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HARDY'S AND MIYACHI'S THEOREMS FOR THE FIRST HANKEL – CLIFFORD TRANSFORM

ТЕОРЕМИ ГАРДІ ТА МІЯЧІ ПРО ПЕРШЕ ПЕРЕТВОРЕННЯ ГАНКЕЛЯ – КЛІФФОРДА

We present an analog of Hardy's and Miyachi's theorems for the first Hankel – Clifford transform.

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1. Introduction. In signal processing, the uncertainty principle states that the signal variances product in the time and frequency domains has a lower bound. The mathematical formulation of this fact is that the uncertainty principles for the Fourier transform relate the variances of a function and its Fourier transform which cannot both be simultaneously sharped localized. One example of this is the Heisenberg uncertainty principle concerning the position and the momentum wave functions in quantum physics. Many mathematical formulations of this general fact can be found in [7]. Namely theorems of Hardy [6], Cowling and Price [3] and Miyachi [11]. In 1933, Hardy [6] demonstrated the following theorem: if $|f(x)| \leq Ce^{-ax^2}$ and $|\widehat{f}(y)| \leq Ce^{-by^2}$ for some positive numbers a, b and C, then $f = 0$ whenever $ab > 1/4$. If $ab = 1/4$, then the function f is a constant multiple of e^{-ax^2}, and if $ab < 1/4$, then are infinitely functions which realise both conditions. In 1997, Miyachi [11] proved the next theorem:

Theorem 1.1. *Let f be an integrable function on* $\mathbb{R}$ *such that*

$$e^{ax^2} f \in L^1(\mathbb{R}) + L^\infty(\mathbb{R}).$$

Further, assume that

$$\int\limits_{\mathbb{R}} \log^+ \left(\frac{|e^{b\lambda^2} \widehat{f}(\lambda)|}{c} \right) d\lambda < \infty$$

for some positive numbers a, b *and* c. *If* $ab = 1/4$, *then f is a constant multiple of the Gaussian* e^{-ax^2}.

The first Hankel – Clifford transform has great importance in solving problems involving cylindrical boundaries. The Hankel – Clifford transformation is useful mathematical tools in solving a certain class of partial differential equations, involving the generalized Kepinsky – Myller – Lebedev differential operator.

In [4] the authors gave a version of Morgan and Cowling – Price in the case of the first Hankel – Clifford transform.

The purpose of this paper is to demonstrate the Hardy's and Miyachi's theorems for the first Hankel – Clifford transform.

First, we would like to mention some main results of the first Hankel – Clifford operator. In the second, we are going to review two principal lemmas of the complex variables theory, which are a version of the Phragmen – Lindelöff theorem. After that, we are going to give a version of Hardy's theorem associated with the first Hankel – Clifford transform. In the last section, we are proving an analog of Miyachi's theorem for the first Hankel – Clifford transform.

In [2, 9, 12], the first Hankel – Clifford transform of order $\mu \geq 0$ was introduced by

$$(h_{1,\mu}\varphi)(\lambda) = \lambda^{\mu} \int\limits_{0}^{+\infty} C_{\mu}(x\lambda)\varphi(x)\, dx,$$

where C_{μ} is the Bessel – Clifford function of the first kind of order μ [5] and defined by

$$C_{\mu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!\,\Gamma(\mu + k + 1)}$$

which is a solution of the differential equation

$$x\frac{\partial^2}{\partial x^2}y + (y+1)\frac{\partial}{\partial x}y + y = 0$$

and is closely related with the Bessel function of the first kind J_{μ} and index μ by

$$C_{\mu}(x) = x^{\frac{-\mu}{2}} J_{\mu}\big(2x^{\frac{1}{2}}\big),$$

where Bessel function J_{μ} is defined in [8] by

$$2^{\mu}x^{-\mu}J_{\mu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\mu + n + 1)} \left(\frac{x}{2}\right)^{2n}.$$

The inversion formula of the first Hankel – Clifford transform is defined by

$$\varphi(x) = (h^{-1}_{1,\mu}h_{1,\mu}\varphi)(x) = x^{\mu} \int\limits_{0}^{+\infty} C_{\mu}(\lambda, x)\varphi(\lambda)\, d\lambda.$$

From [10], the first Hankel – Clifford transform satisfied that

$$h^{-1}_{1,\mu} = h_{1,\mu} \quad \text{for} \quad \mu \geq 0.$$

The demonstration of the basic outcomes relies on the following two complex variable lemmas, which will be presented in this section.

Lemma 1.1. *Let h be an entire function on $\mathbb{C}$ such that*

$$|h(z)| \leq Ce^{a|z|^2} \quad \forall z \in \mathbb{C}$$

and

$$|h(t)| \leq Ae^{-at^2} \quad \forall t \in \mathbb{R}$$

for some positive constants a, B and C. Then $h(z) = \text{const}\; e^{-az^2}$, $z \in \mathbb{C}$.

Proof. See [13] (Lemma 2.1).

Let us define $\log^{+}(x) = \log(x)$ if $x > 1$, and $\log^{+}(x) = 0$ otherwise. We also need the following lemma.

Lemma 1.2. *Let h an entire function and suppose that, there exist constants $A, B > 0$ such that*

$$|h(z)| \leq A e^{B \Re(z)^2} \quad \forall z \in \mathbb{C}$$

and

$$\int\limits_{-\infty}^{+\infty} \log^+ |h(t)| dt < \infty.$$

Then h is a constant function.

Proof. See [11] (Lemma 4).

We need also an estimated results for the Bessel's function, we have the next lemma [1].

Lemma 1.3. *Let $\mu - 1/2$. We have the following results*:

(1) $|j_\mu(x)| \leq 1,$

(2) $1 - j_\mu(x) = O(1),\ x \geq 1,$

(3) $1 - j_\mu(x) = O(x^2),\ 0 \leq x \leq 1,$

(4) $\sqrt{hx} J_\mu(hx) = O(1),\ hx \geq 0.$

Since the last formula and the definition of $j_\mu(x)$, we get

$$j_\mu(x) = O(x^{-\mu - 1/2}).$$

This estimation allows us to conclude that there is a constant κ_μ related to μ satisfying the following inequality:

$$|j_\mu(x)| \leq \kappa_\mu\, x^{-\mu - 1/2}.$$

2. Main results. In this section, we state the Hardy's and Miyachi's theorems. We start by Hardy's theorem for the first Hankel – Clifford transform.

2.1. Hardy's theorem for the first Hankel – Clifford transform.

Theorem 2.1. *Let f be a measurable function on $\mathbb{R}$ such that*

$$|f(x)| \leq C |x|^{\mu+1} e^{-ax^2} \tag{2.1}$$

and

$$|(h_{1,\mu} f)(y)| \leq C |y|^\mu e^{-\frac{y^2}{a}}. \tag{2.2}$$

For some constants $a, C > 0$ the function f is a constant multiple of $x^{\mu+1} e^{-ax^2}$.

Proof. First, we have the function $(h_{1,\mu} f)(z)$ is well defined for all z. Moreover, by using the estimated (4) of Lemma 1.3 and (2.1), for all $z \in \mathbb{C}$, we have

$$\big|(h_{1,\mu} f)(z)\big| \leq 2^\mu |z|^\mu \int\limits_0^{+\infty} \big|\big(2\sqrt{x} z^{\frac{1}{2}}\big)^{-\mu} J_\mu\big(2\sqrt{x} z^{\frac{1}{2}}\big)\big| |f(x)|\, dx \leq$$

$$\leq |z|^\mu \int\limits_0^{+\infty} x^{\mu+1} e^{-ax^2} \sum_{k=0}^{\infty} \frac{\Big(2|z|^{\frac{1}{2}} \sqrt{x}/2\Big)^{2k}}{\Gamma(k + \mu + 1) k!} dx \leq$$

$$\leq |z|^{\mu} \sum_{k=0}^{\infty} \frac{\left(|z|^{\frac{1}{2}}\right)^{2k}}{\Gamma(k+\mu+1)k!} \int\limits_{0}^{+\infty} x^{\mu+k+1} e^{-ax^2} dx \leq$$

$$\leq |z|^{\mu} \sum_{k=0}^{\infty} \frac{\left(|z|^{\frac{1}{2}}\right)^{2k}}{\Gamma(k+\mu+1)k!} \frac{\Gamma(\mu+k+1)}{a^{k+\mu+1}} \leq$$

$$\leq C|z|^{\mu} \sum_{k=0}^{\infty} \frac{\left(|z|^{\frac{1}{2}}/\sqrt{a}\right)^{2k}}{k!}.$$

If $|z| < 1$, then

$$\left|(h_{1,\mu}f)(z)\right| \leq C|z|^{\mu} \sum_{k=0}^{\infty} \frac{(1/\sqrt{a})^{2k}}{k!} = C|z|^{\mu} e^{\frac{1}{a}} \leq$$

$$\leq C|z|^{\mu} e^{\frac{|z|^{\mu}}{a}}.$$

We put $C' = Ce^{\frac{1}{a}}$, so we have, if $|z| < 1$,

$$\left|(h_{1,\mu}f)(z)\right| \leq C'|z|^{\mu} \leq C'|z|^{\mu} e^{\frac{|z|^{\mu}}{a}}.$$

If $|z| \geq 1$, then

$$\left|(h_{1,\mu}f)(z)\right| \leq C|z|^{\mu} \sum_{k=0}^{\infty} \frac{|1/\sqrt{a}|^{2k}}{k!} = C|z|^{\mu} e^{\frac{|z|^2}{a}}.$$

So, for all $z \in \mathbb{C}$,

$$\left|(h_{1,\mu}f)(z)\right| \leq C|z|^{\mu} e^{\frac{|z|^2}{a}}.$$

Then

$$\left|z^{-\mu}(h_{1,\mu}f)(z)\right| \leq Ce^{\frac{|z|^2}{a}}.$$

From assumption (2.2) we obtain

$$\left|y^{-\mu}(h_{1,\mu}f)(y)\right| \leq Ce^{-\frac{y^2}{a}} \quad \forall y \in \mathbb{R}.$$

Thus, $z^{-\mu}h_{1,\mu}(f)(z)$ is an entire function, according to Lemma 1.1, $\lambda^{-\mu}h_{1,\mu}(f)(\lambda)$ must be a multiple of $e^{-\frac{\lambda^2}{a}}$. Or we get

$$h_{1,\mu} = h_{1,\mu}^{-1},$$

then $f(\lambda)$ is a multiple of $\lambda^{\mu}e^{-\frac{\lambda^2}{a}}$.

Theorem 2.1 is proved.

2.2. Miyachi's theorem for the first Hankel – Clifford transform.

Theorem 2.2. *Let $a > 0$. We suppose that f is a function on $\mathbb{R}$ such that*

$$x^{-\frac{\mu}{2}-\frac{1}{4}} e^{ax^2} f \in L^1(\mathbb{R}^+) + L^{\infty}(\mathbb{R}^+)$$

and

$$\int_{-\infty}^{+\infty} \log^+\left(\frac{|h_{1,\mu}(f)(\lambda)\lambda^{-\frac{\mu}{2}-\frac{1}{4}}e^{b\lambda^2}|}{A}\right)d\lambda < \infty$$

for some $A,\ 0 < A < \infty$. *Then f is a constant multiple of* $x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}$.

Proof. By the first assumption

$$x^{-\frac{\mu}{2}-\frac{1}{4}}e^{ax^2}f \in L^1(\mathbb{R}^+) + L^\infty(\mathbb{R}^+),$$

then there are two functions $u \in L^1(\mathbb{R}^+)$ and $v \in L^\infty(\mathbb{R}^+)$ such that

$$x^{-\frac{\mu}{2}-\frac{1}{4}}e^{ax^2}f(x) = u(x) + v(x)$$

and, thus,

$$f(x) = x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}u(x) + x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}v(x),$$

$$h_{1,\mu}(f)(\lambda) = h_{1,\mu}(x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}u)(\lambda) + h_{1,\mu}\big(x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}v\big)(\lambda).$$

If $\lambda \in \mathbb{C}$, we have

$$\left|h_{1,\mu}\big(x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}u\big)(\lambda)\right| \le 2^\mu|\lambda|^\mu \int_0^{+\infty} \left|\big(2\sqrt{x}\lambda^{\frac{1}{2}}\big)^{-\mu}J_\mu\big(2\sqrt{x}\lambda^{\frac{1}{2}}\big)\right| x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}|u(x)|\,dx \le$$

$$\le C|\lambda|^\mu \int_0^{+\infty} \big(2x^{\frac{1}{2}}\lambda^{\frac{1}{2}}\big)^{-\mu-\frac{1}{2}} x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}|u(x)|dx \le$$

$$\le C|\lambda|^{\frac{\mu}{2}-\frac{1}{4}} \int_0^{+\infty} x^{-\frac{1}{2}\mu-\frac{1}{4}}x^{\frac{1}{2}\mu+\frac{1}{4}}e^{-ax^2}|u(x)|\,dx \le$$

$$\le C|\lambda|^{\frac{\mu}{2}-\frac{1}{4}} \int_0^{+\infty} e^{-ax^2}|u(x)|\,dx \le$$

$$\le C|\lambda|^{\frac{\mu}{2}-\frac{1}{4}} \le C|\lambda|^{\frac{\mu}{2}-\frac{1}{4}}e^{2\mathrm{Im}(\lambda)^2},$$

where C is a positive constant. Then

$$\left|h_{1,\mu}\big(x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}u\big)(\lambda)\right| \le C|\lambda|^{\frac{\mu}{2}-\frac{1}{4}}e^{2\mathrm{Im}(\lambda)^2}.$$

So,

$$\left|\lambda^{-\frac{\mu}{2}+\frac{1}{4}}h_{1,\mu}\big(x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}u\big)(\lambda)\right| \le Ce^{2\mathrm{Im}(\lambda)^2},$$

and we have

$$\left|h_{1,\mu}\big(x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}v\big)(\lambda)\right| \le 2^\mu|\lambda|^\mu \int_0^{+\infty} \left|\big(2\sqrt{x}\lambda^{\frac{1}{2}}\big)^{-\mu}J_\mu\big(2\sqrt{x}\lambda^{\frac{1}{2}}\big)\right| x^{\frac{\mu}{2}+\frac{1}{4}}e^{-ax^2}|v(x)|dx \le$$

$$\leq |\lambda|^{\mu} \int_{0}^{+\infty} \left(2x^{\frac{1}{2}}\lambda^{\frac{1}{2}}\right)^{-\mu-\frac{1}{2}} x^{\frac{\mu}{2}+\frac{1}{4}} e^{-ax^2} |v(x)| dx \leq$$

$$\leq |\lambda|^{\frac{\mu}{2}-\frac{1}{4}} \int_{0}^{+\infty} x^{-\frac{1}{2}\mu-\frac{1}{4}} x^{\frac{\mu}{2}+\frac{1}{4}} e^{-ax^2} |v(x)|\, dx \leq$$

$$\leq |\lambda|^{\frac{\mu}{2}-\frac{1}{4}} \|v\|_{\infty} \int_{0}^{+\infty} e^{-ax^2} u u dx \leq$$

$$\leq C|\lambda|^{\frac{\mu}{2}-\frac{1}{4}} \leq C|\lambda|^{\frac{\mu}{2}-\frac{1}{4}} e^{2\mathrm{Im}(\lambda)^2}.$$

So,

$$\left|\lambda^{-\frac{\mu}{2}+\frac{1}{4}} h_{1,\mu}\left(x^{\frac{\mu}{2}+\frac{1}{4}} e^{-ax^2} v\right)(\lambda)\right| \leq C e^{2\mathrm{Im}(\lambda)^2},$$

then

$$\left|\lambda^{-\frac{\mu}{2}+\frac{1}{4}} h_{1,\mu}(f)(\lambda)\right| \leq C e^{2\mathrm{Im}(\lambda)^2}.$$

Since $\lambda^{-\frac{\mu}{2}+\frac{1}{4}} h_{1,\mu}(f)(\lambda)$ is an entire function, then, by the Lemma 1.2 and $h_{1,\mu} = h_{1,\mu}^{-1}$, we can obtain that f is a multiple of $x^{\mu} e^{-ax^2}$.

Theorem 2.2 is proved.

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