{2.4}$$

Together with the left inequality in (2.2), this implies that

$$\lim_{n\to\infty} \nu_p(n) = \frac{1}{p-1}. \tag{2.5}$$

As usual, we set $p' = p/(p-1)$ for $1 < p < \infty$.

Lemma 2.2. *Let $1 < p < \infty$ and let $N \geq 2$ be an integer number. Set*

$$x_n^{(N)} = \begin{cases} n^{-1/p} & \text{for} \quad n = 1, \ldots, N, \\ 0 & \text{for} \quad n > N \end{cases}$$

and $\mathbf{x}^{(N)} = \{x_n^{(N)}\}$. Then

$$\lim_{N\to\infty} \frac{\|\mathbf{x}^{(N)}\|_{\text{ces}(p)}}{\ln N} = p' \tag{2.6}$$

and

$$\lim_{N\to\infty} \frac{\|\mathbf{x}^{(N)}\|_{\text{cop}(p)}}{\ln N} = p. \tag{2.7}$$

Proof. For the sum $s_n^{(N)} = \sum_{k=1}^n x_k^{(N)}$ we have

$$p'\big[(n+1)^{1/p'} - 1\big] \leq s_n^{(N)} \leq p' n^{1/p'}, \qquad 1 \leq n \leq N,$$

and $s_n^{(N)} = s_N^{(N)}$ for $n > N$. It easily follows that

$$(p')^p (1 - \varepsilon_N) \ln N \leq \sum_{n=1}^{\infty} \left(\frac{s_n^{(N)}}{n} \right)^p \leq (p')^p (\ln N + p'),$$

where $\varepsilon_N > 0$, $\varepsilon_N \to 0$ as $N \to \infty$. These inequalities imply (2.6).

Further, for $2 \leq n \leq N$,

$$\eta_n^{(N)} = \sum_{k=n}^{\infty} \frac{x_k^{(N)}}{k} = \sum_{k=n}^{N} k^{-1/p-1} \leq \int_{n-1}^{N} \frac{dx}{x^{1+1/p}} \leq \frac{p}{(n-1)^{1/p}}.$$

Thus,

$$\sum_{n=1}^{\infty} \big(\eta_n^{(N)}\big)^p \leq p^p \ln N + C,$$

where C is a constant. On the other hand,

$$\eta_n^{(N)} \geq p\big(n^{-1/p} - (N+1)^{-1/p}\big), \quad 1 \leq n \leq N.$$

From here, for any $\varepsilon > 0$ and sufficiently big N,

$$\sum_{n=1}^{N} \big(\eta_n^{(N)}\big)^p \geq p^p (1 - \varepsilon) \ln N.$$

These estimates imply (2.7).

Lemma 2.2 is proved.

Now we prove our main lemma. Throughout this paper, for a nonnegative sequence $\{x_n\}$, we denote

$$s_n = \sum_{k=1}^{n} x_k, \qquad \xi_n = \frac{x_n}{n}, \qquad \eta_n = \sum_{k=n}^{\infty} \xi_k. \tag{2.8}$$

Also, we use notation (2.1).

Lemma 2.3. *Let $\{x_n\}$ be a positive sequence and let $1 < p < \infty$. Set*

$$\alpha_n = \int_0^1 \left(\frac{s_n}{n} - y\xi_n\right)^{p-1} dy,$$

$$\beta_n = \int_0^1 (\eta_{n+1} + y\xi_n)^{p-1} dy,$$

and

$$\gamma_n = \int_0^1 \left(\frac{s_n}{n} - y\xi_n\right)(\eta_{n+1} + y\xi_n)^{p-2} dy.$$

Then

$$\sum_{n=1}^{\infty} \left(\frac{s_n}{n}\right)^p = p \sum_{n=1}^{\infty} x_n \alpha_n \nu_p(n) \tag{2.9}$$

and

$$\sum_{n=1}^{\infty} \eta_n^p = p \sum_{n=1}^{\infty} x_n \beta_n = p(p-1) \sum_{n=1}^{\infty} x_n \gamma_n. \tag{2.10}$$

Proof. First, applying summation by parts, we have ($s_0 = 0$)

$$I_p = \sum_{n=1}^{\infty} \left(\frac{s_n}{n}\right)^p = \sum_{n=1}^{\infty} (s_n^p - s_{n-1}^p) \sum_{k=n}^{\infty} \frac{1}{k^p} =$$

$$= p \sum_{n=1}^{\infty} \frac{\nu_p(n)}{n^{p-1}} x_n \int_0^1 (s_n - yx_n)^{p-1} dy = p \sum_{n=1}^{\infty} x_n \alpha_n \nu_p(n),$$

which gives (2.9).

Further,

$$J_p = \sum_{n=1}^{\infty} \eta_n^p = \sum_{n=1}^{\infty} n(\eta_n^p - \eta_{n+1}^p) = p \sum_{n=1}^{\infty} x_n \int_0^1 (\eta_{n+1} + y\xi_n)^{p-1} dy, \tag{2.11}$$

and we obtain the left equality in (2.10).

Next, we apply summation by parts one more time. Set $\varphi_n(y) = \eta_{n+1} + y\xi_n,\ y \in [0, 1]$. Then

$$\varphi_n(y)^{p-1} - \varphi_{n+1}(y)^{p-1} = (\eta_{n+2} + \xi_{n+1} + y\xi_n)^{p-1} - (\eta_{n+2} + y\xi_{n+1})^{p-1} =$$

$$= (p-1) \int_{y\xi_{n+1}}^{\xi_{n+1}+y\xi_n} (\eta_{n+2} + u)^{p-2} du.$$

Thus,

$$\int_0^1 \left[\varphi_n(y)^{p-1} - \varphi_{n+1}(y)^{p-1}\right] dy =$$

$$= (p-1)\left[\int_0^1 \int_{y\xi_{n+1}}^{\xi_{n+1}} (\eta_{n+2}+u)^{p-2}\, dudy + \int_0^1 \int_0^{y\xi_n} (\eta_{n+1}+u)^{p-2}\, dudy\right] =$$

$$= (p-1)\left[\xi_{n+1}\int_0^1 y(\eta_{n+2}+y\xi_{n+1})^{p-2}\, dy + \xi_n \int_0^1 (1-y)(\eta_{n+1}+y\xi_n)^{p-2}\, dy\right].$$

Using this equality and applying summation by parts in (2.11), we obtain

$$J_p = p(p-1)\sum_{n=1}^{\infty} s_n \left[\xi_{n+1}\int_0^1 y(\eta_{n+2}+y\xi_{n+1})^{p-2} dy + \xi_n \int_0^1 (1-y)(\eta_{n+1}+y\xi_n)^{p-2} dy\right].$$

As above, we assume that $s_0 = 0$. We have

$$y\sum_{n=1}^{\infty} \xi_{n+1}(\eta_{n+2}+y\xi_{n+1})^{p-2} s_n + (1-y)\sum_{n=1}^{\infty} \xi_n(\eta_{n+1}+y\xi_n)^{p-2} s_n =$$

$$= y\sum_{n=1}^{\infty} \xi_n(\eta_{n+1}+y\xi_n)^{p-2} s_{n-1} + (1-y)\sum_{n=1}^{\infty} \xi_n(\eta_{n+1}+y\xi_n)^{p-2} s_n =$$

$$= \sum_{n=1}^{\infty} \xi_n(s_n - yx_n)(\eta_{n+1}+y\xi_n)^{p-2}.$$

Thus,

$$J_p = p(p-1)\sum_{n=1}^{\infty} x_n \int_0^1 \left(\frac{s_n}{n} - y\xi_n\right)(\eta_{n+1}+y\xi_n)^{p-2}\, dy,$$

which is the right equality in (2.10).

Lemma 2.3 is proved.

3. Main results. As above, we use notations (2.1) and (2.8). First we obtain the optimal constant in the inequality (1.4).

Theorem 3.1. *Let* $\mathbf{x} = \{x_n\}$ *be a nonnegative sequence and let* $2 \le p < \infty$. *Then*

$$\|\mathbf{x}\|_{\mathrm{cop}(p)} \le (p-1)\|\mathbf{x}\|_{\mathrm{ces}(p)}. \tag{3.1}$$

The constant is optimal.

Proof. We shall use notations introduced in Lemma 2.3. First we observe that by Hölder's inequality

$$\gamma_n = \int\limits_0^1 \left(\frac{s_n}{n} - y\xi_n\right) (\eta_{n+1} + y\xi_n)^{p-2}\, dy \le$$

$$\le \left(\int\limits_0^1 \left(\frac{s_n}{n} - y\xi_n\right)^{p-1} dy\right)^{1/(p-1)} \left(\int\limits_0^1 (\eta_{n+1} + y\xi_n)^{p-1} dy\right)^{(p-2)/(p-1)} =$$

$$= \alpha_n^{1/(p-1)} \beta_n^{(p-2)/(p-1)}.$$

Using the second of equalities (2.10), and applying again the Hölder inequality, we get

$$J_p = \sum_{n=1}^{\infty} \eta_n^p = p(p-1) \sum_{n=1}^{\infty} x_n \gamma_n \le p(p-1) \sum_{n=1}^{\infty} x_n \alpha_n^{1/(p-1)} \beta_n^{(p-2)/(p-1)} \le$$

$$\le (p-1) \left(p \sum_{n=1}^{\infty} x_n \alpha_n\right)^{1/(p-1)} \left(p \sum_{n=1}^{\infty} x_n \beta_n\right)^{(p-2)/(p-1)}.$$

We observe that by (2.2)

$$\sum_{n=1}^{\infty} x_n \alpha_n \le (p-1) \sum_{n=1}^{\infty} x_n \alpha_n \nu_p(n). \tag{3.2}$$

As above, set $I_p = \sum_{n=1}^{\infty} (s_n/n)^p$. Using (3.2), (2.9), and the first equality in (2.10), we have

$$J_p \le (p-1)^{p'} I_p^{1/(p-1)} J_p^{(p-2)/(p-1)}.$$

From here, $J_p \le (p-1)^p I_p$, which yields (3.1). It follows immediately from Lemma 2.2 that the constant in (3.1) is the best possible.

***Remark* 3.1.** It is clear that inequality (3.2) is strict except the case when $x_n = 0$ for all $n \in \mathbb{N}$. Thus, the equality in (3.1) holds if and only if $\mathbf{x} = \mathbf{0}$.

Applying Lemma 2.3, we obtain also the following result.

Theorem 3.2. *Let $\{x_n\}$ be a nonnegative sequence and let $1 < p \le 2$. Set $w_p(n) = \nu_p(n)^{1/p}$. Then*

$$\left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} x_k\right)^p\right)^{1/p} \le$$

$$\le (p-1)^{-1/p'} \left(\sum_{n=1}^{\infty} \left(\sum_{k=n}^{\infty} \frac{x_k}{k} w_p(k)\right)^p\right)^{1/p}. \tag{3.3}$$

The constant is optimal.

Proof. We keep notations (2.8). Also, we set

$$\tilde{x}_n = x_n w_p(n), \qquad \tilde{s}_n = \sum_{k=1}^{n} \tilde{x}_k, \qquad \tilde{\xi}_n = \frac{\tilde{x}_n}{n}, \qquad \text{and} \qquad \tilde{\eta}_n = \sum_{k=n}^{\infty} \tilde{\xi}_k.$$

Since the sequence $\{\nu_p(n)\}$ is decreasing, we have $\tilde{s}_n \geq w_p(n)s_n$. Thus, applying Hölder's inequality with the exponent $p-1 \in (0,1]$, we obtain

$$\tilde{\gamma}_n = \int_0^1 \left(\frac{\tilde{s}_n}{n} - y\tilde{\xi}_n\right)(\tilde{\eta}_{n+1} + y\tilde{\xi}_n)^{p-2}dy \geq$$

$$\geq w_p(n)\left(\int_0^1 \left(\frac{s_n}{n} - y\xi_n\right)^{p-1} dy\right)^{1/(p-1)} \left(\int_0^1 (\tilde{\eta}_{n+1} + y\tilde{\xi}_n)^{p-1}dy\right)^{(p-2)/(p-1)}.$$

As above, we denote

$$\alpha_n = \int_0^1 \left(\frac{s_n}{n} - y\xi_n\right)^{p-1} dy.$$

Further, set

$$\tilde{\beta}_n = \int_0^1 \left(\tilde{\eta}_{n+1} + y\tilde{\xi}_n\right)^{p-1}dy.$$

Applying the right equality in (2.10) to the sequence $\{\tilde{x}_n\}$, we have

$$\tilde{J}_p = \sum_{n=1}^{\infty} \tilde{\eta}_n^p = p(p-1)\sum_{n=1}^{\infty} \tilde{x}_n\tilde{\gamma}_n.$$

Using estimate for $\tilde{\gamma}_n$ obtained above, and applying again Hölder's inequality, we get

$$\tilde{J}_p = p(p-1)\sum_{n=1}^{\infty} \tilde{x}_n\tilde{\gamma}_n \geq p(p-1)\sum_{n=1}^{\infty}(\nu_p(n)x_n\alpha_n)^{1/(p-1)}(\tilde{x}_n\tilde{\beta}_n)^{(p-2)/(p-1)} \geq$$

$$\geq (p-1)\left(p\sum_{n=1}^{\infty} x_n\alpha_n\nu_p(n)\right)^{1/(p-1)} \left(p\sum_{n=1}^{\infty} \tilde{x}_n\tilde{\beta}_n\right)^{(p-2)/(p-1)}.$$

Now we apply the left equality in (2.10) to $\{\tilde{x}_n\}$ and equality (2.9) to $\{x_n\}$. This gives that

$$\tilde{J}_p \geq (p-1)I_p^{1/(p-1)}\tilde{J}_p^{(p-2)/(p-1)},$$

where

$$I_p = \sum_{n=1}^{\infty}\left(\frac{s_n}{n}\right)^p.$$

The latter inequality implies (3.3).

Let now $\mathbf{x}^{(N)} = \left\{x_n^{(N)}\right\}$ be the sequence defined in Lemma 2.2. By (2.6),

$$\lim_{N\to\infty} \frac{\|\mathbf{x}^{(\mathbf{N})}\|_{\mathrm{ces}(\mathbf{p})}}{\ln N} = \frac{p}{p-1}.$$

Further, set $\tilde{x}_n^{(N)} = x_n^{(N)} w_p(n)$ and $\tilde{\mathbf{x}}^{(N)} = \{\tilde{x}_n^{(N)}\}$. It easily follows from (2.5) and (2.7) that

$$\lim_{N\to\infty} \frac{\|\tilde{\mathbf{x}}^{(N)}\|_{\mathrm{cop}(p)}}{\ln N} = \frac{p}{(p-1)^{1/p}}.$$

It shows that the constant in (3.3) cannot be improved.

Theorem 3.2 is proved.

Remark **3.2.** As we have mentioned above, we do not know the value of the best constant A_p in (1.3). By Lemma 2.2,

$$A_p \ge \frac{1}{p-1}. \tag{3.4}$$

On the other hand, we have

$$s_n = \sum_{k=1}^{n} k(\eta_k - \eta_{k+1}) \le \sum_{k=1}^{n} \eta_k \tag{3.5}$$

and by Hardy's inequality

$$\left(\sum_{n=1}^{\infty}\left(\frac{s_n}{n}\right)^p\right)^{1/p} \le \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\eta_k\right)^p\right)^{1/p} \le p'\left(\sum_{n=1}^{\infty}\eta_n^p\right)^{1/p}.$$

Thus,

$$A_p \le p'. \tag{3.6}$$

It follows from (3.4) and (3.6) that

$$\lim_{p\to 1}(p-1)A_p = 1. \tag{3.7}$$

Clearly, estimate (3.6) is too rough if $p \to 2$. Indeed, by (1.1), the best constant $A_2 = \zeta(2) = \pi^2/6 < 2$.

Another upper bound for A_p can be derived from Theorem 3.2. Since $\nu_p(n)$ decreases (see Lemma 2.1), it follows from (3.3) that for $1 < p \le 2$

$$\sum_{n=1}^{\infty}\left(\frac{s_n}{n}\right)^p \le \frac{1}{(p-1)^{p-1}}\sum_{n=1}^{\infty}\nu_p(n)\eta_n^p. \tag{3.8}$$

Further, by (2.2), we have

$$\sum_{n=1}^{\infty}\left(\frac{s_n}{n}\right)^p \le \frac{\zeta(p)}{(p-1)^{p-1}}\sum_{n=1}^{\infty}\eta_n^p, \quad 1 < p \le 2. \tag{3.9}$$

For $p = 2$ this inequality coincides with (1.1). Furthermore, if $\tilde{A}_p = \zeta(p)^{1/p}(p-1)^{-1/p'}$, then

$$\lim_{p\to 1}(p-1)\tilde{A}_p = 1,$$

which agrees with (3.7). However, for $1 < p < 2$ the constant in (3.9) is not optimal. It can be easily shown, using (3.8) and (3.5).

Finally, concerning the best constant A_p in (1.3), we can only state that it satisfies the inequalities

$$\frac{1}{p-1} \le A_p \le \frac{\zeta(p)^{1/p}}{(p-1)^{1/p'}}.$$

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