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RESONANT EQUATIONS WITH CLASSICAL ORTHOGONAL POLYNOMIALS. I

РЕЗОНАНСНІ РІВНЯННЯ З КЛАСИЧНИМИ ОРТОГОНАЛЬНИМИ ПОЛІНОМАМИ. I

In the present paper, we study some resonant equations related to the classical orthogonal polynomials and propose an algorithm of finding their particular and general solutions in the explicit form. The algorithm is especially suitable for the computer algebra tools, such as Maple. The resonant equations form an essential part of various applications e.g. of the efficient functional-discrete method aimed at the solution of operator equations and eigenvalue problems. These equations also appear in the context of supersymmetric Casimir operators for the di-spin algebra, as well as for the square operator equations $A^2u = f$; e.g., for the biharmonic equation.

Вивчаються деякі резонансні рівняння, що мають відношення до класичних ортогональних поліномів. Запропоновано алгоритм знаходження їхніх частинних та загальних розв'язків у явному вигляді. Цей алгоритм найкраще підходить для методів комп'ютерної алгебри, таких як Maple. Резонансні рівняння складають суттєву частину багатьох застосувань, зокрема ефективного функціонально-дискретного методу, що застосовується при розв'язанні операторних рівнянь та задач на власні значення. Такі рівняння також з'являються в контексті суперсиметричних операторів Казиміра для ді-спінової алгебри, а також для квадратичних операторних рівнянь $A^2u = f$, наприклад для бігармонічного рівняння.

1. Introduction. Polynomials, especially the orthogonal polynomials [8, 9, 26] is a very important and often used mathematical tool. One of the application fields for polynomials are differential equations. Some of them possess polynomial solutions and the solution of other ones can be approximated by polynomials. The topic of the present paper is a special class of resonant differential equations with differential operators related to the classical orthogonal polynomials.

There are various definitions of resonant equations (see, e.g., [1, 2]), where a boundary-value problem is called resonant, when the operator, defined by the differential equation and by the boundary conditions does not possess the inverse. In the present paper we follow the definition from [6, 17, 19] and call an equation of the form $Lf = g$ with $Lg = 0$ resonant. In other words, the right-hand side of the resonant equation belongs to the kernel $K(L)$ of the operator L. These equations are interesting both from theoretical point of view and from the practical side in various applications. For example, in [18] was proposed the so-called functional-discrete method (FD-method) for solving of operator equations and of eigenvalue problems. The method is based on the ideas of perturbation of the operator involved and on the homotopy idea. This approach was applied to various problems in particulary to eigenvalue problems in [10 – 14] and has been proven to possess a super exponential convergence rate. An essential part of the algorithm are some inhomogeneous equations with a resonant component in the sense of the definition above.

A simple but profound example showing the principally different behaviors of the solutions in the resonant and in the non-resonant cases gives the following simple differential equation (so-called vibration equation):

$$\frac{d^2y}{dt^2} + \mu^2 y = \sin(\nu t). \tag{1.1}$$

There exists a particular solution of the form

$$y(t) = \begin{cases} \dfrac{1}{\mu^2 - \nu^2} \sin(\nu t), & \text{if} \quad \nu \neq \pm\mu, \\ -\dfrac{t}{2\mu} \cos(\mu t), & \text{if} \quad \nu = \pm\mu, \end{cases} \tag{1.2}$$

which is resonant for $\nu = \pm\mu$ $\Big($in this case the right-hand side $\sin(\mu t)$ solves the homogeneous equation $\dfrac{d^2y}{dt^2} + \mu^2 y = 0\Big)$ and non-resonant otherwise. The vibration amplitude in the resonant case tends to infinity if the stimulating vibration frequency ν on the right-hand side of the equation tends to the resonant eigenfrequency μ of the system described by the differential operator on the left-hand side. The depth of difference between the non-resonant and the resonant solutions is discussed in [7].

The example above can be embedded into the following abstract framework. Let some system be described by an operator equation $Au - \lambda u = f$ in some Hilbert space H, where the operator A is completely defined by its spectrum, i.e., by the eigenvalues λ_j, $j = 1, 2, \ldots$, and the corresponding eigenvectors u_j, $j = 1, 2, \ldots$. Here λ is a parameter characterizing the system. If the right-hand side is of the form $f = \alpha u_k$ for the fixed α, k, i.e., f solves the equation $(A - \lambda_k)f = 0$, then the solution of the corresponding operator equation is

$$u = \frac{\alpha}{\lambda_k - \lambda} u_k.$$

We have $\|u\| \to \infty$ ($\|u\|$ can be interpreted as the amplitude in the example above) in two cases: 1) if $\alpha \to \infty$ (the stimulating amplitude tends to infinity) and 2) if the system parameter tends to an eigenvalue λ_k of the operator, i.e., $\lambda \to \lambda_k$. The second case is called resonance and the value λ_k of the parameter λ is called the resonant value. In this case we deal with the resonant equation in the sense of our definition above and of the example equation (1.1). It is clear, that a system can possess various resonant parameter values.

The resonant equations arise also when solving the quadratic operator equation

$$A^2 u = 0 \tag{1.3}$$

with a given operator A. Denoting $Au = v$ we reduce equation (1.3) to the “simpler” pair of equations $Av = 0$, $Au = v$ where the last equation is resonant.

The resonant phenomena play a very important role in the natural world, in various technical applications, e.g. magnetic resonance imaging (nuclear spin tomography) [24], fluid dynamics [15, 16] etc. The resonant equations arose also in the context of supersymmetric Casimir operators for the di-spin algebra (see, e.g., [6, 7] and the literature cited therein). These equations often require specific solution and investigation techniques [1 – 3]. The solvability condition of a resonant equation in a Hilbert space is the orthogonality of the right-hand side to the kernel of the operator. The situation for other spaces is more complicated (see, e.g., [1 – 3]), where some sufficient solvability conditions has been proven. For example, in [2] the following Liénard equation, describing vibrations and various dynamical systems, was considered as an illustration of the presented solvability theory:

$$\ddot{x}(t) + g(x)\dot{x}(t) + ax(t) = f(t), \quad x(0) = x(1), \quad \dot{x}(0) = \dot{x}(1), \quad t \in [0,1]. \tag{1.4}$$

Here a is a constant, the function $g\colon R^1 \to R^1$ is supposed to be continuous and the solution is looking for in the class of twice continuously differentiable on $[0,1]$ functions. It was shown that this problem possesses at least one solution for arbitrary function

$$f(t)\colon \int\limits_0^1 f(t)dt = 0$$

provided that

$$|g(x)| \le b, \quad a \in R^1, \quad b + 4|a|/3 < 1. \tag{1.5}$$

The simple counterexample with $f(t) = \dfrac{4}{3\pi}\cos(2\pi t) + \sin(2\pi t),\ g(x) \equiv 0,\ a = 4\pi^2$ (here the first summand is a resonant component) shows that the differential equation is resonant in the sense of our definition, conditions (1.5) due to $b = 0,\ a = 4\pi^2$ as well as the condition $\displaystyle\int_0^1 f(t)dt = 0$ are not fulfilled, but there exists a set of solutions given by

$$u(t) = C\cos(2\pi t) + D\sin(2\pi t) + \frac{t}{3\pi^2}\sin(2\pi t) + \frac{1}{3\pi^2}\sin(\pi t) \quad \forall C, D \in R^1,$$

i.e., conditions (1.5) are rather coarse.

In the present paper we consider resonant equations with the differential operators of the hypergeometric type which define the classical orthogonal polynomials. The solutions of such homogeneous differential equation is the corresponding orthogonal polynomial (or the solution of the first kind) and the second linear independent solution is the so-called function of the second kind, so that the general solution is a linear combination of both. The inhomogeneous differential equations with the corresponding orthogonal polynomial or the function of the second kind on the right-hand side are resonant equations correspondingly of the first and of the second kind. We need their particular solutions to write down the general solution of the inhomogeneous resonant equation.

We propose a general algorithm to find such particular solutions explicitly, therefore, we can obtain the general solutions of the inhomogeneous resonant equations of the first and of the second kind in explicit form. This algorithm is especially suitable for the computer algebra tools like Maple etc. Besides, it provides a constructive proof of the existence of the solutions too.

The paper consists of two parts and is organized as follows. In Section 2 we show that the resonant equations are a natural part of the FD-method. The main result of the Section 3 is Theorem 3.1, giving a formula for particular solutions of a resonant operator equation depending on some parameter. This section contains also the description of the general Algorithm 3.1 to compute the particular solutions of the inhomogeneous resonant equations with the differential operators related to the classical orthogonal polynomials. Theorem 3.1 plays the crucial role for the justification of our algorithm. Each of the next two sections consists of two subsections devoted to the corresponding resonant equations of the first and of the second kind with the differential operators related to the classical orthogonal polynomials of Legendre and Jacobi types. The explicit formulas for the general solutions of the corresponding inhomogeneous resonant differential equations are given. The classical orthogonal polynomials defined on the infinite intervals, namely the Hermite and the Laguerre polynomials are the topics of part II. With the aim to emphasize the advantages of our algorithm we give the particular solutions through the hypergeometric or confluent hypergeometric functions too.

2. The homotopy based method for the eigenvalue problems. Let us briefly explain the ideas of perturbation and homotopy for the eigenvalue problem

$$(A+B)u_n - \lambda_n u_n = \theta \tag{2.1}$$

in a Hilbert space X with a scalar product $(\cdot,\cdot)$ and with the null-element θ under the assumption that the spectrum of the operator $A+B$ is discrete and we are looking for the eigenpair with a given fixed index n.

Let $\overline{B}$ be an approximating operator for B in the sense that the eigenvalue problem

$$(A+\overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta \tag{2.2}$$

is “simpler” then problem (2.1).

Formally, a homotopy between two problems P_1 and P_2 with solutions u_1 and u_2 from some topological space X is defined to be a parametric problem $P_H(t)$ with a solution $u(t)$ continuously depending on the parameter $t \in [0,1]$ and such that $u(0) = u_1$ and $u(1) = u_2$ (compare with http://en.wikipedia.org/wiki/Homotopy).

Following to the homotopy idea for a given eigenpair number n we imbed our problem into the parametric family of problems

$$(A+W(t))u_n(t) - \lambda_n(t)u_n(t) = \theta, \quad t \in [0,1], \tag{2.3}$$

with $W(t) = \overline{B} + t\varphi(B)$, $\varphi(B) = B - \overline{B}$, where $\overline{B}$ is some approximation of B. This family contains the both problems (2.1) and (2.2), so that we obviously have

$$u_n(0) = u_n^{(0)}, \quad \lambda_n(0) = \lambda_n^{(0)}, \quad u_n(1) = u_n, \quad \lambda_n(1) = \lambda_n. \tag{2.4}$$

This suggests the idea to look for the solution of (2.3) as the Taylor series

$$\lambda_n(t) = \sum_{j=0}^{\infty} \lambda_n^{(j)} t^j, \quad u_n(t) = \sum_{j=0}^{\infty} u_n^{(j)} t^j, \tag{2.5}$$

where formally

$$\lambda_n^{(j)} = \frac{1}{j!}\frac{d^j \lambda_n(t)}{dt^j}\bigg|_{t=0}, \quad u_n^{(j)} = \frac{1}{j!}\frac{d^j u_n(t)}{dt^j}\bigg|_{t=0}. \tag{2.6}$$

Setting $t = 1$ in (2.5) we obtain

$$\lambda_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \quad u_n = \sum_{j=0}^{\infty} u_n^{(j)} \tag{2.7}$$

provided that series (2.5) converge for all $t \in [0,1]$. The truncated series

$$\overset{m}{\lambda}_n = \sum_{j=0}^{\infty} \lambda_n^{(j)}, \quad \overset{m}{u}_n = \sum_{j=0}^{\infty} u_n^{(j)} \tag{2.8}$$

represent a computational algorithm of rank m.

The formulas (2.6) are not suitable for a numerical algorithm, therefore we need an other way to compute the corrections $\lambda_n^{(j)}$, $u_n^{(j)}$ which we describe below.

Substituting (2.5) into (2.3) and matching the coefficients in front of the same powers of t we arrive at the following recurrence sequence of equations:

$$(A+\overline{B})u_n^{(j+1)} - \lambda_n^{(0)} u_n^{(j+1)} = F_n^{(j+1)}, \quad j=-1,0,1,\ldots, \tag{2.9}$$

with $F_n^{(0)}=0$ and

$$F_n^{(j+1)} = F_n^{(j+1)}(\lambda_n^{(0)},\ldots,\lambda_n^{(j+1)}; u_n^{(0)},\ldots,u_n^{(j)}) = -\varphi(B)u_n^{(j)} + \sum_{p=0}^{j} \lambda_n^{(j+1-p)} u_n^{(p)} =$$

$$= \lambda_n^{(j+1)} u_n^{(0)} - \varphi(B)u_n^{(j)} + \sum_{p=1}^{j} \lambda_n^{(j+1-p)} u_n^{(p)}, \ j=-1,0,1,\ldots. \tag{2.10}$$

For the pair $\lambda_n^{(0)}$, $u_n^{(0)}$ corresponding to the index $j=-1$ we have the so-called base eigenvalue problem

$$(A+\overline{B})u_n^{(0)} - \lambda_n^{(0)} u_n^{(0)} = \theta \tag{2.11}$$

in a Hilbert space which is assumed to have not multiple eigenvalues and to be "simpler" then the original one and produces the initial data for problems (2.9), (2.10). We suppose that $u_n^{(0)}$, $n=1,2,\ldots,$ is a basis of the corresponding Hilbert space. The case of the base problems with multiple eigenvalues was studied in [10, 20, 21].

Each problem (2.10) contains in the right-hand side the summand $\lambda_n^{(j+1)} u_n^{(0)}$ which solves the homogeneous equation with the same operator, i.e., the solution $u_n^{(j+1)}$ of (2.10) contains a component which is the solution of the corresponding resonant equation.

Problems (2.9) for higher indices $j \geq 0$ are solvable provided that

$$(F_n^{(j+1)}, u_n^{(0)}) = 0, \quad j=0,1,\ldots. \tag{2.12}$$

Supposing additionally (for uniqueness)

$$(u_n^{(j+1)}, u_n^{(0)}) = 0, \quad j=0,1,\ldots, \tag{2.13}$$

we obtain

$$\lambda_n^{(j+1)} = (\varphi(B)u_n^{(j)}, u_n^{(0)}), \quad j=0,1,\ldots. \tag{2.14}$$

Under these conditions we obtain the particular solution

$$u_n^{(j+1)} = \sum_{p=1, p\neq n}^{\infty} \frac{((F_n^{(j)}, u_p^{(0)}))}{\lambda_p^{(0)} - \lambda_n^{(0)}} u_p^{(0)} \tag{2.15}$$

satisfying condition (2.13). The start values $\lambda_n^{(0)}$, $u_n^{(0)}$ for the recursion (2.9), (2.14) is the solution of the base problem.

The next theorem [18] gives the error estimates of the method above and its convergence as $m \to \infty$.

Theorem 2.1. *Let A be a closed operator in a Hilbert space H, problem* (2.11) *possesses a discrete spectrum of eigenvalues $0 \le \lambda_1^{(0)} < \lambda_2^{(0)} < \ldots$ and the corresponding eigenvectors $u_n^{(0)}$, $n = 1, 2, \ldots,$ represent a basis of H. Let the inequality*

$$q_n = 4M_n \|\varphi(B)\| < 1 \tag{2.16}$$

with

$$M_n = \max\left\{\frac{1}{\lambda_n^{(0)} - \lambda_{n-1}^{(0)}}, \frac{1}{\lambda_{n+1}^{(0)} - \lambda_n^{(0)}}\right\} \tag{2.17}$$

holds true. Then the series (2.7) *converge to the solution λ_n, u_n of problem* (2.1) *and the accuracy of algorithm* (2.8) *is given by the estimates*

$$\begin{aligned} \|u_n - \overset{m}{u}_n\| &\le \alpha_{m+1} \frac{q_n^{m+1}}{1 - q_n}, \\ \|\lambda_n - \overset{m}{\lambda}_n\| &\le \|\varphi(B)\| \alpha_m \frac{q_n^m}{1 - q_n}, \end{aligned} \tag{2.18}$$

where

$$\alpha_m = 2\frac{(2m-1)!!}{(2m+21)!!}. \tag{2.19}$$

3. Representation of particular solutions. This section deals with particular solutions of the resonant equations. We give a representation of particular solutions of a resonant equation in a Banach space. Besides we propose an algorithm to compute particular solutions of resonant equations with the differential operators related to the classical orthogonal polynomials.

The following result has been proven in [19].

Theorem 3.1. *Let $A \colon X \to X$ be a linear operator acting in a Banach space X, the set $K(A) \subset X$ be the kernel of A and a connected set $\Sigma(A)$ in the complex plane be the spectral set of A. If $f(\lambda) \in K(A - \lambda E)$, $\lambda \in \Sigma(A)$ is a differentiable function, then the solution of the resonant equation*

$$(A - \lambda E)u = f(\lambda) \tag{3.1}$$

can be represented by

$$u(\lambda) = \frac{df(\lambda)}{d\lambda}. \tag{3.2}$$

The proof of this theorem is based on the equivalent equation $(A - \lambda_0 E)\dfrac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f(\lambda)$ with some fixed λ_0 and on passing to the limit $\lambda \to \lambda_0$.

Now, let

$$\mathcal{A}_n = \sigma(x)\frac{d^2}{dx^2} + \tau(x)\frac{d}{dx} + \lambda_n \tag{3.3}$$

be a differential operator of the hypergeometric type with a polynomial $\sigma(x)$ of the degree not greater then two, a polynomial $\tau(x)$ of the degree not greater then one and a constant λ_n and $P_n(x)$ be some classical orthogonal polynomial satisfying the homogeneous differential equation

$$\mathcal{A}_n P_n(x) = 0 \tag{3.4}$$

(see, e.g., [5, 23, 25]). We call the polynomial solution $P_n(x)$ of this homogeneous differential equation the function of the first kind. Let $Q_n(x)$ be the second linear independent solution of the homogeneous differential equation, which is called the function of the second kind.

Let us consider the resonant equations of the type

$$\mathcal{A}_n u_n(x) = R_n(x). \tag{3.5}$$

In the case when $R_n(x)$ is the classical orthogonal polynomial $P_n(x)$ (the function of the first kind), the inhomogeneous differential equation (3.5) is called the resonant equation of the first kind. The inhomogeneous differential equation of the type (3.5) with the right-hand side $Q_n(x)$ instead of $R_n(x)$ is called the resonant differential equation of the second kind. Both functions $P_n(x)$ and $Q_n(x)$ satisfy the same homogeneous differential equation (3.4) and the same recurrence equation

$$R_{n+1}(x) = (\alpha_n x + \beta_n)R_n(x) - \gamma_n R_{n-1}(x), \quad n = 1, 2, \ldots, \tag{3.6}$$

with some constants $\alpha_n,\ \beta_n,\ \gamma_n$ (see, e.g., [5, 22, 23, 25]). Since our algorithm below for particular solutions of the resonant differential equations of the first and of the second kind (3.5) is based on the same recurrence relation (3.6) it is valid for the resonant equations of both types and we use the notation $R_n(x)$ below for both $P_n(x)$ and $Q_n(x)$.

Algorithm 3.1. 1. Using Theorem 3.1 we find some particular solutions of (3.5) for $n = 0, 1$, i.e.,

$$\chi_0(x) = -\frac{1}{\lambda'(\nu)}\left.\frac{d\,R_\nu(x)}{d\nu}\right|_{\nu=0}, \qquad \chi_1(x) = -\frac{1}{\lambda'(\nu)}\left.\frac{d\,R_\nu(x)}{d\nu}\right|_{\nu=1}. \tag{3.7}$$

Note that here and in what follows the differentiation with respect to a natural parameter $n \in \mathbb{N}$ means: 1) the switch to a real parameter $\nu \in \mathbb{R}$, i.e., the use of the hypergeometrical or confluent hypergeometrical functions, 2) the differentiation by ν, 3) the substitution of n instead of ν in the derivative.

2. The set of functions

$$u_0(x) = \chi_0(x) + c_0P_0(x) + d_0Q_0(x), \quad u_1(x) = \chi_1(x) + c_1P_1(x) + d_1Q_1(x) \tag{3.8}$$

with arbitrary coefficients $c_0,\ c_1,\ d_0,\ d_1$ represents particular solutions of the inhomogeneous resonant equation too. These coefficients can be chosen at the next step of the algorithm so that the following particular solutions $u_k(x),\ k = 2, 3, \ldots,$ obtained by the recursion below satisfy the corresponding resonant equation.

3. Differentiating the recurrence equation (3.6) for R_n by n we obtain

$$u_{n+1}(x) = -\frac{1}{\lambda'(n+1)}\left[-\frac{d\lambda(n)}{dn}(\alpha_n x + \beta_n)\,u_n(x) + \frac{d\lambda(n-1)}{dn}\gamma_n u_{n-1}(x) + \right.$$

$$\left. + \left(\frac{d\alpha_n}{dn}x + \frac{d\beta_n}{dn}\right)R_n(x) - \frac{d\gamma_n}{dn}R_{n-1}(x)\right], \quad n = 1, 2, \ldots. \tag{3.9}$$

We set here $n = 1$ and demand that $u_2(x)$ obtained from (3.9), (3.8), satisfies the resonant differential equation (3.5). From this condition we determine the coefficients $c_0,\ c_1,\ d_0,\ d_1$ and, therefore, the initial values (3.8) for the recursive algorithm (3.9). Using Theorem 3.1 we prove below that $u_n(x)$ then satisfy the resonant equation for all $n = 0, 1, 2, \ldots$.

4. Resonant equation of the Legendre type. ***4.1. The Legendre resonant equation of the first kind.*** Let us consider the following inhomogeneous equation with the Legendre differential operator on the left-hand and the Legendre polynomial on the right-hand side:

$$\frac{d}{dx}\left[(1-x^2)\frac{du(x)}{dx}\right] + n(n+1)u(x) = P_n(x). \tag{4.1}$$

It is the resonant equation of the first kind since the Legendre polynomial $P_n(x)$ satisfies the corresponding homogeneous differential equation. The second linear independent solution of the homogeneous differential equation $Q_n(x)$ is called the Legendre function of the second kind. The general solution of the homogeneous differential equation (4.1) is given by

$$u(x) = c_1 P_n(x) + c_2 Q_n(x),$$

where $c_1,\ c_2$ are arbitrary constants.

The explicit representation of the Legendre function of the second kind can be presented through the hypergeometric function (see, e.g., [5], § 10.10):

$$Q_n(x) = Q_0(x)P_n(x) - \sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\frac{2n-4k+3}{(2k-1)(n-k+1)}P_{n-2k+1}(x) =$$

$$= \frac{2^n (n!)^2}{(2n+1)!(1+x)^{n+1}}F\left(n+1, n+1; 2n+2; \frac{2}{1+x}\right) =$$

$$= (-1)^{n+1}\frac{2^n (n!)^2}{(2n+1)!(1-x)^{n+1}}F\left(n+1, n+1; 2n+2; \frac{2}{1-x}\right) =$$

$$= \frac{1}{2}\left[F\left(n+1, n+1; 2n+2; \frac{2}{1+x}\right) + \right.$$

$$\left. +(-1)^{n+1}\frac{2^n (n!)^2}{(2n+1)!(1-x)^{n+1}}F\left(n+1, n+1; 2n+2; \frac{2}{1-x}\right)\right],$$

$$Q_0(x) = \frac{1}{2}\ln\frac{x+1}{x-1}. \tag{4.2}$$

Here $F(a,b;c;z) = \sum_{p=0}^{n} \frac{(a)_p(b)_p z^n}{(c)_p p!}$ is the hypergeometric function of z, $(a)_0 = 1$, $(a)_p = \frac{\Gamma(a+p)}{\Gamma(a)}$ is the Pochhammer symbol, and $\Gamma(x)$ is the Gamma function. We remember, that many of the well known mathematical functions can be expressed in terms of the hypergeometric function, or as limiting cases of it. Two typical examples are

$$\ln(1+z) = z\,F(1,1;2;-z),$$

$$(1-z)^{-a} = F(a,1;1;z).$$

The Legendre functions as well as several orthogonal polynomials, including Jacobi polynomials $P_n^{(\alpha,\beta)}$ and their special cases Legendre polynomials ($\alpha = 0, \beta = 0$), Chebyshev polynomials $T_n(x)$ ($\alpha = -1/2, \beta = -1/2$), Gegenbauer polynomials $C_n^{\lambda}(x)$ ($\alpha = \beta = \lambda - 1/2$) can be written in terms of hypergeometric functions in many ways, for example (see, e.g., [7], § 10.8),

$$\begin{gathered}
P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n} F_n\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) = \\
= (-1)^n \binom{n+\beta}{n} F_n\left(-n, n+\alpha+\beta+1; \beta+1; \frac{1+x}{2}\right), \\
P_n(x) = P_n^{(0,0)}(x) = \frac{1}{2}\left[F_n\left(-n, n+1; 1; \frac{1-x}{2}\right) + \right. \\
\left. +(-1)^n\, F_n\left(-n, n+1; 1; \frac{1+x}{2}\right)\right].
\end{gathered} \tag{4.3}$$

The use of the hypergeometric functions to obtain a solution of a resonant equation represents a direct way to solve the resonant equations. This way is due to the fact that the hypergeometric differential equation

$$z(1-z)\frac{d^2u}{dz^2} + [c - (a+b+1)z]\frac{du}{dz} - abu = 0 \tag{4.4}$$

with an appropriate choice of their parameters can be transformed to the following Legendre equation [4] (§ 3.2)

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \nu(\nu+1)w = 0. \tag{4.5}$$

Due to Theorem 3.1 and (4.3) we have a particular solution of (4.1) in the form

$$u_n(x) = \frac{1}{2}\left[\tilde{u}_n(x) + (-1)^n \tilde{u}_n(-x)\right], \tag{4.6}$$

where (see (4.3) as well as [5], § 10.8, formula (16) with $\alpha = 0, \beta = 0$)

$$\tilde{u}_n(x) = k_n \frac{d}{d\nu} P_\nu(x)|_{\nu=n} = k_n \frac{d}{d\nu} F_\nu\left(1+\nu, -\nu; 1; \frac{1-x}{2}\right)\Big|_{\nu=n} =$$

$$= k_n \left[\sum_{p=1}^{n} \frac{d}{dn} \frac{(1+n)_p(-n)_p}{(p!)^2} \left(\frac{1-x}{2}\right)^p + \right.$$

$$\left. +(-1)^{n+1} n! \sum_{p=n+1}^{\infty} \frac{(1+n)_p(p-n-1)!}{(p!)^2} \left(\frac{1-x}{2}\right)^p \right] \tag{4.7}$$

with $k_n = -\dfrac{1}{2n+1}$. The last formula can be transformed in the following way

$$\tilde{u}_n(x) = k_n \left[\sum_{p=1}^{n} \frac{-1}{(p!)^2} \left[2n + p - 2n^2(n+p) \sum_{i=1}^{p-1} \frac{1}{i^2 - n^2} \right] \prod_{i=1}^{p-1} (i^2 - n^2) \left(\frac{1-x}{2}\right)^p + \right.$$

$$\left. +(-1)^{n+1} \sum_{p=n+1}^{\infty} \frac{1}{p} \prod_{i=1}^{n} \frac{p+i}{p-i} \left(\frac{1-x}{2}\right)^p \right]. \tag{4.8}$$

Using the formulas

$$\frac{1}{p} \prod_{i=1}^{n} \frac{p+i}{p-i} = \sum_{i=0}^{n} \frac{a_{n,i}}{p-i}, \quad a_{n,i} = (-1)^{n+i} \frac{(n+i)!}{(n-i)!(i!)^2},$$

$$(-1)^n \sum_{i=0}^{n} a_{n,i} \left(\frac{1-x}{2}\right)^i \equiv F_n \left(1+n, -n; 1; \frac{1-x}{2}\right) = P_n(x) \tag{4.9}$$

the sum of the last series can be transformed to

$$\sum_{p=n+1}^{\infty} \frac{1}{p} \prod_{i=1}^{n} \frac{p+i}{p-i} \left(\frac{1-x}{2}\right)^p =$$

$$= -\sum_{i=0}^{n} a_{n,i} \left(\frac{1-x}{2}\right)^i \ln\left(\frac{1+x}{2}\right) - \sum_{p=0}^{n-1} a_{n,p} \left(\frac{1-x}{2}\right)^p \sum_{i=1}^{n-p} \frac{1}{i} \left(\frac{1-x}{2}\right)^i =$$

$$= (-1)^{n+1} P_n(x) \ln\left(\frac{1+x}{2}\right) - \sum_{i=1}^{n} \left(\frac{1-x}{2}\right)^i \sum_{p=0}^{i-1} \frac{a_{n,p}}{i-p}. \tag{4.10}$$

Thus, for the function (4.7) we have

$$\tilde{u}_n(x) = -\frac{1}{2n+1} P_n(x) \ln\left(\frac{1+x}{2}\right) + \sum_{i=1}^{n} \left(\frac{1-x}{2}\right)^i b_{n,i},$$

$$b_{n,i} = -\frac{1}{2n+1} \left[\frac{1}{i!} \frac{d}{dn} (1+n)_i(-n)_i + (-1)^n \sum_{p=0}^{i-1} \frac{a_{n,p}}{i-p} \right]. \tag{4.11}$$

Such direct way to obtain a particular solution as described above is rather awkward. Below we propose an algorithmic way based on Theorem 3.1 and on the recursion formula for the corresponding orthogonal polynomials. This algorithm can be easily implemented by the computer algebra tools, for example, by Maple.

Actually, for $n = 0, 1$ using Theorem 3.1 we obtain from (4.6) the following particular solutions:

$$\chi_0(x) = -\frac{1}{2}\ln(1-x^2), \quad \chi_1(x) = -\frac{x}{6}\ln(1-x^2) + \frac{11}{18}x. \tag{4.12}$$

Differentiating the recurrence relation for $P_n(x)$ by n we arrive at the following recurrence equation for particular solutions:

$$u_{n+1}(x) = -\frac{1}{2n+3}\left[-\frac{(2n+1)^2 x}{n+1}u_n(x) + \frac{n(2n-1)}{n+1}u_{n-1}(x) + \right.$$

$$\left. + \frac{x}{(n+1)^2}P_n(x) - \frac{1}{(n+1)^2}P_{n-1}(x)\right], \quad n = 1, 2, \ldots. \tag{4.13}$$

The Legendre polynomials $P_0(x) = 1$ and $P_1(x) = x$ as well as the Legendre functions of the second kind $Q_0(x)$ and $Q_1(x)$ satisfy the corresponding homogeneous Legendre differential equation, that's why in accordance with our algorithm and in regard of (4.12) we can use the ansatzes for the initial values

$$\begin{gathered} u_0(x) = -\frac{1}{2}\ln(1-x^2) + c_0 P_0(x) + d_0 Q_0(x), \\ u_1(x) = -\frac{x}{6}\ln(1-x^2) + \frac{11}{18}x + c_1 P_1(x) + d_1 Q_1(x) \end{gathered} \tag{4.14}$$

with undefined coefficients $c_0,\ c_1,\ d_0,\ d_1$. After substituting these into (4.13) with $n = 1$ we demand that $u_2(x)$ satisfies the resonant differential equation (4.1), from where we obtain

$$\begin{gathered} d_0 = 0, \quad d_1 = 0, \\ c_0 = \frac{17}{6} + 3c_1. \end{gathered} \tag{4.15}$$

Setting, for example, $c_0 = 0$ we obtain $c_1 = -\dfrac{17}{18}$ and arrive at the representations

$$\begin{gathered} u_1(x) = -\frac{1}{6}P_1(x)\ln(1-x^2) - \frac{1}{3}x, \\ u_2(x) = -\frac{1}{10}P_2(x)\ln(1-x^2) - \frac{7}{20}x^2 + \frac{1}{20}. \end{gathered} \tag{4.16}$$

In general, we have

$$u_n(x) = -\frac{1}{2(2n+1)}P_n(x)\ln(1-x^2) + v_n(x), \tag{4.17}$$

where $v_n(x)$ satisfies the recurrence equation

$$v_{n+1}(x) = -\frac{1}{2n+3}\left[-\frac{(2n+1)^2 x}{n+1}v_n(x) + \frac{n(2n-1)}{n+1}v_{n-1}(x) + \right.$$
$$\left. + \frac{x}{(n+1)^2}P_n(x) - \frac{1}{(n+1)^2}P_{n-1}(x)\right], \quad n = 1, 2, \ldots,$$
$$v_0(x) = 0, v_1(x) = -\frac{x}{3}. \tag{4.18}$$

This recurrence equation together with (4.14), (4.17) provides, for example, the following particular solutions:

$$u_3(x) = -\frac{1}{14}P_3(x)\ln(1-x^2) + \frac{5}{28}x - \frac{37}{84}x^3,$$
$$u_4(x) = -\frac{1}{18}P_4(x)\ln(1-x^2) - \frac{7}{288} + \frac{59}{144}x^2 - \frac{533}{864}x^4. \tag{4.19}$$

The next theorem shows that the functions $u_n(x)$ obtained by our recursive algorithm satisfy the resonant Legendre differential equation of the first kind for all $n = 0, 1, \ldots.$

Theorem 4.1. *The functions $u_n(x)$ obtained by the recursive algorithm* (4.13) *satisfy the resonant Legendre differential equation of the first kind* (4.1) *for each $n = 0, 1, 2, \ldots.$*

Proof. These functions for $n = 0, 1, 2$ satisfy the resonant Legendre differential equation by construction. Let us assume that $u_p(x),\ p = 0, 1, \ldots, n,$ satisfy this differential equation and prove that it is the case for $p = n + 1$. Differentiating the classical relation [7] (§ 10.10)

$$(1-x^2)\frac{d\,P_n(x)}{dx} = n\left[P_{n-1}(x) - xP_n(x)\right] \tag{4.20}$$

by n and using Theorem 3.1 we arrive at the equation

$$-(2n+1)(1-x^2)\frac{du_n(x)}{dx} = -n\,(2n-1)\,u_{n-1}(x) + x\,n\,(2n+1)\,u_n(x) + P_{n-1}(x) - xP_n(x). \tag{4.21}$$

Applying to (4.13) the Legendre differential operator

$$\mathcal{A}_n = (1-x^2)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + n(n+1) \tag{4.22}$$

we obtain

$$\mathcal{A}_{n+1}u_{n+1}(x) = P_{n+1}(x) + \frac{2(2n+1)}{(2n+3)(n+1)}\left[(2n+1)(1-x^2)\frac{du_n(x)}{dx} - \right.$$
$$\left. -n\,(2n-1)\,u_{n-1}(x) + x\,n\,(2n+1)\,u_n(x) + P_{n-1}(x) - xP_n(x)\right]. \tag{4.23}$$

It follows from (4.21) that the expression in the square brackets is equal to zero which proves the theorem.

Now, the general solution of the inhomogeneous equation (4.1) can be represented by

$$u(x) = u_h + u_n(x) = c_1P_n(x) + c_2Q_n(x) + u_n(x), \tag{4.24}$$

where c_1, c_2 are arbitrary constants.

Remark **4.1.** The basis of the proof of Theorem 4.1 are a recurrence equation $xp_n(x) = \alpha_n p_{n-1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x)$ for the corresponding orthogonal with the weight $\sigma(x)$ polynomial $p_n(x)$ and the differentiation formula $\sigma(x)p_n'(x) = \alpha_n^{(1)} p_{n+1}(x) + (\beta_n^{(1)} + \gamma_n^{(1)} x)p_n(x)$ (see (4.20) for the Legendre polynomials) which represents the weighted derivative of the polynomial under consideration through two neighboring polynomials (see, e.g., [22], § 9). But the second linear independent solution of the corresponding homogeneous equation, which is called the function of the second kind (in the case above, the Legendre function of the second kind $Q_n(x)$, which is not polynomial!) satisfies the same recurrence equation and the same differentiation formula (see [22, p. 67]). Thus, a similar theorem for the particular solutions of the resonant equations of the second kind obtained by the corresponding recursive algorithm is valid too.

4.2. The Legendre resonant equation of the second kind. In this subsection we consider the equation (4.1) with the Legendre function of the second kind

$$Q_n(x) = \frac{2^n(1+x)^{-n-1}(n!)^2}{(2n+1)!} F\left(n+1, n+1; 2n+2; \frac{2}{1+x}\right)$$

as the right-hand side, i.e., we have again a resonant equation.

The general solution of such resonant equation is

$$u(x) = c_1 P_n(x) + c_2 Q_n(x) + u_n(x), \tag{4.25}$$

where the linear independent Legendre polynomial $P_n(x)$ and the Legendre function of the second kind $Q_n(x)$ satisfy the homogeneous Legendre equation, c_1, c_2 are arbitrary constants and $u_n(x)$ is a particular solution of the inhomogeneous resonant equation.

Note, that in [7] a solution is obtained for the case $n = 0$ only and there was pointed out that it is very difficult to obtain the solution for other n in a closed form. But our Theorem 3.1 allows one to obtain the particular solution for arbitrary n as

$$u_n(x) = -\frac{1}{(2n+1)} \left[2\psi(n+1) - 2\psi(2n+2) + \ln(2) - \ln(1+x)\right] Q_n(x) -$$

$$- \frac{2^n(n!)^2(1+x)^{-n-1}}{(2n+1)(2n+1)!} \frac{d}{d\nu} \left[F\left(\nu+1, \nu+1; 2\nu+2; \frac{2}{1+x}\right)\right]_{\nu=n}, \tag{4.26}$$

where $\psi(z) = \dfrac{d}{dz} \ln\Gamma(z) = \dfrac{\Gamma'(z)}{\Gamma(z)}$ is the logarithmic derivative of the Gamma function. Various representations of this function can be found in [5] (§ 1.7). For $n = 0, 1$ we obtain the particular solutions

$$\chi_0(x) = -P_0(x)w(x),$$

$$\chi_1(x) = -\frac{1}{3}P_1(x)w(x) - \frac{1}{6}\ln\left(x^2-1\right) - \frac{2}{3},$$

$$w(x) = -\text{polylog}\left(2, \frac{2}{1+x}\right) - \frac{1}{2}\ln^2(x+1) + \frac{1}{2}\ln(x+1)\ln(x-1) =$$

$$= -\text{dilog}\left(\frac{2}{1+x}\right) - \frac{1}{2}\ln^2(x+1) + \frac{1}{2}\ln(x+1)\ln(x-1), \ x > 1, \tag{4.27}$$

where polylog is the so-called polylogarithm-function of order s and of the argument z (Jonquiere's function):

$$\mathrm{polylog}(s,z) = \mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

(dilog or Spence's function, denoted also by $\mathrm{Li}_2(z)$ is a special case of polylog for $s = 2$).

Other explicit representations of particular solutions can be obtain by Algorithm 3.1. Differentiating the recurrence relation for the Legendre function of the second kind

$$Q_{\nu+1}(x) = \frac{x(2\nu+1)}{(\nu+1)} Q_\nu(x) - \frac{\nu}{(\nu+1)} Q_{\nu-1}(x) \tag{4.28}$$

with respect to ν and taking into account Theorem 3.1 we obtain the recurrence formula

$$u_{n+1}(x) = -\frac{1}{2n+3}\left[-\frac{(2n+1)^2 x}{n+1} u_n(x) + \frac{n(2n-1)}{n+1} u_{n-1}(x) + \frac{x}{(n+1)^2} Q_n(x) - \right.$$
$$\left. - \frac{1}{(n+1)^2} Q_{n-1}(x)\right], \quad n = 1, 2, \ldots. \tag{4.29}$$

In accordance with our algorithm and in regard of (4.27) we use the following ansatzes for initial values:

$$\begin{gathered} u_0(x) = -P_0(x)w(x) + c_0 P_0(x) + d_0 Q_0(x), \\ u_1(x) = -\frac{1}{3} P_1(x)w(x) - \frac{1}{6}\ln\left(x^2-1\right) - \frac{2}{3} + c_1 P_1(x) + d_1 Q_1(x), \quad x > 1, \end{gathered} \tag{4.30}$$

with undefined coefficients $c_0,\ c_1,\ d_0,\ d_1$. After substitution into (4.29) with $n = 1$ we demand that $u_2(x)$ satisfies the resonant differential equation of the second kind. Then we obtain

$$\begin{gathered} c_0 = 0, c_1 = 0, \\ -d_0 + 3d_1 + 1 = 0. \end{gathered} \tag{4.31}$$

Setting, for example, $d_0 = -\frac{1}{2}$ we get $d_1 = -\frac{1}{2}$ and herewith the particular solution

$$u_2(x) = -\frac{1}{5} P_2(x)w(x) - \frac{3x}{20}\ln(x^2-1) - \frac{1}{30}\ln\left(\frac{x+1}{x-1}\right) - \frac{3x}{5} - \frac{1}{3} Q_2(x), \quad x > 1. \tag{4.32}$$

The particular solutions $u_n(x),\ n = 3, 4, \ldots,$ can be obtained using (4.29) and the initial conditions (4.30), (4.32).

The next theorem shows that the functions $u_n(x)$ obtained by our recursive algorithm satisfy the resonant Legendre differential equation of the second kind for all $n = 0, 1, \ldots.$

Theorem 4.2. *The functions $u_n(x)$ obtained by the recursive algorithm* (4.29) *satisfy the resonant Legendre differential equation of the second kind* (4.1) *for each $n = 0, 1, 2, \ldots.$*

The proof is completely analogous to that of Theorem 4.1 in regard of the fact that the Legendre functions of the second kind (which are not polynomials!) satisfy the same recurrence equation as the Legendre polynomials and the differentiating formula (4.20) (see [7], § 10.10).

5. Resonant equation of the Jacobi type. ***5.1. The Jacobi resonant equation of the first kind.*** In this section we consider the resonant equation of the Jacobi type

$$(1-x^2)\frac{d^2u_n(x)}{dx^2}+$$

$$+[\beta-\alpha-(\alpha+\beta+2)x]\frac{du_n(x)}{dx}+n(n+\beta+1)u_n(x)=P_n^{(\alpha,\beta)}(x), \tag{5.1}$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial [5] (§ 10.8) satisfying the homogeneous differential equation. The general solution of this equation is

$$u(x)=c_1P_n^{(\alpha,\beta)}(x)+c_2Q_n^{(\alpha,\beta)}(x)+u_n(x), \tag{5.2}$$

where

$$Q_n^{(\alpha,\beta)}(x)=\frac{2^{n+\alpha+\beta}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(x-1)^{n+\alpha+1}(x+1)^{\beta}\Gamma(2n+\alpha+\beta+2)}\times$$

$$\times F\left(n+1,n+\alpha+1;2n+\alpha+\beta+2;\frac{2}{1-x}\right) \tag{5.3}$$

is the Jacobi function of the second kind [5] (§ 10.8), $c_1,\ c_2$ are arbitrary constants and $u_n(x)$ is a particular solution of the inhomogeneous equation.

Due to Theorem 3.1 we have for a particular solution

$$u_n(x)=-\frac{1}{2n+\alpha+\beta+1}\left[\frac{\partial}{\partial\nu}P_\nu^{(\alpha,\beta)}(x)\right]_{\nu=n}=$$

$$=-\frac{1}{2n+\alpha+\beta+1}\left[\frac{\partial}{\partial\nu}\frac{\Gamma(\nu+\alpha)}{\Gamma(\alpha)\Gamma(\nu+1)}F\left(-\nu,\nu+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right)\right]_{\nu=n}=$$

$$=-\frac{1}{2n+\alpha+\beta+1}\left\{[\Psi(n+\alpha)-\Psi(n+1)]P_n^{(\alpha,\beta)}(x)+\right.$$

$$\left.+\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}\frac{\partial}{\partial\nu}\,F\left(-\nu,\nu+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right)\Bigg|_{\nu=n}\right\}=$$

$$=-\frac{1}{2n+\alpha+\beta+1}\left\{[\Psi(n+\alpha)-\Psi(n+1)]P_n^{(\alpha,\beta)}(x)+\right.$$

$$+\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}\left[\sum_{p=1}^{n}\frac{d}{dn}\frac{(\alpha+\beta+1+n)_p(-n)_p}{p!(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p+\right.$$

$$\left.\left.+(-1)^{n+1}n!\sum_{p=n+1}^{\infty}\frac{(\alpha+\beta+1+n)_p(p-n-1)!}{p!(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p\right]\right\}. \tag{5.4}$$

For example, we have the particular solutions

$$\chi_0(x) = u_0(x) = \frac{1}{\alpha+\beta+1}\left[-\psi(\alpha)+\psi(1)+\sum_{p=1}^{\infty}\frac{(\alpha+\beta+1)_p}{p(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p\right],$$

$$\chi_1(x) = u_1(x) = -\frac{1}{\alpha+\beta+3}\Bigg[(\psi(\alpha+1)-\psi(2))P_1^{(\alpha,\beta)}(x)-$$

$$-\alpha\frac{(\alpha+\beta+3)}{(\alpha+1)}\left(\frac{1-x}{2}\right)+\alpha\sum_{p=2}^{\infty}\frac{(\alpha+\beta+2)_p}{p(p-1)(\alpha+1)_p}\left(\frac{1-x}{2}\right)^p\Bigg]. \tag{5.5}$$

Differentiating (with respect to n) the recurrence formula for the Jacobi polynomials

$$P_{n+1}^{(\alpha,\beta)}(x) = (a(n)x+b(n))P_n^{(\alpha,\beta)}(x) - c(n)P_{n-1}^{(\alpha,\beta)}(x),$$

$$a(n) = \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)},$$

$$b(n) = \frac{\alpha^2-\beta^2}{2(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}, \tag{5.6}$$

$$c(n) = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}$$

with taking into account (5.4) we arrive at the recursion

$$u_{n+1}(x) = -\frac{1}{2n+\alpha+\beta+3}\Big[-(2n+\alpha+\beta+1)(a(n)x+b(n))u_n(x)+$$

$$+(2n+\alpha+\beta-1)c(n)u_{n-1}(x)+(a'(n)x+b'(n))P_n^{(\alpha,\beta)}(x)-c'(n)P_{n-1}^{(\alpha,\beta)}(x)\Big], \tag{5.7}$$

$$n = 1, 2, \ldots,$$

and with the initial conditions (5.5) we can obtain $u_n(x)$ for arbitrary n.

It is rather complicated to obtain an explicit formula for the solution of the Jacobi resonant equation for arbitrary α, β, therefore we consider an example only.

***Example* 5.1.** Let us consider the case of the Jacobi resonant equation of the first kind with $\alpha = 1,\ \beta = 2$. From (5.4) with $n = 0, 1$ we have the particular solutions

$$\chi_0(x) = -\frac{1}{64}\left[5\ln(x+1)+11\ln(x-1)\right]+$$

$$+\frac{5(x-1)+10(x^2-1)-11(x+1)^2}{96(x+1)^2(x-1)},$$

$$\chi_1(x) = -\frac{5x-1}{384}\left[10\ln(x+1)+17\ln(x-1)\right]+ \tag{5.8}$$

$$+\frac{1075x^4+1298x^3-1842x^2-1918x+487}{96(x+1)^2(x-1)}.$$

The initial values for recursion (5.7) are chosen in the form

$$u_0(x) = \chi_0(x) + d_0 Q_0^{(1,2)}(x) + c_0 P_0^{(1,2)}(x),$$
$$u_1(x) = \chi_1(x) + d_1 Q_1^{(1,2)}(x) + c_1 P_1^{(1,2)}(x), \quad x > 1, \tag{5.9}$$

where the undefined coefficients are determined so that $u_2(x)$ satisfies the resonant differential equation. We substitute (5.9) into (5.1) with $n = 1$ and then into the resonant differential equation, which yields

$$c_0 = 0, \quad d_0 = -\frac{7}{24},$$
$$c_1 = -\frac{47}{2880}, \quad d_1 = 0 \tag{5.10}$$

and further proceed in accordance with our Algorithm 3.1.

5.2. The Jacobi resonant equation of the second kind. In this subsection we consider the resonant equation

$$(1 - x^2)\frac{d^2u(x)}{dx^2} + [\beta - \alpha - (\alpha + \beta + 2)x]\frac{du(x)}{dx} +$$
$$+ n(n + \alpha + \beta + 1)u(x) = Q_n^{(\alpha,\beta)}(x), \tag{5.11}$$

where $Q_n^{(\alpha,\beta)}(x)$ is the Jacobi function of the second kind [5] (§ 10.8) given by formula (5.3).

Due to Theorem 3.1 we have for a particular solution the general formula

$$u_n(x) = -\frac{1}{2n + \alpha + \beta + 1}\left[\frac{\partial}{\partial \nu} Q_\nu^{(\alpha,\beta)}(x)\right]_{\nu=n} =$$
$$= -\frac{1}{2n + \alpha + \beta + 1}\Bigg\{[\ln(2) + \Psi(n + \alpha + 1) + \Psi(n + \beta + 1) - \ln(x - 1) -$$
$$-2\Psi(2n + \alpha + \beta + 1)]Q_n^{(\alpha,\beta)}(x) +$$
$$+ \frac{2^{n+\alpha+\beta}\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{(x - 1)^{n+\alpha+1}(x + 1)^{\beta}\Gamma(2n + \alpha + \beta + 2)} \times$$
$$\times \frac{\partial}{\partial \nu} F\left(\nu + 1, \nu + \alpha + 1; 2\nu + \alpha + \beta + 1; \frac{2}{1 - x}\right)\Bigg|_{\nu=n}\Bigg\}. \tag{5.12}$$

This formula is rather complicated for practical use, therefore we use our recursive algorithm. By differentiation of the recurrence equation we obtain the following recurrence formula:

$$u_{n+1}(x) = -\frac{1}{2n + \alpha + \beta + 3}\Big[-(2n + \alpha + \beta + 1)(a(n)x + b(n))u_n(x) +$$
$$+(2n + \alpha + \beta - 1)c(n)u_{n-1}(x) + (a'(n)x + b'(n))Q_n^{(\alpha,\beta)}(x) - c'(n)Q_{n-1}^{(\alpha,\beta)}(x)\Big], \tag{5.13}$$

$$n = 1, 2, \ldots,$$

which together with $\chi_0(x) = u_0(x),\ \chi_1(x) = u_1(x)$ from (5.12) and with the corresponding ansatz for the initial values provides an algorithm for $u_n(x)$ for any $n = 2, 3, \ldots$. Since the formulas in the general case are rather cumbersome, we restrict ourself to an example.

***Example* 5.2.** Let $\alpha = 1,\ \beta = 2$, then we have for the Jacobi functions of the second kind

$$Q_0^{(1,2)}(x) = -\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + \frac{3x^2+3x-2}{3(x+1)^2(x-1)},$$

$$Q_1^{(1,2)}(x) = -\frac{5x-1}{4}\ln\left(\frac{x+1}{x-1}\right) + \frac{15x^3+12x^2-13x-8}{6(x+1)^2(x-1)}, \tag{5.14}$$

$$Q_2^{(1,2)}(x) = -\frac{21x^2-6x-3}{8}\ln\left(\frac{x+1}{x-1}\right) + \frac{105x^4+75x^3-115x^2-65x+16}{20(x+1)^2(x-1)}, \ldots.$$

The general formula is the following:

$$Q_n^{(1,2)}(x) = -P_n^{(1,2)}(x)\frac{1}{2}\ln\left(\frac{x+1}{x-1}\right) + q_n^{(1,2)}(x), \tag{5.15}$$

where the functions $q_n^{(1,2)}(x)$ satisfy the recurrence equation for the Jacobi polynomials but with the initial conditions, which are given by the second summands in $Q_0^{(1,2)}(x),\ Q_1^{(1,2)}(x)$. From (5.12) we obtain the following particular solutions of the resonant equation of the second kind for $n = 0, 1$

$$\chi_0(x) = -\frac{1}{4}w(x) + \frac{15x-11}{48(x-1)}\ln(x+1) - \frac{15x^2+22x+3}{48(x+1)^2}\ln(x-1) -$$

$$-\frac{9x^2+3x-4}{72(x+1)^2\,(x-1)},$$

$$\chi_1(x) = -\frac{1}{12}\,(5x-1)\,w(x) +$$

$$+\frac{-135x^3-159x^2+42x+56}{360(x+1)^2}\ln(x-1) + \frac{45x^2-32x-8}{120(x-1)}\ln(x+1) +$$

$$+\frac{22025x^4+18340x^3-25422x^2-18452x+3413}{8640\,(x+1)^2\,(x-1)}, \tag{5.16}$$

$$w(x) = \operatorname{dilog}\left(\frac{x+1}{2}\right) + \frac{1}{2}\ln(x+1)\,\ln(x-1) - \ln(2)\ln(x-1).$$

For the initial values in our recursive algorithm we use the ansatzes

$$u_0(x) = \chi_0(x) + c_0P_0^{(1,2)}(x) + d_0Q_0^{(1,2)}(x),$$

$$u_1(x) = \chi_1(x) + c_1P_1^{(1,2)}(x) + d_1Q_1^{(1,2)}(x), \quad x > 1, \tag{5.17}$$

with undefined coefficients $c_0,\ d_0,\ c_1,\ d_1$. Substituting these into (5.13) with $n=1$ and demanding that the result satisfies the resonant differential equation, we obtain

$$d_0=\frac{3}{2}c_0+\frac{7}{80},\quad d_1=\frac{3}{2}c_1+\frac{881}{516} \tag{5.18}$$

and herewith the particular solution of the resonant equation

$$u_n^{(1,2)}(x)=-\frac{1}{2n+4}P_n^{(1,2)}(x)w(x)+p_n^{(1,2)}(x)\ln(x+1)+r_n^{(1,2)}(x)\ln(x-1)+v_n^{(1,2)}(x). \tag{5.19}$$

Here the functions $p_n^{(1,2)}(x),\ r_n^{(1,2)}(x)$ satisfy the recurrence relation for the Jacobi polynomials with the initial conditions

$$\begin{gathered}p_0^{(1,2)}(x)=\frac{15x-11}{48(x-1)},\quad p_1^{(1,2)}(x)=\frac{45x^2-32x-8}{120(x-1)},\\ r_0^{(1,2)}(x)=-\frac{15x^2+22x+3}{48(x+1)^2},\quad r_1^{(1,2)}(x)=\frac{-135x^3-159x^2+42x+56}{360(x+1)^2}.\end{gathered} \tag{5.20}$$

The function $v_n^{(1,2)}(x)$ satisfies the recurrence equation

$$\begin{gathered}v_{n+1}(x)=-\frac{1}{2n+\alpha+\beta+3}\Big[-(2n+\alpha+\beta+1)(a(n)x+\\ +b(n))v_n(x)+(2n+\alpha+\beta-1)c(n)v_{n-1}(x)+\\ +(a'(n)x+b'(n))Q_n^{(\alpha,\beta)}(x)-c'(n)Q_{n-1}^{(\alpha,\beta)}(x)\Big],\\ n=1,2,\ldots,\end{gathered} \tag{5.21}$$

with $\alpha=1,\beta=2$ and with the initial conditions

$$\begin{gathered}v_0(x)=-\frac{9x^2+3x-4}{72(x+1)^2\,(x-1)}+d_0Q_0^{(1,2)}(x),\\ v_1(x)=\frac{22025x^4+18340x^3-25422x^2-18452x+3413}{8640\,(x+1)^2\,(x-1)}+d_1Q_1^{(1,2)}(x).\end{gathered} \tag{5.22}$$

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