UDC 519.21

S. Meradji, H. Boutabia, S. Stihi (LaPS Laboratory, Badji-Mokhtar Univ., Algeria)

STOCHASTIC DIFFERENTIAL EQUATIONS FOR EIGENVALUES AND EIGENVECTORS OF A G-WISHART PROCESS WITH DRIFT

СТОХАСТИЧНІ ДИФЕРЕНЦІАЛЬНІ РІВНЯННЯ ДЛЯ ВЛАСНИХ ЗНАЧЕНЬ І ВЛАСНИХ ВЕКТОРІВ G-ПРОЦЕСУ ВІШАРТА ЗІ ЗНОСОМ

We propose a system of G-stochastic differential equations for the eigenvalues and eigenvectors of the G-Wishart process defined according to a G-Brownian motion matrix as in the classical case. Since we do not necessarily have the independence between the entries of the G-Brownian motion matrix, we assume in our model that their quadratic covariations are zero. An intermediate result, which states that the eigenvalues never collide is also obtained. This extends Bru's results obtained for the classical Wishart process (1989).

Запропоновано систему G-стохастичних диференціальних рівнянь для власних значень і власних векторів G-процесу Вішарта, визначену, як і у класичному випадку, через G-броунівську матрицю руху. З огляду на те, що елементи G-броунівської матриці руху не є обов'язково незалежними, в нашій моделі ми припускаємо, що їхні квадратні коваріації дорівнюють нулю. Отримано також проміжний результат про те, що власні значення ніколи не стикаються. Цей факт узагальнює результати Брю (1989), що отримані для класичного процесу Вішарта.

1. Introduction. Random matrices have been widely developed in recent years as a branch of mathematics, but also as applications in many fields of sciences such as physics, biology, population genetics, finance, meteorology and oceanography. The earliest studied ensemble of random matrices is the Wishart ensemble, introduced by Wishart [7] in 1928 in the context of multivariate data analysis, much before Wigner introduced the standard Gaussian ensembles of random matrices in the physics literature. In physics, Wishart matrices have appeared in multiple areas: In nuclear physics, quantum gravity and also in several problems in statistical physics. On the other hand, many studies have been done on the asymptotic behavior of the eigenvalues of random matrices, in particular by L. Pastur, M. Shcherbina [10]. In another context, Girko [4] used the perturbation technique to give the stochastic differential equations (SDEs) of eigenvalues and eigenvectors for a matrix-valued process with independent increments.

However, the notion of sublinear expectation space was introduced by Peng [3], which is a generalization of classical probability space. The G-expectation, a type of sublinear expectation, has played an important role in the researches of sublinear expectation space recently. Together with the notion of G-expectations Peng also introduced the related G-normal distribution and the G-Brownian motion. The G-Brownian motion is a stochastic process with stationary and independent increments and its quadratic variation process is, unlike the classical case, a non deterministic process. Moreover, an Itô calculus for the G-Brownian motion has been developed recently in [13 – 15].

The aim of this paper is to derive from the SDE of G-Wishart matrix with drift, a system of SDE for its eigenvalues, eigenvectors and prove that the eigenvalues never collide. As in the classical case, the G-Wishart matrix with drift is defined by $X_t = (B_t + \eta t)^T (B_t + \eta t)$, where η is a deterministic matrix and B_t is a G-Brownian motion matrix of dimension $n \times n$, the matrix stochastic process X_t takes values in the space of symmetric $n \times n$ matrices. In fact, our results are a generalization of the works obtained by Bru [1] and by E. Mayerhofer [8] in the sense that the classical Brownian motion is replaced by a G-Brownian motion. The main difficulties lie in the fact that the G-expectation is not linear and that the quadratic variation $\langle B\rangle$ is not a deterministic process. The notion of independence of random variables with respect to a non linear expectation being delicate, so we assume in our model that $\langle B^{ij}, B^{kl}\rangle = 0$ if $(i,j) \neq (k,l)$ and $\langle B^{ij}\rangle$ depend only on j.

The remainder of this paper is organized as follows. In Section 2, we recall some notions and properties in the G-expectation space which will be useful in this paper. Section 3 deals with G-Brownian motion matrix and G-Wishart process with drift. In Section 4, we state the main results of this paper, that is the study of SDEs satisfied by the eigenvalues and the eigenvectors processes and the fact that the eigenvalues never collide. Section 5 is devoted to the proof of the main results.

2. Basic settings. For the convenience of the reader, we review some basic notions and results of the G-expectation, the related spaces of random variables and the SDEs driven by a G-Brownian motion (for more details see [3, 11, 12]).

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on Ω, such that $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. $\mathcal{H}$ is considered as the space of random variables.

Definition 1. *A sublinear expectation on $\mathcal{H}$ is a functional $\hat{E}\colon \mathcal{H} \to \mathbb{R}$ satisfying the following properties: For all $X,\ Y \in \mathcal{H}$, we have:*

1) *monotonicity: if $X \geq Y$, then $\hat{E}[X] \geq \hat{E}[Y]$;*
2) *preservation of constants: $\hat{E}[c] = c$ for all $c \in \mathbb{R}$;*
3) *subadditivity: $\hat{E}[X] - \hat{E}[Y] \leq \hat{E}[X-Y]$;*
4) *positive homogeneity: $\hat{E}[\lambda X] = \lambda \hat{E}[X]$ for all $\lambda \geq 0$.*

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

We denote by $C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ the space of real continuous functions defined on $\mathbb{R}^n$ such that

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)\,|x-y| \qquad \text{for all} \qquad x, y \in \mathbb{R}^n,$$

where $k \in \mathbb{N}$ and $C > 0$ depend only on φ.

Definition 2. *In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$, a random vector $Y \in \mathcal{H}^n$ is said to be independent from another random vector $X \in \mathcal{H}^m$ under $\hat{E}$, if*

$$\hat{E}[\varphi(X,Y)] = \hat{E}\left[\hat{E}[\varphi(x,Y)]\Big|_{x=X}\right] \qquad \forall \varphi \in C_{l,\mathrm{Lip}}\left(\mathbb{R}^{m+n}\right).$$

Let X_1 and X_2 be two n-dimensional random vectors defined respectively in the sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if

$$\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)] \qquad \textit{for all} \qquad \varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n).$$

If $\overline{X}$ is independent from X and $\overline{X} \overset{d}{=} X$, then $\overline{X}$ is said to be an independent copy of X.

After the above basic definition we introduce now the central notion of G-normal distribution.

Definition 3. *Let be given two reals $\overline{\sigma}$, $\underline{\sigma}$ with $0 \leq \underline{\sigma} \leq \overline{\sigma}$. A random variable ξ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{E})$ is called $G_{\overline{\sigma},\underline{\sigma}}$-normally distributed, denoted by $\xi \sim \mathcal{N}\left(0; \left[\underline{\sigma}^2; \overline{\sigma}^2\right]\right)$, if for each $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R})$, the following function defined by*

$$u(t,x) = \hat{E}\left[\varphi\left(x + \sqrt{t}\xi\right)\right], \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

is the unique viscosity solution of the parabolic partial differential equation

$$\partial_t u(t,x) = G\left(\partial^2_{xx} u(t,x)\right), \quad (t,x) \in [0,\infty) \times \mathbb{R},$$

$$u(0,x) = \varphi(x).$$

Here $G = G_{\overline{\sigma},\underline{\sigma}}$ is the following sublinear function parameterized by $\underline{\sigma}$ and $\overline{\sigma}$:

$$G(\alpha) = \frac{1}{2}\left(\overline{\sigma}^2 \alpha^+ - \underline{\sigma}^2 \alpha^-\right), \quad \alpha \in \mathbb{R}$$

(*recall that $\alpha^+ = \max\{0,\alpha\}$ and $\alpha^- = -\min\{0,\alpha\}$). In fact $\overline{\sigma}^2 = \hat{E}\left[\xi^2\right]$ and $\underline{\sigma}^2 = -\hat{E}\left[-\xi^2\right]$.*

Definition 4. *A stochastic process $(B_t)_{t\geq 0}$ in a sublinear expectation space $\left(\Omega,\mathcal{H},\hat{E}\right)$ is called a G-Brownian motion if the following properties are satisfied*:

(i) $B_0 = 0$;

(ii) *for each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $\mathcal{N}\left(0;\left[\underline{\sigma}^2 s;\overline{\sigma}^2 s\right]\right)$-distributed and independent from $(B_{t_1},\ldots,B_{t_n})$ for all $n \in \mathbb{N}$ and $0 \leq t_1 \leq \ldots \leq t_n \leq t$.*

Definition 5. *A d-dimensional random vector $X = (X_1,\ldots,X_d)$ in a sublinear expectation space $\left(\Omega,\mathcal{H},\hat{E}\right)$ is called G-normal distributed if for each $a,\ b \geq 0$*:

$$aX + b\overline{X} \overset{d}{=} \sqrt{a^2+b^2}X,$$

where $\overline{X}$ is an independent copy of X, and

$$G(A) := \frac{1}{2}\hat{E}\left[\langle AX, X\rangle\right] : \mathbb{S}_d \to \mathbb{R},$$

here $\mathbb{S}_d$ denotes the collection of $d \times d$ symmetric matrices.

By [12] we know that $X = (X_1,\ldots,X_d)$ is G- normal distributed if and only if $u(t,x) := := \hat{E}\left[\varphi\left(x+\sqrt{t}X\right)\right]$, $(t,x) \in [0,\infty) \times \mathbb{R}^d$, $\varphi \in C_{l,\mathrm{Lip}}(\mathbb{R}^n)$ is the unique viscosity solution of the following G-heat equation:

$$\partial_t u(t,x) = G(Du(t,x)), (t,x) \in [0,\infty) \times \mathbb{R},$$

$$u(0,x) = \varphi(x),$$

where $Du(t,x)$ is the Hessian of $u(t,x)$.

The function $G(.) : \mathbb{S}_d \to \mathbb{R}$ is a monotonic, sublinear functional on $\mathbb{S}_d$, from which we can deduce that there exists a bounded, convex and closed subset $\Sigma \in \mathbb{S}_d^+$ the collection of nonnegative matrices in $\mathbb{S}_d$ such that

$$G(A) = \frac{1}{2}\sup_{B\in\Sigma} \mathrm{tr}(AB).$$

We write $X \sim \mathcal{N}(0;\Sigma)$.

We now give the definition of d-dimensional G-Brownian motion.

Definition 6. *A d-dimensional process $(B_t)_{t\geq 0}$ defined on $\left(\Omega,\mathcal{H},\widehat{\mathbb{E}}\right)$ is called a d- dimensional G-Brownian motion if the following properties are satisfied*:

(i) $B_0 = 0$;

(ii) *for all $t,\ s \geq 0$, the increment $B_{t+s} - B_t$ is $\mathcal{N}(0; s\sum)$-distributed and independent of $(B_{t_1},\ldots,B_{t_n})$, for each $n \in \mathbb{N}$ and each sequence $0 \leq t_1 \leq \ldots \leq t_n \leq t$.*

Note that $\langle a, B_t\rangle$ is a real $G_{\underline{\sigma_a},\overline{\sigma_a}}$-Brownian motion for each $a \in \mathbb{R}^d$, where $\langle .,.\rangle$ is tihe Euclidean inner product of $\mathbb{R}^d$, $\overline{\sigma_a}^2 = \hat{E}\left(\langle a, B_1\rangle^2\right)$ and $\underline{\sigma_a}^2 = -\hat{E}\left(-\langle a, B_1\rangle^2\right)$ (for more details see [12]).

3. G-Whishart process with drift. In the following we will identify each $n \times n$ matrix to a vector of n^2 dimension. Let us consider $\Omega = C_0(\mathcal{M}_n)$ be the set of all $\mathcal{M}_n$-valued continuous functions $(\omega_t)_{t\in\mathbb{R}^+}$ with $\omega_0 = \mathbf{0}$, where $\mathcal{M}_n$ is the space of $n \times n$ matrices, equipped with the distance

$$\rho\left(\omega^1, \omega^2\right) = \sum_{i=1}^{\infty} 2^{-i}\left[\left(\max_{t\in[0,i]}\left|\omega_t^1 - \omega_t^2\right|\right) \wedge 1\right], \quad \omega^1, \omega^2 \in \Omega.$$

We denote by $\mathcal{B}(\Omega)$ the Borel σ-algebra on Ω. We also set, for each $t \in [0,\infty)$, $\Omega_t := \{\omega_{.\wedge t} : \omega \in \Omega\}$. The spaces of Lipschitzian functions on Ω are denoted by

$$\mathrm{Lip}(\Omega_t) = \{\varphi(B_{t_1\wedge t}, \ldots, B_{t_n\wedge t}) : t_1, \ldots, t_n \in [0;\infty),\ \varphi \in C_{l,\mathrm{Lip}}(\mathcal{M}_n)\},$$

$$\mathrm{Lip}(\Omega) = \bigcup_{n=1}^{\infty} \mathrm{Lip}(\Omega_n).$$

Here we use $C_{l,\mathrm{Lip}}(\mathcal{M}_n)$ in our framework only for convenience. In general $C_{l,\mathrm{Lip}}(\mathcal{M}_n)$ can be replaced by the following spaces of functions defined on $\mathcal{M}_n$:

$L^\infty(\mathcal{M}_n)$: the space of all bounded Borel-measurable functions;

$C_{\mathrm{unif}}(\mathcal{M}_n)$: the space of all bounded and uniformly continuous functions;

$C_{b.\mathrm{Lip}}(\mathcal{M}_n)$: the space of all bounded and Lipschitz continuous functions;

$\mathrm{Lip}(\mathcal{M}_n)$: the space of all Lipschitzian functions on $\mathcal{M}_n$.

As in Peng [11,12], we can construct a nonlinear expectation $\hat{E}$ on $\mathrm{Lip}(\Omega)$, under which the coordinate process (B_t) (i.e., $B_t(\omega) = \omega_t$) is a G-Brownian motion matrix. Thus $\left(B_t^{ij}\right)$ is a $G_{\underline{\sigma_{ij}},\overline{\sigma_{ij}}}$-Brownian motion where $\overline{\sigma_{ij}}^2 = \hat{E}\left[\left(B_1^{ij}\right)^2\right]$ and $\underline{\sigma_{ij}}^2 = -\hat{E}\left[-\left(B_1^{ij}\right)^2\right]$ for each i, $j \in \overline{1,n}$.

Following Peng [3] (see also [11] for a simple proof), there exists a weakly compact family $\mathcal{P}$ of probability measures defined on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\hat{E}[X] = \sup_{P\in\mathcal{P}} E^P[X] \ \text{ for each } X \in \mathrm{Lip}(\Omega),$$

where E^P stands for the linear expectation under P. We say that a property holds quasi surely (q.s.) if it holds P a.s. for each $P \in \mathcal{P}$.

Let $T > 0$ be a fixed time. We denote by $L_G^p(\Omega_T)$, $p \geq 1$, the completion of G-expectation space $\mathrm{Lip}(\Omega_T)$ under the norm $\|X\|_{p,G} := \left(\hat{E}[|X|^p]\right)^{\frac{1}{p}}$. Peng [12] defined also the conditional expectation $\hat{E}(. \mid \Omega_t)$, which is continuous on $L_G^p(\Omega_T)$.

Definition 7 [12]. *A process $(M_t)_{t\geq 0}$ is called a G-martingale if for each $t \in [0;T]$, $M_t \in L_G^1(\Omega_t)$ and for each $s \in [0,t]$ we have $\hat{E}[M_t \mid \Omega_s] = M_s$ q.s.*

For each partition $\{t_0, \ldots, t_N\} = \pi_T$ of $[0,T]$, such that $0 = t_0 < t_1 < \ldots < t_N = T$, we set

$$\mu(\pi_T) = \max\{|t_{i+1} - t_i| : i = 0, \ldots, N-1\}.$$

Definition 8. *Let $M_G^{p,0}(0,T;\mathbb{R})$ be the collection of processes in the following form: for a given sequence $\left(\pi_T^N\right)$ of partitions of $[0,T]$ such that $\lim_{N\to\infty}\mu\left(\pi_T^N\right) = 0$,*

$$\eta_t(\omega)=\sum_{j=0}^{N-1}\xi_j(\omega)\mathbf{1}_{[t_j,t_{j+1})}(t),$$

where $\xi_j\in \mathrm{Lip}\left(\Omega_{t_j}\right),\ j=0,\ldots,N-1.$

Definition 9. *The quadratic variation of* $\left(B_t^{ij}\right)_{t\geq 0}$ *is defined by*

$$\left\langle B^{ij}\right\rangle_t:=\lim_{\mu\left(\pi_t^N\right)\to 0}\sum_{l=0}^{N-1}\left(B_{t_{l+1}}^{ij}-B_{t_l}^{ij}\right)^2=\left(B_t^{ij}\right)^2-2\int_0^t B_s^{ij}dB_s^{ij}.$$

It was proved in [3,12] that $\underline{\sigma_{ij}}^2t\leq\left\langle B^{ij}\right\rangle_t\leq\overline{\sigma_{ij}}^2t$. We denote by $M_G^p(0,T;\mathbb{R})$ the completion of $M_G^{p,0}(0,T;\mathbb{R})$ under the norm $\|\eta\|_{M_G^p}=\left(\hat{E}\left[\int_0^T|\eta_s|^p\,ds\right]\right)^{1/p}$ for $p\geq 1$.

Definition 10. *For each* $\eta\in M_G^{2,0}(0,T;\mathbb{R})$, *we define*

$$I(\eta)=\int_0^T\eta_t dB_t^{ij}:=\sum_{l=0}^{N-1}\xi_j\left(B_{t_{l+1}}^{ij}-B_{t_l}^{ij}\right).$$

The mapping $I\colon M_G^{2,0}(0,T;\mathbb{R})\to L_G^2(\Omega_T)$ *is continuous and thus can be continuously extended to* $M_G^2(0,T;\mathbb{R})$.

Definition 11. *The integral of a process* $\eta\in M_G^{1,0}(0,T;\mathbb{R})$ *with respect to* $\left\langle B^{ij}\right\rangle_t$ *is defined by*

$$Q(\eta)=\int_0^T\eta_t d\left\langle B^{ij}\right\rangle_t:=\sum_{l=0}^{N-1}\xi_l\left(\left\langle B^{ij}\right\rangle_{t_{l+1}}-\left\langle B^{ij}\right\rangle_{t_l}\right).$$

The mapping $Q\colon M_G^{1,0}(0,T;\mathbb{R})\to L_G^1(\Omega_T)$ *is continuous and thus can be continuously extended to* $M_G^1(0,T;\mathbb{R})$.

Unlike the classical theory, the quadratic covariation process of B is not always a deterministic process and it can be formulated in $L_G^2(\Omega_t)$ by

$$\left\langle B^{ij},B^{kl}\right\rangle_t:=B_t^{ij}B_t^{kl}-\int_0^t B_s^{ij}dB_s^{kl}-\int_0^t B_s^{kl}dB_s^{ij}.$$

For the following generalized Itô formula (see [11] for the vectorial case), we use Einstein's notation.

Theorem 1. *Let* $\varphi\in C^2(\mathcal{M}_n)$ *and its first and second derivatives are in* $C_{b,\mathrm{Lip}}(\mathcal{M}_n)$. *Let* $X=\left(X^{ij}\right)$ *be a matrix process on* $[0,T]$ *with the form*

$$X_t^{pq}=X_0^{pq}+\int_0^t\alpha^{pq}(s)\,ds+\int_0^t\theta_{ijkl}^{pq}(s)\,d\left\langle B^{ij},B^{kl}\right\rangle_s+\int_0^t\beta_{kl}^{pq}(s)\,dB_s^{kl},$$

where $\alpha^{pq},\theta_{ijkl}^{pq}\in M_G^1(0,T)$ *and* $\beta_{kl}^{pq}\in M_G^2(0,T)$. *Then for each* $t\in[0,T]$, *we have*

$$\varphi(X_t) - \varphi(X_0) =$$

$$= \int_0^t \partial_{x^{pq}} \varphi(X_u)\, \beta_{kl}^{pq}(u)\, dB_u^{kl} + \int_0^t \partial_{x^{pq}} \varphi(X_u)\, \alpha^{pq}(u)\, du +$$

$$+ \int_0^t \left[\partial_{x^{pq}} \varphi(X_u)\, \theta_{ijkl}^{pq}(u) + \frac{1}{2} \partial^2_{x^{p'q'} x^{pq}} \varphi(X_u)\, \beta_{ij}^{pq}(u)\, \beta_{kl}^{p'q'}(u) \right] d\left\langle B^{ij}, B^{kl} \right\rangle_u \quad q.s.$$

Note that this formula remains valid if X is not a square matrix.

***Remark* 1.** By a direct application of the G-Itô's formula, we find the integration formula by parts, that is

$$d\left(X_t^{pq} X_t^{mn}\right) = dX_t^{pq} X_t^{mn} + X_t^{pq} dX_t^{mn} + dX_t^{pq} dX_t^{mn},$$

where $dX_t^{pq} dX_t^{mn} = \sum_{i,j,k,l} \beta_t^{pqij} \beta_t^{mnkl} d\left\langle B^{ij}, B^{kl}\right\rangle_t$.

Then we have $d\left\langle B^{ij}, B^{kl}\right\rangle = dB^{ij} dB^{kl}$.

In the rest of this paper, we write $\left\langle B^{ij}\right\rangle_t$ instead of $\left\langle B^{ij}, B^{ij}\right\rangle_t$ and we assume that B satisfies the following assumption:

(A) There exist an increasing real process b^j such that $\left\langle B^{ij}, B^{kl}\right\rangle_t = \delta_{ik}\delta_{jl} b_t^j$ q.s. for each i, j, $k, l \in \overline{1, n}$, where δ_{uv} is the Kronecker symbol.

Then we get $\underline{\sigma}^2 t \le b_t^j \le \overline{\sigma}^2 t$, where $\overline{\sigma} := \max_{i,j} \overline{\sigma_{ij}}$ and $\underline{\sigma} := \min_{i,j} \underline{\sigma_{ij}}$. Note that in the classical case the assumption (A) is satisfied with $b_t^j = t$.

Definition 12. *A G-Wishart process with drift is defined by* $X_t = (B_t + \eta t)^T\, (B_t + \eta t)$, *where* η *is a deterministic matrix, and* Y^T *denotes the transpose of a matrix* Y.

Note that if B is the classical Brownian motion, then X is the classical Wishart process with drift, which appeared in many different applications such as communication technology, nuclear physics, quantum chromodynamics, statistical physics of directed polymers in random media.

As in the classical case, we define the *Stratonovich differential* $\circ$ for two matrices X and Y:

$$X \circ dY = XdY + \frac{1}{2} dXdY \text{ and } dX \circ Y = dXY + \frac{1}{2} dXdY,$$

where $dXdY$ is the matricial product. The following proposition holds.

Proposition 1. *For each matrices* X, Y *defined as in the above theorem, we have:*

(i) *the integration formula by parts*:

$$d(XY) = XdY + dXY + dXdY,$$

(ii) *the formulae*

$$d(XY) = dX \circ Y + X \circ dY,$$

$$dX \circ (YZ) = (dX \circ Y) \circ Z$$

and

$$(X \circ dY)^T = dY^T \circ X^T.$$

Proof. (i) follows from the Remark 1.

(ii) follows from (i) and the definition of the Stratonovich differential.

4. Main results. We wish to find SDEs of eigenvalues and eigenvectors for a G-Wishart process with drift. Our approach is the same as that used in [1]. The idea is simply to set

$$M_t = B_t + \eta t.$$

Then we have

$$X_t = M_t^T M_t,$$

and

$$dX_t = dB_t^T M_t + M_t^T dB_t + \left(\eta^T M_t + M_t^T \eta\right) dt + dB_t^T dB_t.$$

Note that $dB_t^T dB_t = nd\langle B\rangle_t$. Since $dX^{ii}dX^{ii} = 4X^{ii}db^i$, then

$$dX^{ii} = 2\sqrt{X^{ii}}d\kappa^i + 2\left(\eta^T M\right)^{ii} dt + ndb^i, \qquad i = 1, \ldots, n,$$

where κ^i are G-Brownian motions, such that $\left\langle \kappa^i, \kappa^j \right\rangle = \delta_{ij} b^j$ for each $i,\ j$.

In what follows, let $H_t^T X_t H_t = \Lambda_t := \operatorname{diag}\left(\lambda_i\left(t\right)\right)$ be the diagonalization form of X_t, where H_t is an orthonormal matrix. The following result is the equivalent of Theorem 18.5.1 [4], proved in the linear context for a matrix-valued process with independent increments.

Theorem 2. *Suppose that at time $t = 0$ all the eigenvalues are distinct. Then the following G-stochastic differential system holds*:

$$d\lambda_i = 2\sqrt{\lambda_i}\sum_l H^{li}d\nu^l + 2\left(\eta H\right)^{ii}\sqrt{\lambda_i}dt + \sum_l \left(H^{li}\right)^2 db^l \left[\sum_{p\neq i}\frac{\lambda_p}{\lambda_i - \lambda_p} + n\right] +$$

$$+\lambda_i \sum_{p\neq i}\frac{1}{\lambda_i - \lambda_p}\sum_l \left(H^{lp}\right)^2 db^l, \tag{1}$$

where $\nu^1, \nu^2, \ldots, \nu^n$ are G-Brownian motions, satisfying $\left\langle \nu^i, \nu^j \right\rangle = \delta_{ij} b^j$ for each $i, j \in \overline{1, n}$.

***Corollary* 1.** *Assume that $b^i = b$ for each $i = 1, \ldots, n$. Then we have the following SDEs*: *for $t < T_0 \colon = \inf\left\{t \colon \det\left(X_t\right) = 0\right\}$*

$$d\left(\operatorname{tr}\left(X_t\right)\right) = 2\sqrt{\operatorname{tr}\left(X_t\right)}d\gamma_t + 2\operatorname{tr}\left(\eta^T\sqrt{X_t}\right)dt + n^2 db_t, \tag{2}$$

$$d\left(\det\left(X_t\right)\right) = 2\det X_t\sqrt{\operatorname{tr}\left(X_t^{-1}\right)}d\beta_t + 2\det X_t \operatorname{tr}\left(\eta H X^{-\frac{1}{2}} H\right)dt +$$

$$+\det X_t \operatorname{tr}\left(X_t^{-1}\right)db_t, \tag{3}$$

$$d\left(\log\left(\det\left(X_t\right)\right)\right) = 2\sqrt{\operatorname{tr}\left(X_t^{-1}\right)}d\beta_t + 2\operatorname{tr}\left(\eta H X^{-\frac{1}{2}} H\right)dt - \operatorname{tr}\left(X_t^{-1}\right)db_t, \tag{4}$$

$$d\left(\det\left(X_t\right)^r\right) = 2r\left(\det X_t\right)^r\sqrt{\operatorname{tr}\left(X_t^{-1}\right)}d\beta_t + 2r\left(\det X_t\right)^r \operatorname{tr}\left(\eta H X^{-\frac{1}{2}} H\right)dt +$$

$$+r\left(2r - 1\right)\left(\det X_t\right)^r \operatorname{tr}\left(X_t^{-1}\right)db_t \quad \textit{for} \quad r \in \mathbb{R}, \tag{5}$$

where γ (resp. β) is a G-real Brownian motion, such that $\langle\gamma\rangle = b$ (resp. $\langle\beta\rangle = b$).

***Remark* 2.** It is easy to derive from this corollary, the corresponding of these SDEs in the classical case which are obtained by Demni [2].

As in [1], the matrix H_t satisfy

$$dH = H \circ dA = HdA + \frac{1}{2}HdAdA,$$

where A is the skew-symmetric $n \times n$ matrix, such that

$$dA = H^T \circ dH.$$

Theorem 3. *The eigenvectors satisfy the following equations*:

$$dH^{ij} = \sum_{k \neq j} H^{ik} \frac{1}{\lambda_j - \lambda_k} \left(\sqrt{\lambda_k} \sum_l H^{lj} d\beta^{jl} + \sqrt{\lambda_j} \sum_l H^{lk} d\beta^{kl} \right) + \frac{1}{2} \sum_k H^{ik} dV^{kj}, \qquad (6)$$

where

$$dV^{kj} = \sum_{p \neq j, p \neq k} \frac{1}{(\lambda_p - \lambda_k)(\lambda_j - \lambda_p)} \left[\lambda_p \sum_l H^{lk} H^{lj} db^l + \delta_{kj} \lambda_k \sum_l H^{lp} H^{lp} db^l \right],$$

and $\left(\beta^{ij}\right)$ *is a G-Brownian motion matrix satisfying the assumption* (A).

Collision time. Let us consider the first collision time $\tau = \inf\left\{t \geq 0 \colon \lambda_j(t) - \lambda_i(t) = 0 \text{ for some } i \neq j\right\}$.

Corollary 2. *If, at time* $t = 0$, *the eigenvalues of* X *are distinct* $\lambda_1 < \lambda_2 < \ldots < \lambda_n$, *then they will never collide, that is,* $\tau = +\infty$ *q.s.*

Proof. Let $(\lambda_j(t) - \lambda_i(t))_{t < \tau}$ be the $\mathbb{R}_+ \backslash \{0\}$ valued stochastic process. As in [1], by applying the G-Itô formula to $U = -\sum_{i<j} \log(\lambda_j - \lambda_i)$ and by using the fact that the quadratic covariation of λ_i, λ_j is equal to $4\delta_{ij}\sqrt{\lambda_i}\sqrt{\lambda_j}db^i$, we get

$$dU = \sum_{i<j} \frac{d\lambda_i - d\lambda_j}{\lambda_j - \lambda_i} + 2\sum_{i<j} \frac{\lambda_i db^i + \lambda_j db^j}{(\lambda_j - \lambda_i)^2}.$$

It follows from the SDE satisfied by λ_i that

$$\langle U \rangle_t = 4 \sum_{i<j} \int_0^t \frac{\lambda_i(s) \sum_l \left(H_s^{li}\right)^2 db_s^l + \lambda_j(s) \sum_l \left(H_s^{lj}\right)^2 db_s^l}{(\lambda_j(s) - \lambda_i(s))^2}.$$

Obviously U_t is a classical local martingale with respect to its natural filtration under each probability measure $P \in \mathcal{P}$. If two eigenvalues collide, then there exists $P^0 \in \mathcal{P}$ such that $P^0(\tau = +\infty) < 1$, $\lim_{t \uparrow \tau} U_t = -\infty$ and U is continuous on $[0, \tau]$. Let ξ_t be the inverse of $\langle U \rangle_t$. By the argument of McKean [9, p. 47], and Bru [1], the process $\widetilde{B}_t = U_{\xi_t}$ is a Brownian motion on $[0, \langle U \rangle_\tau[$ P^0 a.s. on $\{\tau < +\infty\}$ [5, p. 92]. Thus,

$$\lim_{t \uparrow \langle U \rangle_{(\tau)}} \widetilde{B}_t = \lim_{\xi_t \uparrow \tau} U_{\xi_t} = \lim_{t \uparrow \tau} U_t = -\infty,$$

which is impossible for a Brownian motion. Hence $\tau = +\infty$ q.s.

5. Proofs of theorems. ***Proof of Theorem* 2.** Since

$$\left(dM^T dM\right)^{ij} = \sum_k dB^{ki} dB^{kj} = n\delta_{ij} db^i,$$

then

$$dX^{ij} = \sum_k \left(M^{kj} dB^{ki} + M^{ki} dB^{kj}\right) + \sum_k \left(\eta^{ki} M^{kj} + M^{ki}\eta^{kj}\right) dt + n\delta_{ij} db^i,$$

which implies that

$$dX^{ij} dX^{km} = \sum_p \left(M^{pj} dB^{pi} + M^{pi} dB^{pj}\right) \sum_q \left(M^{qm} dB^{qk} + M^{qk} dB^{qm}\right) =$$

$$= \left(X^{jm}\delta_{ik} + X^{jk}\delta_{im}\right) db^i + \left(X^{im}\delta_{jk} + X^{ik}\delta_{jm}\right) db^j.$$

As in [1], we have $d\Lambda = dN - dA \circ \Lambda + \Lambda \circ dA$, where $dN\colon = H^T \circ dX \circ H$ and introduce the skew-symmetric matrix A such that, $A_0 = 0,\ dA = H^T \circ dH$. Then we obtain

$$dH = HdA + \frac{1}{2} HdAdA.$$

The process $\Lambda \circ dA - dA \circ \Lambda$ is zero on the diagonal, consequently $d\lambda_i = dN^{ii}$ and $0 = dN^{ij} + (\lambda_i - \lambda_j)\, dA^{ij}$, when $i \neq j$. Thus, for $\{t < \tau\}$,

$$dA^{ij} = \frac{1}{\lambda_j - \lambda_i} dN^{ij}.$$

In fact $dN = H^T dXH + \frac{1}{2} H^T dXdH + \frac{1}{2} dH^T dXH$, which implies that the G-martingale part of dN equals the G-martingale part of $H^T dXH$ given by

$$dN^{ij} dN^{km} = \sum_{p,q} H^{pi} dX^{pq} H^{qj} \sum_{p',q'} H^{p'k} dX^{p'q'} H^{q'm} =$$

$$= \sum_{p,q,p',q'} H^{pi} H^{qj} H^{p'k} H^{q'm} \left[\left(X^{qq'}\delta_{pp'} + X^{qp'}\delta_{pq'}\right) db^p + \right.$$

$$\left. + \left(X^{pq'}\delta_{qp'} + X^{pp'}\delta_{qq'}\right) db^q\right] =$$

$$= \sum_p \left[H^{pi} H^{pk} db^p \sum_{q,q'} H^{qj} X^{qq'} H^{q'm} + \right.$$

$$\left. + H^{pi} H^{pm} db^p \sum_{q,p'} H^{qj} X^{qp'} H^{p'k}\right] +$$

$$+ \sum_q \left[H^{qj} H^{qk} db^q \sum_{p,q'} H^{pi} X^{pq'} H^{q'm} + \right.$$

$$\left.+H^{qj}H^{qm}db^q\sum_{p,p'}H^{pi}X^{pp'}H^{p'k}\right] =$$

$$=\sum_p H^{pi}H^{pk}db^p\Lambda^{jm}+\sum_p H^{pi}H^{pm}db^p\Lambda^{jk}+$$

$$+\sum_q H^{qj}H^{qk}db^q\Lambda^{im}+\sum_q H^{qj}H^{qm}db^q\Lambda^{ik} \tag{7}$$

and

$$dN^{ii}dN^{jj}=4\Lambda^{ij}\sum_p H^{pi}H^{pj}db^p=$$

$$=4\lambda_i\delta_{ij}\sum_p\left(H^{pi}\right)^2db^p.$$

The finite-variation part of dN

$$dF=H^T\left(\eta^TM+M^T\eta\right)Hdt,$$

since $M=\Lambda^{\frac{1}{2}}H^T$, then

$$dF=\left(H^T\eta^T\Lambda^{\frac{1}{2}}H^TH+H^TH\Lambda^{\frac{1}{2}}\eta H\right)dt=$$

$$=\left(H^T\eta^T\Lambda^{\frac{1}{2}}+\Lambda^{\frac{1}{2}}\eta H\right)dt.$$

It follows that

$$dF^{ii}=2\left(\eta H\right)^{ii}\sqrt{\lambda_i}dt.$$

Now we compute the integral part dQ of dN, with respect to db^i:

$$dQ=H^TdB^TdBH+\frac{1}{2}H^TdXdH+\frac{1}{2}dH^TdXH=$$

$$=H^TdB^TdBH+\frac{1}{2}\left(\left(dH^TH\right)\left(H^TdXH\right)+\left(H^TdXH\right)\left(H^TdH\right)\right)=$$

$$=H^TdB^TdBH+\frac{1}{2}\left(dNdA+(dNdA)^T\right).$$

Since

$$\left(H^TdB^TdBH\right)^{ij}=\sum_{p,q}H^{pi}\left(dB^TdB\right)^{pq}H^{qj}$$

and

$$\left(dB^TdB\right)^{pq}=\sum_l dB^{lp}dB^{lq}=\sum_l\delta_{pq}db^p=n\delta_{pq}db^p,$$

then

$$\left(H^TdB^TdBH\right)^{ij}=\sum_{p,q}H^{pi}n\delta_{pq}db^pH^{qj}=n\sum_p H^{pi}H^{pj}db^p. \tag{8}$$

On the other hand,

$$(dNdA)^{ij} = \sum_p dN^{ip} dA^{pj} = \sum_{p\neq j} dN^{ip} \frac{1}{\lambda_j - \lambda_p} dN^{pj} =$$

$$= \sum_{p\neq j} \frac{1}{\lambda_j - \lambda_p} \left[\sum_l H^{li} H^{lp} db^l \Lambda^{pj} + \sum_l H^{li} H^{lj} db^l \Lambda^{pp} + \right.$$

$$\left. + \sum_l H^{lp} H^{lp} db^l \Lambda^{ij} + \sum_l H^{lp} H^{lj} db^l \Lambda^{ip} \right] =$$

$$= \sum_{p\neq j} \frac{1}{\lambda_j - \lambda_p} \left[\lambda_p \sum_l H^{li} H^{lj} db^l + \lambda_i \delta_{ij} \sum_l \left(H^{lp} \right)^2 db^l + \right.$$

$$\left. + \lambda_i \delta_{ip} \sum_l H^{lp} H^{lj} db^l \right]. \tag{9}$$

It now follows from (8) and (9), that

$$dQ^{ii} = \sum_l \left(H^{li} \right)^2 db^l \left[\sum_{p\neq i} \frac{\lambda_p}{\lambda_i - \lambda_p} + n \right] + \lambda_i \sum_{p\neq i} \frac{1}{\lambda_i - \lambda_p} \sum_l \left(H^{lp} \right)^2 db^l.$$

Then

$$d\lambda_i = 2\sqrt{\lambda_i} \sum_l H^{li} d\nu^l + dF^{ii} + dQ^{ii},$$

where ν^l is a G-Brownian motion such that $d\nu^i d\nu^j = \delta_{ij} db^i$.

Theorem 2 is proved.

***Proof of Corollary* 1.** We have

$$d\left(\mathrm{tr}\left(X_t\right)\right) = \sum_i dX_t^{ii} =$$

$$= \sum_i \left(2\sqrt{X_t^{ii}} d\kappa_t^i + 2\left(\eta^T \sqrt{X_t} \right)^{ii} dt + n db_t \right).$$

On the other hand, since the quadratic variation of $\mathrm{tr}\,(X)$ is $4\sum_i X^{ii} db = 4\mathrm{tr}\,(X)\,db$, then

$$d\left(\mathrm{tr}\left(X_t\right)\right) = 2\sqrt{\mathrm{tr}\left(X_t\right)} d\gamma_t + 2\mathrm{tr}\left(\eta^T \sqrt{X_t} \right) dt + n^2 db_t.$$

Firstly, observe that, by using the formula (1), we obtain

$$d\lambda_i = 2\sqrt{\lambda_i} dv^i + 2\left(\eta H\right)^{ii} \sqrt{\lambda_i} dt + \left[\sum_{p\neq i} \frac{\lambda_p + \lambda_i}{\lambda_i - \lambda_p} + n \right] db.$$

By using the G-Itô formula, we have

$$d\left(\det X\right)=\sum_{i}\frac{\det X}{\lambda_i}d\lambda_i+\frac{1}{2}\sum_{i\neq j}\frac{\det X}{\lambda_i\lambda_j}d\lambda_i d\lambda_j.$$

It follows, from the fact that $d\lambda_i d\lambda_j=4\sqrt{\lambda_i}\sqrt{\lambda_j}\delta_{ij}db$, that

$$d\left(\det X\right)=\det X\sum_{i}\frac{d\lambda_i}{\lambda_i}=$$

$$=2\det X\sum_{i}\frac{dv^i}{\sqrt{\lambda_i}}+2\det X\sum_{i}\frac{\left(\eta H\right)^{ii}}{\sqrt{\lambda_i}}dt+$$

$$+\det X\sum_{i}\frac{1}{\lambda_i}\left[\sum_{p\neq i}\frac{\lambda_p+\lambda_i}{\lambda_i-\lambda_p}+n\right]db.$$

The formula (3) follows from the following facts:

$$\sum_{i}\frac{\left(\eta H\right)^{ii}}{\sqrt{\lambda_i}}=\mathrm{tr}\left(\eta H\Lambda^{-\frac{1}{2}}\right)=\mathrm{tr}\left(\eta HX^{-\frac{1}{2}}H\right),$$

$$\sum_{i}\frac{1}{\lambda_i}\left[\sum_{p\neq i}\frac{\lambda_p+\lambda_i}{\lambda_i-\lambda_p}+n\right]=\sum_{i}\frac{n}{\lambda_i}+\sum_{i}\sum_{p\neq i}\left(\frac{-1}{\lambda_i}+\frac{2}{\lambda_i-\lambda_p}\right)=$$

$$=n\mathrm{tr}\left(X^{-1}\right)-\left(n-1\right)\mathrm{tr}\left(X^{-1}\right)+2\sum_{i}\sum_{p\neq i}\frac{1}{\lambda_i-\lambda_p}=$$

$$=\mathrm{tr}\left(X^{-1}\right),$$

and the quadratic variation of $\det X$ is $4\left(\det X\right)^2\sum_{i,j}\frac{1}{\sqrt{\lambda_i}}\frac{1}{\sqrt{\lambda_j}}\delta_{ij}db=4\det X^2\mathrm{tr}\left(X^{-1}\right)db$. Equations (4) and (5) follows from (3), the G-Itô formula and the quadratic variation of $\det X$.

***Proof of Theorem* 3.** In order to find SDEs for H_t on $\{t<\tau\}$, we deduce from the definition of dA that

$$dH=H\circ dA=HdA+\frac{1}{2}HdAdA.$$

By using the formula (7), we have, for $i\neq j$,

$$dN^{ij}dN^{ij}=\lambda_i\sum_{l}\left(H^{lj}\right)^2db^l+\lambda_j\sum_{l}\left(H^{li}\right)^2db^l,$$

which implies that

$$dN^{ij}=\sqrt{\lambda_i}\sum_{l}H^{lj}d\beta^{jl}+\sqrt{\lambda_j}\sum_{l}H^{li}d\beta^{il},$$

where $\left(\beta^{il}\right)$ is a G-Brownian motion matrix satisfying the assumption (A). It follows that

$$dA^{ij}=\frac{1}{\lambda_j-\lambda_i}\left(\sqrt{\lambda_i}\sum_{l}H^{lj}d\beta^{jl}+\sqrt{\lambda_j}\sum_{l}H^{li}d\beta^{il}\right).$$

Now we compute $(dAdA)^{ij}$:

$$(dAdA)^{ij} = \sum_p dA^{ip} dA^{pj} =$$

$$= \sum_{p\neq j, p\neq i} \frac{1}{(\lambda_p - \lambda_i)(\lambda_j - \lambda_p)} dN^{ip} dN^{pj} =$$

$$= \sum_{p\neq j, p\neq i} \frac{1}{(\lambda_p - \lambda_i)(\lambda_j - \lambda_p)} \left[\sum_l H^{li} H^{lp} db^l \Lambda^{pj} + \sum_l H^{li} H^{lj} db^l \Lambda^{pp} + \right.$$

$$\left. + \sum_l H^{lp} H^{lp} db^l \Lambda^{ij} + \sum_l H^{lp} H^{lj} db^l \Lambda^{ip} \right] =$$

$$= \sum_{p\neq j, p\neq i} \frac{1}{(\lambda_p - \lambda_i)(\lambda_j - \lambda_p)} \left[\lambda_p \sum_l H^{li} H^{lj} db^l + \delta_{ij} \lambda_i \sum_l H^{lp} H^{lp} db^l \right].$$

Similarly as in [1], we obtain the formula (6).

Theorem 3 is proved.

***Example* 1.** We consider the case of the classical Wishart process, which corresponds to $\eta = 0$ and $X_t = B_t^T B_t$, where (B_t) is the classical Brownian motion matrix. It was shown in [1] that

$$d\lambda_i = 2\sqrt{\lambda_i} d\nu^i + \left[\sum_{p\neq i} \frac{\lambda_p + \lambda_i}{\lambda_i - \lambda_p} + n \right] dt,$$

where ν^i are classical Brownian motions. We can obtain this formula by the formula (1) with $b_t^i = t$ and the fact that $\sum_l \left(H^{li}\right)^2 = 1$. The same is true for the SDE of the eigenvectors.

***Remark* 3.** If we consider the G-Wishart process $X_t = Y_t^T Y_t$ where Y is the G-Ornstein–Uhlenbeck matrix, that is the solution of the G-SDE

$$dY_t = -\frac{1}{2} Y_t dt + a dB_t, \quad a > 0,$$

where B is a G-Brownian motion satisfying the assumption (A), we can obtain with the same manner (see [6]) that

$$d\lambda_i = 2a\sqrt{\lambda_i} \sum_l H^{li} d\nu^l - \lambda_i dt + a^2 \left[\sum_{p\neq i} \frac{\lambda_p}{\lambda_i - \lambda_p} + n \right] \sum_l \left(H^{li}\right)^2 db^l +$$

$$+ a^2 \lambda_i \sum_{p\neq i} \frac{1}{\lambda_i - \lambda_p} \sum_l \left(H^{lp}\right)^2 db^l,$$

and

$$dH^{ij} = a \sum_{k\neq j} H^{ik} \frac{1}{\lambda_j - \lambda_k} \left(\sqrt{\lambda_k} \sum_l H^{lj} d\beta^{jl} + \sqrt{\lambda_j} \sum_l H^{lk} d\beta^{kl} \right) + \frac{a^2}{2} \sum_k H^{ik} dV^{kj}.$$

6. Conclusion. In this paper, the system of the SDEs of eigenvalues and eigenvectors for a G-Wishart process with drift defined by using a G-Brownian motion matrix was given. This system has been difficult to obtain because of the fact that the quadratic variation of the G-Brownian motion is not deterministic. Added to that, our main difficulty lies in the fact that the entries of the G-Brownian motion matrix are not independent in general. To avoid these difficulties, it was assumed in our model, that the quadratic covariations of the entries of the G-Brownian motion matrix are zero and that the quadratic variations depend only on the index of column. The G-formula of integration by parts was the key of this work. An intermediate result of the non collision of the eigenvalues was also proven.

References

1. *Bru M. F.* Diffusions of perturbed principal component analysis // J. Multivar. Anal. – 1989. – **29**, № 1. – P. 127 – 136.
2. *Demni N.* The Laguerre process and generalized Hartman – Watson law // Bernoulli. – 2007. – **13**, № 2. – P. 556 – 580.
3. *Denis L., Hu M., Peng S.* Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion paths // Potential Anal. – 2011. – **34**, № 2. – P. 139 – 161.
4. *Girko V. L.* Theory of random determinants. – Dordrecht: Kluwer Acad. Publ., 1990.
5. *Ikeda N., Watanabe S.* Stochastic differential equation and diffusion processes. – Amsterdam: North-Holland, 1981.
6. *Katori M., Tanemura H.* Complex Brownian motion representation of the Dyson model // Electron. Commun. Probab. – 2013. – **18**, № 4. – P. 1 – 16.
7. *Majumdar S. N.* Handbook of random matrix theory. – Oxford Univ. Press, 2011.
8. *Mayerhofer E., Pfaffel O., Stelzer R.* On strong solutions for positive definite jump diffusions // Stochast. Process. and Appl. – 2011. – **121**, № 9. – P. 2072 – 2086.
9. *McKean H. P.* Stochastic integrals. – New York: Acad. Press, 1969.
10. *Pastur L., Shcherbina M.* Eigenvalue distribution of large random matrices // Math. Surveys Monogr. – 2011. – **171**.
11. *Peng L., Falei W.* On the comparison theorem for multi-dimensional G-SDEs // Statist. Probab. Lett. C. – 2015. – **96**. – P. 38 – 44.
12. *Peng S.* Nonlinear expectations and stochastic calculus under uncertainty with robust central limit theorem and G-Brownian motion // *arXiv:1002:4546v1* (2010).
13. *Soner H. M., Touzi N., Zhang J.* Martingale representation theorem for the G-expectation // Stochast. Process. and Appl. – 2011. – **121**, № 2. – P. 265 – 287.
14. *Sun Z., Zhang X., Guo J.* A stochastic maximum principle for processes driven by G-Brownian motion and applications to finance // *arxiv:1402.6793v2 [math.OC]* (2014).
15. *Zhang H.* A complex version of G-expectation and its application to conformal martingale // *arxiv: 1502.02787v1 [math.PR]* (2015).

Received 25.04.16