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EXISTENCE RESULTS FOR DOUBLY NONLINEAR PARABOLIC EQUATIONS WITH TWO LOWER ORDER TERMS AND L^1-DATA

РЕЗУЛЬТАТИ ПРО ІСНУВАННЯ РОЗВ'ЯЗКІВ ДВІЧІ НЕЛІНІЙНИХ ПАРАБОЛІЧНИХ РІВНЯНЬ З ДВОМА ЧЛЕНАМИ НИЖЧОГО ПОРЯДКУ ТА L^1-ДАНИМИ

We investigate the existence of a renormalized solution for a class of nonlinear parabolic equations with two lower order terms and L^1-data.

Вивчається проблема існування перенормованого розв'язку для класу нелінійних параболічних рівнянь з двома членами нижчого порядку та L^1-даними.

1. Introduction. We consider the following nonlinear parabolic problem:

$$\begin{aligned}
&\frac{\partial b(x,u)}{\partial t} - \operatorname{div}(a(x,t,u,\nabla u)) + g(x,t,u,\nabla u) + H(x,t,\nabla u) = f \quad \text{in} \quad Q_T,\\
&b(x,u)(t=0) = b(x,u_0) \quad \text{in} \quad \Omega, \\
&u = 0 \quad \text{on} \quad \partial\Omega \times (0,T),
\end{aligned} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, $T > 0$, $p > 1$ and Q_T is the cylinder $\Omega \times (0,T)$. The operator $-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray–Lions operator which is coercive and grows like $|\nabla u|^{p-1}$ with respect to ∇u, the function $b(x,u)$ is an unbounded on u, and $b(x,u_0) \in L^1(\Omega)$. The functions g and H are two Carathéodory functions with suitable assumptions see below. Finally the datum $f \in L^1(Q_T)$.

The problem (1.1) is encountered in a variety of physical phenomena and applications. For instance, when $b(x,u) = u$, $a(x,t,u,\nabla u) = |\nabla u|^{p-2}\nabla u$, $g = f = 0$, $H(x,t,\nabla u) = \lambda|\nabla u|^q$, where q and λ are positive parameter, the equation in problem (1.1) can be viewed as the viscosity approximation of Hamilton–Jacobi-type equation from stochastic control theory [18]. In particular, when $b(x,u) = u$, $a(x,t,u,\nabla u) = \nabla u$, $g = f = 0$, $H(x,t,\nabla u) = \lambda|\nabla u|^2$, where λ is positive parameter, the equation in problem (1.1) appears in the physical theory of growth and roughening of surfaces, where it is known as the Kardar–Parisi–Zhang equation [14]. We introduce the definition of the renormalized solutions for problem (1.1) as follows. This notion was introduced by P.-L. Lions and Di Perna [12] for the study of Boltzmann equation (see also P.-L. Lions [17] for a few applications to fluid mechanics models). This notion was then adapted to an elliptic version of (1.1) by Boccardo et al. [9] when the right-hand side is in $W^{-1,p'}(\Omega)$, by Rakotoson [24] when the right-hand side being a in $L^1(\Omega)$, and by Dal Maso, Murat, Orsina and Prignet [10] for the case of right-hand side being a general measure data, see also [19, 20].

For $b(x,u) = u$ and $H = 0$, the existence of a weak solution to problem (1.1) (which belongs to $L^m(0,T;W_0^{1,m}(\Omega))$ with $p > 2 - \dfrac{1}{N+1}$ and $m < \dfrac{p(N+1)-N}{N+1}$ was proved in [8] (see also [7]) where $g = 0$, and in [23] where $g = 0$, and in [11, 21, 22]. When the function $g(x, t, u, \nabla u) \equiv g(u)$ is independent on the $(x, t, \nabla u)$ and g is continuous, the existence of a renormalized solution to problem (1.1) is proved in [5]. Otherwise, recently in [1] is proved the existence of a renormalized solution to problem (1.1) where the variational case.

The scope of the present paper is to prove an existence result for renormalized solutions to a class of problems (1.1) with two lower order terms and L^1-data. The difficulties connected to our problem (1.1) are due to the presence of the two terms g and H which induce a lack of coercivity, noncontrolled growth of the function $b(x, s)$ with respect to s, the functions $a(x, t, u, \nabla u)$ do not belong to $(L^1_{\mathrm{loc}}(Q_T))^N$ in general, and the data $b(x, u_0)$, f are only integrable.

The rest of this article is organized as follows. In Section 2 we make precise all the assumptions on b, a, g, H, u_0, we also give the concept of a renormalized solution for the problem (1.1). In Section 3 we establish the existence of our main results.

2. Essential assumptions and different notions of solutions. Throughout the paper, we assume that the following assumptions hold true. Let Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 1$, $T > 0$ is given and we set $Q_T = \Omega \times (0, T)$, and

$$b \colon \Omega \times \mathbb{R} \to \mathbb{R} \quad \text{is a Carathéodory function,}$$

such that for every $x \in \Omega$, $b(x, .)$ is a strictly increasing C^1-function with $b(x, 0) = 0$. Next, for any $k > 0$, there exists $\lambda_k > 0$ and functions $A_k \in L^\infty(\Omega)$ and $B_k \in L^p(\Omega)$ such that

$$\lambda_k \leq \frac{\partial b(x, s)}{\partial s} \leq A_k(x) \qquad \text{and} \qquad \left| \nabla_x \left(\frac{\partial b(x, s)}{\partial s} \right) \right| \leq B_k(x), \tag{2.1}$$

for almost every $x \in \Omega$, for every s such that $|s| \leq k$, we denote by $\nabla_x \left(\dfrac{\partial b(x, s)}{\partial s} \right)$ the gradient of $\dfrac{\partial b(x, s)}{\partial s}$ defined in the sense of distributions.

Let $a \colon Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, such that

$$\big| a(x, t, s, \xi) \big| \leq \beta \big[k(x, t) + |s|^{p-1} + |\xi|^{p-1} \big], \tag{2.2}$$

for a.e. $(x, t) \in Q_T$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, some positive function $k(x, t) \in L^{p'}(Q_T)$ and $\beta > 0$,

$$\big[a(x, t, s, \xi) - a(x, t, s, \eta) \big] (\xi - \eta) > 0 \quad \text{for all} \quad (\xi, \eta) \in \mathbb{R}^N \times \mathbb{R}^N, \quad \text{with} \quad \xi \neq \eta, \tag{2.3}$$

$$a(x, t, s, \xi) \xi \geq \alpha |\xi|^p, \quad \text{where} \quad \alpha \quad \text{is a strictly positive constant.} \tag{2.4}$$

Furthermore, let $g(x, t, s, \xi) \colon Q_T \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $H(x, t, \xi) \colon Q_T \times \mathbb{R}^N \to \mathbb{R}$ are two Carathéodory functions which satisfy, for almost every $(x, t) \in Q_T$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$, the following conditions:

$$\big| g(x, t, s, \xi) \big| \leq L_1(|s|) \big(L_2(x, t) + |\xi|^p \big), \tag{2.5}$$

$$g(x, t, s, \xi) s \geq 0, \tag{2.6}$$

where $L_1 : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function, while $L_2(x, t)$ is positive and belongs to $L^1(Q_T)$,

$$\exists\delta > 0, \quad \nu > 0 \quad \forall |s| \geq \delta : \big|g(x,t,s,\xi)\big| \geq \nu|\xi|^p, \tag{2.7}$$

$$\big|H(x,t,\xi)\big| \leq h(x,t)|\xi|^{p-1}, \quad \text{where} \quad h(x,t) \quad \text{is positive and belongs to} \quad L^p(Q_T). \tag{2.8}$$

We recall that, for $k > 1$ and s in $\mathbb{R}$, the truncation is defined as $T_k(s) = \max(-k, \min(k,s))$.

We shall use the following definition of renormalized solution for problem (1.1) in the following sense.

Definition 1. *Let $f \in L^1(Q_T)$ and $b(\cdot, u_0(\cdot)) \in L^1(\Omega)$. A renormalized solution of problem* (1.1) *is a function u defined on Q_T, satisfying the following conditions*:

$$T_k(u) \in L^p\big(0,T; W_0^{1,p}(\Omega)\big) \quad \textit{for all} \quad k \geq 0 \quad \textit{and} \quad b(x,u) \in L^\infty\big(0,T; L^1(\Omega)\big), \tag{2.9}$$

$$\int\limits_{\{m \leq |u| \leq m+1\}} a(x,t,u,\nabla u)\nabla u \, dx\, dt \to 0 \quad \textit{as} \quad m \to +\infty, \tag{2.10}$$

$$\begin{gathered}\frac{\partial B_S(x,u)}{\partial t} - \operatorname{div}\Big(S'(u)a(x,t,u,\nabla u)\Big) + S''(u)a(x,t,u,\nabla u)\nabla u + \\ + g(x,t,u,\nabla u)S'(u) + H(x,t,\nabla u)S'(u) = fS'(u) \quad \textit{in} \quad \mathcal{D}'(Q_T),\end{gathered} \tag{2.11}$$

for all functions $S \in W^{2,\infty}(\mathbb{R})$ which are piecewise $\mathcal{C}^1(\mathbb{R})$, such that S' has a compact support in $\mathbb{R}$ and

$$B_S(x,u)(t=0) = B_S(x,u_0) \quad \textit{in} \quad \Omega, \quad \textit{where} \quad B_S(x,z) = \int\limits_0^z \frac{\partial b(x,r)}{\partial r} S'(r)\, dr. \tag{2.12}$$

***Remark* 1.** Equation (2.11) is formally obtained through pointwise multiplication of (1.1) by $S'(u)$. However, while $a(x,t,u,\nabla u)$, $g(x,t,u,\nabla u)$, and $H(x,t,\nabla u)$ does not in general make sense in $\mathcal{D}'(Q_T)$, all the terms in (2.11) have a meaning in $\mathcal{D}'(Q_T)$.

Indeed, if M is such that $\operatorname{supp} S' \subset [-M, M]$, the following identifications are made in (2.11):

$|B_S(x,u)| = |B_S(x, T_M(u))| \leq M\|S'\|_{L^\infty(\mathbb{R})} A_M(x)$ belongs to $L^\infty(\Omega)$ since A_M is a bounded function;

$S'(u)a(x,t,u,\nabla u)$ identifies with $S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)$ a.e. in Q_T; since $|T_M(u)| \leq$ $\leq M$ a.e. in Q_T and $S'(u) \in L^\infty(Q_T)$, we obtain from (2.2) and (2.9) that

$$S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big) \in \big(L^{p'}(Q_T)\big)^N;$$

$S''(u)a(x,t,u,\nabla u)\nabla u$ identifies with $S''(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)\nabla T_M(u)$ and $S''(u)a\big(x, t, T_M(u), \nabla T_M(u)\big)\nabla T_M(u) \in L^1(Q_T)$;

$S'(u)\Big(g\big(x,t,u,\nabla u\big) + H(x,t,\nabla u)\Big)$ identifies with $S'(u)\Big(g\big(x,t,T_M(u),\nabla T_M(u)\big) + H\big(x,t, \nabla T_M(u)\big)\Big)$ a.e. in Q_T; since $|T_M(u)| \leq M$ a.e. in Q_T and $S'(u) \in L^\infty(Q_T)$, we obtain from (2.2), (2.5), and (2.8) that

$$S'(u)\Big(g\big(x,t,T_M(u),\nabla T_M(u)\big) + H(x,t,\nabla T_M(u))\Big) \in L^1(Q_T);$$

$S'(u)f$ belongs to $L^1(Q_T)$.

The above considerations show that (2.11) holds in $\mathcal{D}'(Q_T)$ and

$$\frac{\partial B_S(x,u)}{\partial t} \in L^{p'}\big(0,T;W^{-1,\,p'}(\Omega)\big) + L^1(Q_T). \tag{2.13}$$

The properties of S, assumptions (2.1) and (2.10) imply that

$$\big|\nabla B_S(x,u)\big| \le \|A_M\|_{L^\infty(\Omega)}\big|\nabla T_M(u)\big|\|S'\|_{L^\infty(\mathbb{R})} + M\|S'\|_{L^\infty(\mathbb{R})}B_M(x) \tag{2.14}$$

and

$$B_S(x,u) \quad \text{belongs to} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big). \tag{2.15}$$

Then (2.13) and (2.15) imply that $B_S(x,u)$ belongs to $C^0\big([0,T];L^1(\Omega)\big)$ (for a proof of this trace result see [21]), so that the initial condition (2.12) makes sense.

Also remark that, for every $S \in W^{1,\infty}(\mathbb{R})$, nondecreasing function such that $\operatorname{supp} S' \subset [-M,M]$, in view of (2.1) we have

$$\lambda_M|S(r) - S(r')| \le \Big|B_S(x,r) - B_S(x,r')\Big| \le$$

$$\le \|A_M\|_{L^\infty(\Omega)}|S(r) - S(r')|, \quad \text{a.e.} \quad x \in \Omega, \quad \forall r,r' \in \mathbb{R}.$$

3. Statements of results. The main results of this article are stated as follows.

Theorem 1. *Let $f \in L^1(Q_T)$ and u_0 is a measurable function such that $b(\cdot,u_0) \in L^1(\Omega)$. Assume that* (2.1) – (2.8) *hold true. Then there exists a renormalized solution u of problem* (1.1) *in the sense of Definition* 1.

Proof. The proof of Theorem 1 is done in five steps.

Step 1: Approximate problem and a priori estimates. For $n > 0$, let us define the following approximation of b, f and u_0.

First, set $b_n(x,r) = b(x,T_n(r)) + \dfrac{1}{n}rb_n$ is a Carathéodory function and satisfies (2.1), there exist $\lambda_n > 0$ and functions $A_n \in L^\infty(\Omega)$ and $B_n \in L^p(\Omega)$ such that $\lambda_n \le \dfrac{\partial b_n(x,s)}{\partial s} \le A_n(x)$ and $\left|\nabla_x\left(\dfrac{\partial b_n(x,s)}{\partial s}\right)\right| \le B_n(x)$, a.e. in Ω, $s \in \mathbb{R}$.

Next, set

$$g_n(x,t,s,\xi) = \frac{g(x,t,s,\xi)}{1 + \dfrac{1}{n}|g(x,t,s,\xi)|} \qquad \text{and} \qquad H_n(x,t,\xi) = \frac{H(x,t,\xi)}{1 + \dfrac{1}{n}|H(x,t,\xi)|}.$$

Note that $\big|g_n(x,t,s,\xi)\big| \le \max\big\{|g(x,t,s,\xi)|;n\big\}$ and $\big|H_n(x,t,\xi)\big| \le \max\big\{|H(x,t,\xi)|;n\big\}$. Moreover, since $f_n \in L^{p'}(Q_T)$ and $f_n \to f$ a.e. in Q_T and strongly in $L^1(Q_T)$ as $n \to \infty$,

$$u_{0n} \in \mathcal{D}(\Omega), \qquad b_n(x,u_{0n}) \to b(x,u_0) \quad \text{a.e. in} \quad \Omega \quad \text{and strongly in} \quad L^1(\Omega) \quad \text{as} \quad n \to \infty. \tag{3.1}$$

Let us now consider the approximate problem

$$\begin{aligned} &\frac{\partial b_n(x,u_n)}{\partial t} - \operatorname{div}(a(x,t,u_n,\nabla u_n)) + g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n) = f_n \quad \text{in} \quad Q_T, \\ &b_n(x,u_n)(t=0) = b_n(x,u_{0n}) \quad \text{in} \quad \Omega, \\ &u_n = 0 \quad \text{in} \quad \partial\Omega \times (0,T). \end{aligned} \tag{3.2}$$

Since $f_n \in L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)$, proving existence of a weak solution $u_n \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ of (3.2) is an easy task (see, e.g., [16, p. 271]), i.e.,

$$\int_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t}, v \right\rangle dt + \int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla v\, dx\, dt +$$

$$+ \int_{Q_T} g_n(x,t,u_n,\nabla u_n) v\, dx\, dt + \int_{Q_T} H_n(x,t,\nabla u_n) v\, dx\, dt =$$

$$= \int_{Q_T} f_n v\, dx\, dt \qquad \text{for all} \quad v \in L^p\big(0,T;W^{1,p}(\Omega)\big) \cap L^\infty(Q_T).$$

Now, we prove the solution u_n of problem (3.2) is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$.

Lemma 1. *Let* $u_n \in L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ *be a weak solution of* (3.2). *Then the following estimates hold*:

$$\|u_n\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)} \leq D, \tag{3.3}$$

where D *depend only on* Ω, T, N, p, p', f, *and* $\|h\|_{L^p(Q_T)}$.

Proof. To get (3.3), we divide the integral $\int_{Q_T} |\nabla u_n|^p\, dx\, dt$ in two parts and we prove the following estimates: for all $k \geq 0$,

$$\int_{\{|u_n| \leq k\}} \big|\nabla u_n\big|^p\, dx\, dt \leq M_1 k, \tag{3.4}$$

and

$$\int_{\{|u_n| > k\}} \big|\nabla u_n\big|^p\, dx\, dt \leq M_2, \tag{3.5}$$

where M_1 and M_2 are positive constants. In what follows we will denote by M_i, $i = 3,4,\ldots$, some generic positive constants. We suppose $p < N$ (the case $p \geq N$ is similar). For $\varepsilon > 0$ and $s \geq 0$, we define

$$\varphi_\varepsilon(r) = \begin{cases} \operatorname{sign}(r) & \text{if} \quad |r| > s+\varepsilon, \\ \dfrac{\operatorname{sign}(r)(|r|-s)}{\varepsilon} & \text{if} \quad s < |r| \leq s+\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

We choose $v = \varphi_\varepsilon(u_n)$ as test function in (3.2), we have

$$\left[\int_\Omega B^n_{\varphi_\varepsilon}(x,u_n)\, dx\right]_0^T + \int_{Q_T} a(x,t,u_n,\nabla u_n)\nabla(\varphi_\varepsilon(u_n))\, dx\, dt +$$

$$+\int\limits_{Q_T} g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt + \int\limits_{Q_T} H_n(x,t,\nabla u_n)\varphi_\varepsilon(u_n)\,dx\,dt =$$

$$= \int\limits_{Q_T} f_n\varphi_\varepsilon(u_n)\,dx\,dt,$$

where

$$B^n_{\varphi_\varepsilon}(x,r) = \int\limits_0^r \frac{\partial b_n(x,s)}{\partial s}\varphi_\varepsilon(s)\,ds.$$

By using $B^n_{\varphi_\varepsilon}(x,r) \geq 0,\ g_n(x,t,u_n,\nabla u_n)\varphi_\varepsilon(u_n) \geq 0$, (2.4), (2.8), Hölder inequality and letting ε go to zero, we obtain

$$-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} \alpha|\nabla u_n|^p\,dx\,dt \leq$$

$$\leq \int\limits_{\{s<|u_n|\}} |f_n|\,dx\,dt + \int\limits_s^{+\infty}\left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}} h^p\,dx\,dt\right)^{\frac{1}{p}}\left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} d\sigma,$$

where $\{s < |u_n|\}$ denotes the set $\big\{(x,t) \in Q_T, s < |u_n(x,t)|\big\}$ and $\mu(s)$ stands for the distribution function of u_n, that is $\mu(s) = \big|\big\{(x,t) \in Q_T,\ |u_n(x,t)| > s\big\}\big|$ for all $s \geq 0$.

On the other hand, from Fleming–Rishel coarea formula and isoperimetric inequality, we have, for almost every $s > 0$,

$$NC_N^{\frac{1}{N}}\big(\mu(s)\big)^{\frac{N-1}{N}} \leq -\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p dx\,dt, \tag{3.6}$$

where C_N is the measure of the unit ball in $\mathbb{R}^N$. By using the Hölder's inequality, we obtain that, for almost every $s > 0$,

$$-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt \leq \big(-\mu'(s)\big)^{\frac{1}{p'}}\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p}}. \tag{3.7}$$

Then, combining (3.6) and (3.7), we obtain, for almost every $s > 0$,

$$1 \leq \Big(NC_N^{\frac{1}{N}}\Big)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p dx\,dt\right)^{\frac{1}{p}}. \tag{3.8}$$

By using (3.8), we have

$$\alpha\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} \leq$$

$$\leq \left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\left(\int\limits_{\{s<|u_n|\}}|f_n|\,dx\,dt\right)+$$

$$+\left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\times$$

$$\times\int\limits_s^{+\infty}\left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}}h^p\,dx\,dt\right)^{\frac{1}{p}}\left(-\frac{d}{d\sigma}\int\limits_{\{\sigma<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}d\sigma. \tag{3.9}$$

Now, we consider two functions B and ψ (see Lemma 2.2 of [2]) defined by

$$\int\limits_{\{s<|u_n|\}}h^p(x,t)\,dx\,dt=\int\limits_0^{\mu(s)}B^p(\sigma)\,d\sigma \tag{3.10}$$

and

$$\psi(s)=\int\limits_{\{s<|u_n|\}}|f_n|\,dx\,dt. \tag{3.11}$$

We have $\|B\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)}\leq\|h\|_{L^p\left(0,T;W_0^{1,p}(\Omega)\right)}$ and $|\psi(s)|\leq\|f_n\|_{L^1(Q_T)}$. From (3.9), (3.10), and (3.11) we get

$$\alpha\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\leq$$

$$\leq\left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\psi(s)+\left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\times$$

$$\times\big(-\mu'(s)\big)^{\frac{1}{p'}}\int\limits_s^{+\infty}B\big(\mu(\nu)\big)\big(-\mu'(\nu)\big)^{\frac{1}{p}}\left(-\frac{d}{d\nu}\int\limits_{\{\nu<|u_n|\}}|\nabla u_n|^p dxdt\right)^{\frac{1}{p'}}d\nu.$$

From Gronwall's lemma (see [3]), we obtain

$$\alpha\left(-\frac{d}{ds}\int\limits_{\{s<|u_n|\}}|\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}}\leq$$

$$\leq\left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\big(-\mu'(s)\big)^{\frac{1}{p'}}\psi(s)+(NC_N^{\frac{1}{N}})^{-1}\big(\mu(s)\big)^{\frac{1}{N}-1}\times$$

$$\times\big(-\mu'(s)\big)^{\frac{1}{p'}}\int\limits_s^{+\infty}\Big[\left(NC_N^{\frac{1}{N}}\right)^{-1}\big(\mu(\sigma)\big)^{\frac{1}{N}-1}\psi(\sigma)\Big]B\big(\mu(\sigma)\big)\big(-\mu'(\sigma)\big)\times$$

$$\times \exp\left(\int_s^\sigma \left(NC_N^{\frac{1}{N}}\right)^{-1} B\big(\mu(r)\big)\big(\mu(r)\big)^{\frac{1}{N}-1}\big(-\mu'(r)\big)dr\right)d\sigma. \tag{3.12}$$

Now, by a variable of change and by Hölder inequality, we estimate the argument of the exponential function on the right-hand side of (3.12):

$$\int_s^\sigma B\big(\mu(r)\big)\big(\mu(r)\big)^{\frac{1}{N}-1}\big(-\mu'(r)\big)dr = \int_s^\sigma B(z) z^{\frac{1}{N}-1} dz \leq$$

$$\leq \int_0^{|\Omega|} B(z) z^{\frac{1}{N}-1} dz \leq \|B\|_{L^p}\left(\int_0^{|\Omega|} z^{(\frac{1}{N}-1)p'}\right)^{\frac{1}{p'}}.$$

Raising to the power p' in (3.12) and we can write

$$-\frac{d}{ds}\int_{\{s<|u_n|\}} |\nabla u_n|^p \, dx\, dt \leq M_1,$$

where M_1 depend only on Ω, N, p, p', f, α, and $\|h\|_{L^p(Q_T)}$, integrating between 0 and k, (3.4) is proved.

We now give the proof of (3.5), using $T_k(u_n)$ as test function in (3.2), gives

$$\left[\int_\Omega B_k^n(x,u_n)\, dx\right]_0^T + \int_\Omega a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\, dx\, dt +$$

$$+\int_\Omega (g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n))T_k(u_n)\, dx\, dt =$$

$$= \int_\Omega f_n T_k(u_n)\, dx\, dt,$$

where

$$B_k^n(x,r) = \int_0^r \frac{\partial b_n(x,s)}{\partial s} T_k(s)\, ds.$$

By using (2.8), we deduce that

$$\left[\int_\Omega B_k^n(x,u_n)\, dx\right]_0^T + \int_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n)\nabla u_n\, dx\, dt +$$

$$+\int_{\{|u_n|\leq k\}} g_n(x,t,u_n,\nabla u_n)u_n\, dx + \int_{\{|u_n|>k\}} g_n(x,t,u_n,\nabla u_n)T_k(u_n)\, dx\, dt \leq$$

$$\leq \int_\Omega f_n T_k(u_n)\, dx\, dt + \int_\Omega h(x,t)|\nabla u_n|^{p-1}|T_k(u_n)|\, dx\, dt,$$

and by using the fact that $B_k^n(x,r) \geq 0,\ g_n(x,t,u_n,\nabla u_n)u_n \geq 0$ and (2.4), we have

$$\alpha \int\limits_{\{|u_n|\leq k\}} |\nabla u_n|^p \, dx\, dt + \int\limits_{\{|u_n|>k\}} g(x,u_n,\nabla u_n) T_k(u_n)\, dx\, dt \leq$$

$$\leq k\|f\|_{L^1} + k \int\limits_{\{|u_n|\leq k\}} h(x,t)|\nabla u_n|^{p-1}\, dx\, dt +$$

$$+ k \int\limits_{\{|u_n|\geq k\}} h(x,t)|\nabla u_n|^{p-1}\, dx\, dt.$$

By Hölder inequality and (3.4), (2.7) and applying Young's inequality, we get, for all $k > \delta$,

$$\nu k \int\limits_{\{|u_n|>k\}} |\nabla u_n|^p \, dx\, dt \leq$$

$$\leq k\|f\|_{L^1(Q_T)} + k^{1+\frac{1}{p'}} M_1 \|h\|_{L^p Q_T)} + k \int\limits_{\{|u_n|>k\}} h(x,t)|\nabla u_n|^{p-1}\, dx\, dt \leq$$

$$\leq k\|f\|_{L^1(Q_T)} + k^{1+\frac{1}{p'}} M_1 \|h\|_{L^p Q_T)} + M_6 k \|h\|_{L^p}^p + \frac{1}{p'} \nu k \int\limits_{\{|u_n|>k\}} |\nabla u_n|^p \, dx\, dt.$$

Hence,

$$\left(1 - \frac{1}{p'}\right) \int\limits_{\{|u_n|>k\}} |\nabla u_n|^p \, dx\, dt \leq M_3 \|f\|_{L^1(Q_T)} + k^{\frac{1}{p'}} M_5 \|h\|_{L^p(Q_T)} + M_7 \|h\|_{L^p}^p. \tag{3.13}$$

Lemma 1 is proved.

Then there exists $u \in L^p\big(0,T; W_0^{1,p}(\Omega)\big)$ such that, for some subsequence

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^p\big(0,T; W_0^{1,p}(\Omega)\big) \tag{3.14}$$

we conclude that

$$\big\|T_k(u_n)\big\|_{L^p(0,T;W_0^{1,p}(\Omega))}^p \leq c_2 k. \tag{3.15}$$

We deduce from the above inequalities, (2.1) and (3.15), that

$$\int\limits_{\Omega} B_k^n(x,u_n)\, dx \leq Ck, \tag{3.16}$$

where $B_k^n(x,z) = \displaystyle\int_0^z \frac{\partial b_n(x,s)}{\partial s} T_k(s)\, ds$.

Now, we turn to prove the almost every convergence of u_n and $b_n(x,u_n)$. Consider now a function nondecreasing $\xi_k \in C^2(\mathbb{R})$ such that $\xi_k(s) = s$ for $|s| \leq \dfrac{k}{2}$ and $\xi_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation by $\xi_k'(u_n)$, we obtain

$$\frac{\partial B_\xi^n(x,u_n)}{\partial t} - \operatorname{div}\Big(a(x,t,u_n,\nabla u_n)\xi_k'(u_n)\Big) + a(x,t,u_n,\nabla u_n)\xi_k''(u_n)\nabla u_n +$$
$$+\Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big)\xi_k'(u_n) =$$
$$= f_n\xi_k'(u_n), \qquad (3.17)$$

in the sense of distributions, where

$$B_\xi^n(x,z) = \int\limits_0^z \frac{\partial b_n(x,s)}{\partial s}\xi_k'(s)\,ds.$$

As a consequence of (3.15), we deduce that $\xi_k(u_n)$ is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ and $\dfrac{\partial B_\xi^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q_T)+L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)$. Due to the properties of ξ_k and (2.1), we conclude that $\dfrac{\partial \xi_k(u_n)}{\partial t}$ is bounded in $L^1(Q_T)+L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)$, which implies that $\xi_k(u_n)$ strongly converges in $L^1(Q_T)$ (see [21]).

Due to the choice of ξ_k, we conclude that for each k, the sequence $T_k(u_n)$ converges almost everywhere in Q_T, which implies that u_n converges almost everywhere to some measurable function u in Q_T. Thus, by using the same argument as in [4, 5, 25], we can show

$$u_n \to u \quad \text{a.e. in} \quad Q_T, \qquad (3.18)$$
$$b_n(x,u_n) \to b(x,u) \quad \text{a.e. in} \quad Q_T.$$

We can deduce from (3.15) that

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big),$$

which implies, by using (2.2), for all $k>0$, that there exists a function $\overline{a}\in(L^{p'}(Q_T))^N$, such that

$$a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) \rightharpoonup \overline{a} \quad \text{weakly in} \quad \big(L^{p'}(Q_T)\big)^N. \qquad (3.19)$$

We now establish that $b(.,u)$ belongs to $L^\infty\big(0,T;L^1(\Omega)\big)$. Using (3.18) and passing to the limit-inf in (3.16) as n tends to $+\infty$, we obtain

$$\frac{1}{k}\int\limits_\Omega B_k(x,u)(\tau)\,dx \le C$$

for almost any τ in $(0,T)$. Due to the definition of $B_k(x,s)$ and the fact that $\dfrac{1}{k}B_k(x,u)$ converges pointwise to $b(x,u)$, as k tends to $+\infty$, shows that $b(x,u)$ belong to $L^\infty\big(0,T;L^1(\Omega)\big)$.

Lemma 2. *Let u_n be a solution of the approximate problem* (3.2). *Then*

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int\limits_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \qquad (3.20)$$

Proof. We use $T_1\big(u_n - T_m(u_n)\big)^+ = \alpha_m(u_n) \in L^p\big(0,T; W_0^{1,p}(\Omega)\big) \cap L^\infty(Q_T)$ as test function in (3.2). Then we have

$$\int_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t}; \alpha_m(u_n) \right\rangle dt + \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n \alpha_m'(u_n)\,dx\,dt +$$

$$+ \int_{Q_T} \big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\big)\alpha_m(u_n)\,dx\,dt \le$$

$$\le \int_{Q_T} \big|f_n \alpha_m(u_n)\big|\,dx\,dt,$$

which, by setting $B_m^n(x,r) = \displaystyle\int_0^r \frac{\partial b_n(x,s)}{\partial s}\alpha_m(s)\,ds$, (2.6) and (2.8) gives

$$\int_\Omega B_m^n(x,u_n)(T)\,dx + \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le$$

$$\le \int_{\{m \le u_n\}} |f_n|\,dx\,dt + \int_{Q_T} h(x,t)|\nabla u_n|^{p-1}\,dx\,dt.$$

Now we use Hölder's inequality and (3.3), in order to deduce

$$\int_\Omega B_m^n(x,u_n)(T)\,dx + \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \le$$

$$\le \int_{\{m \le u_n\}} |f_n|\,dx\,dt + c_1\left(\int_{\{m \le u_n\}} |h(x,t)|^p\,dx\,dt\right)^{\frac{1}{p'}}.$$

Since $B_m^n(x,u_n)(T) \ge 0$ and the strong convergence of f_n in $L^1(Q_T)$, by Lebesgue's theorem, we have

$$\lim_{m\to\infty}\lim_{n\to\infty} \int_{\{m \le u_n\}} |f_n|\,dx\,dt = 0.$$

Similarly, since $h \in L^p(Q_T)$, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty}\left(\int_{\{m \le u_n\}} |h(x,t)|^p\,dx\,dt\right)^{\frac{1}{p'}} = 0.$$

We conclude that

$$\lim_{m\to\infty}\limsup_{n\to\infty} \int_{\{m \le u_n \le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.21}$$

On the other hand, using $T_1\big(u_n - T_m(u_n)\big)^-$ as test function in (3.2) and reasoning as in the proof of (3.21) we deduce that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\int\limits_{\{-(m+1)\le u_n\le -m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{3.22}$$

Thus, (3.20) follows from (3.21) and (3.22).

Step 2: Almost everywhere convergence of the gradients. This step is devoted to introduce for $k \ge 0$ fixed a time regularization of the function $T_k(u)$ in order to perform the monotonicity method (the proof of this steps is similar the Step 4 in [5]). This kind of regularization has been first introduced by R. Landes (see Lemma 6 and Proposition 3 [15, p. 230] and Proposition 4 in [15, p. 231]). For $k > 0$ fixed, and let $\varphi(t) = te^{\gamma t^2}$, $\gamma > 0$. It is well known that when $\gamma > \left(\dfrac{L_1(k)}{2\alpha}\right)^2$, one has

$$\varphi'(s) - \left(\frac{L_1(k)}{\alpha}\right)|\varphi(s)| \ge \frac{1}{2} \quad \text{for all} \quad s \in \mathbb{R}. \tag{3.23}$$

Let $\{\psi_i\} \subset \mathcal{D}(\Omega)$ be a sequence which converge strongly to u_0 in $L^1(\Omega)$. Set $w^i_\mu = (T_k(u))_\mu + e^{-\mu t}\,T_k(\psi_i)$, where $(T_k(u))_\mu$ is the mollification with respect to time of $T_k(u)$. Note that w^i_μ is a smooth function having the following properties:

$$\frac{\partial w^i_\mu}{\partial t} = \mu(T_k(u) - w^i_\mu), \qquad w^i_\mu(0) = T_k(\psi_i), \quad \left|w^i_\mu\right| \le k, \tag{3.24}$$

$$w^i_\mu \to T_k(u) \quad \text{strongly in} \quad L^p\big(0,T;W^{1,p}_0(\Omega)\big) \quad \text{as} \quad \mu\to\infty. \tag{3.25}$$

We introduce the following function of one real variable:

$$h_m(s) = \begin{cases} 1 & \text{if} \quad |s| \le m, \\ 0 & \text{if} \quad |s| \ge m+1, \\ m+1-|s| & \text{if} \quad m \le |s| \le m+1, \end{cases}$$

where $m > k$. Let $\theta^{\mu,i}_n = T_k(u_n) - w^i_\mu$ and $z^{\mu,i}_{n,m} = \varphi\big(\theta^{\mu,i}_n\big)h_m(u_n)$. By using in (3.2) the test function $z^{\mu,i}_{n,m}$, we obtain since $g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n) - w^i_\mu\big)h_m(u_n) \ge 0$ on $\{|u_n| > k\}$:

$$\int\limits_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t};\, \varphi(T_k(u_n) - w^i_\mu)h_m(u_n)\right\rangle dt +$$

$$+\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w^i_\mu\big)\varphi'\big(\theta^{\mu,i}_n\big)h_m(u_n)\,dx\,dt +$$

$$+\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n\varphi\big(\theta^{\mu,i}_n\big)h'_m(u_n)\,dx\,dt +$$

$$+\int\limits_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n) - w^i_\mu\big)h_m(u_n)\,dx\,dt \le$$

$$\leq \int\limits_{Q_T} \left|f_n z_{n,m}^{\mu,i}\right| dx\,dt + \int\limits_{Q_T} \left|H_n(x,t,\nabla u_n) z_{n,m}^{\mu,i}\right| dx\,dt. \tag{3.26}$$

In the rest of this paper, we will omit for simplicity the denote $\varepsilon(n,\mu,i,m)$ all quantities (possibly different) such that

$$\lim_{m\to\infty} \lim_{i\to\infty} \lim_{\mu\to\infty} \lim_{n\to\infty} \varepsilon(n,\mu,i,m) = 0,$$

and this will be the order in which the parameters we use will tend to infinity, that is, first n, then μ, i and finally m. Similarly we will write only $\varepsilon(n)$, or $\varepsilon(n,\mu),\ldots$ to mean that the limits are made only on the specified parameters.

We will deal with each term of (3.26). First of all, observe that

$$\int\limits_{Q_T} \left|f_n z_{n,m}^{\mu,i}\right| dx\,dt + \int\limits_{Q_T} \left|H_n(x,t,\nabla u_n) z_{n,m}^{\mu,i}\right| dx\,dt = \varepsilon(n,\mu),$$

since $\varphi\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)$ converges to $\varphi\big(T_k(u) - (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)\big)h_m(u)$ strongly in $L^p(Q_T)$ and weakly $-*$ in $L^\infty(Q_T)$ as $n \to \infty$ and finally $\varphi\big(T_k(u) - (T_k(u))_\mu + e^{-\mu t}T_k(\psi_i)\big)h_m(u)$ converges to 0 strongly in $L^p(Q_T)$ and weakly $-*$ in $L^\infty(Q_T)$ as $\mu \to \infty$. Thanks to (3.20) the third and fourth integrals on the right-hand side of (3.26) tend to zero as n and m tend to infinity, and by Lebesgue's theorem and $F \in (L^{p'}(Q_T))^N$, we deduce that the right-hand side of (3.26) converges to zero as n, m and μ tend to infinity. Since $(T_k(u_n) - w_\mu^i)h_m(u_n) \rightharpoonup (T_k(u) - w_\mu^i)h_m(u)$ weakly* in $L^1(Q_T)$ and strongly in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ and $(T_k(u) - w_\mu^i)h_m(u) \rightharpoonup 0$ weakly* in $L^1(Q_T)$ and strongly in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$ as $\mu \to +\infty$.

On the one hand, the definition of the sequence w_μ^i makes it possible to establish the following lemma.

Lemma 3. *For $k \geq 0$ we have*

$$\int\limits_0^T \left\langle \frac{\partial b_n(x,u_n)}{\partial t}; \varphi(T_k(u_n) - w_\mu^i)h_m(u_n)\right\rangle dt \geq \varepsilon(n,m,\mu,i). \tag{3.27}$$

Proof (see Blanchard and Redwane [6]).

On the other hand, the second term of the left-hand side of (3.26) can be written as

$$\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt =$$

$$= \int\limits_{\{|u_n|\leq k\}} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt +$$

$$+ \int\limits_{\{|u_n|> k\}} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)h_m(u_n)\,dx\,dt =$$

$$= \int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)\,dx\,dt +$$

$$+\int\limits_{\{|u_n|>k\}} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n)-\nabla w_\mu^i\big)\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt,$$

since $m>k$ and $h_m(u_n)=1$ on $\{|u_n|\le k\}$, we deduce that

$$\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n)-\nabla w_\mu^i\big)\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt=$$

$$=\int\limits_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)\times$$

$$\times\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\varphi'\big(T_k(u_n)-w_\mu^i\big)\,dx\,dt+$$

$$+\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\times$$

$$\times\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt+$$

$$+\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u)\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt-$$

$$-\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\nabla w_\mu^i\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt=$$

$$=K_1+K_2+K_3+K_4. \tag{3.28}$$

By using (2.2), (3.19) and Lebesgue's theorem, we have $a\big(x,t,T_k(u_n),\nabla T_k(u)\big)$ converges to $a\big(x,t,T_k(u),\nabla T_k(u)\big)$ strongly in $\big(L^{p'}(Q_T)\big)^N$ and $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ weakly in $\big(L^p(Q_T)\big)^N$. Then

$$K_2=\varepsilon(n). \tag{3.29}$$

By using (3.19) and (3.25), we have

$$K_3=\int\limits_{Q_T}\overline{a}\,\nabla T_k(u)\,dx\,dt+\varepsilon(n,\mu). \tag{3.30}$$

For what concerns K_4 we can write, since $h_m(u_n)=0$ on $\{|u_n|>m+1\}$:

$$K_4=-\int\limits_{Q_T} a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla w_\mu^i\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt=$$

$$=-\int\limits_{\{|u_n|\le k\}} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla w_\mu^i\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt-$$

$$-\int\limits_{\{k<|u_n|\le m+1\}} a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla w_\mu^i\times$$

$$\times\varphi'\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt,$$

and, as above, by letting $n \to \infty$,

$$K_4 = -\int\limits_{\{|u|\le k\}} \overline{a}\,\nabla w_\mu^i \varphi'\big(T_k(u) - w_\mu^i\big)\,dx\,dt -$$

$$-\int\limits_{\{k<|u|\le m+1\}} \overline{a}\,\nabla w_\mu^i \varphi'\big(T_k(u) - w_\mu^i\big) h_m(u)\,dx\,dt + \varepsilon(n),$$

so that, by letting $\mu \to \infty$,

$$K_4 = -\int\limits_{Q_T} \overline{a}\,\nabla T_k(u)\,dx\,dt + \varepsilon(n,\mu). \tag{3.31}$$

In view of (3.28), (3.29), (3.30), and (3.31), we conclude that

$$\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\big(\nabla T_k(u_n) - \nabla w_\mu^i\big)\varphi'\big(T_k(u_n) - w_\mu^i\big) h_m(u_n)\,dx\,dt =$$

$$= \int\limits_{Q_T} \Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)\times$$

$$\times\big(\nabla T_k(u_n) - \nabla T_k(u)\big)\varphi'\big(T_k(u_n) - w_\mu^i\big)\,dx\,dt + \varepsilon(n,\mu). \tag{3.32}$$

To deal with the third term of the left-hand side of (3.26), observe that

$$\left|\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n \varphi\big(\theta_n^{\mu,i}\big) h_m'(u_n)\,dx\,dt\right| \le \varphi(2k) \int\limits_{\{m\le|u_n|\le m+1\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt.$$

Thanks to (3.20), we obtain

$$\left|\int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n \varphi\big(\theta_n^{\mu,i}\big) h_m'(u_n)\,dx\,dt\right| \le \varepsilon(n,m). \tag{3.33}$$

We now turn to fourth term of the left-hand side of (3.26), we can write

$$\left|\int\limits_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n) - w_\mu^i\big) h_m(u_n)\,dx\,dt\right| \le$$

$$\le \int\limits_{\{|u_n|\le k\}} L_1(k)L_2(x,t) + \big|\nabla T_k(u_n)\big|^p \big|\varphi\big(T_k(u_n) - w_\mu^i\big)\big| h_m(u_n)\,dx\,dt \le$$

$$\le L_1(k)\int\limits_{Q_T} L_2(x,t)\big|\varphi\big(T_k(u_n) - w_\mu^i\big)\big|\,dx\,dt +$$

$$+ \frac{L_1(k)}{\alpha} \int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt, \qquad (3.34)$$

since $L_2(x,t)$ belong to $L^1(Q_T)$ it is easy to see that

$$L_1(k)\int\limits_{Q_T} L_2(x,t)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt = \varepsilon(n,\mu).$$

On the other hand, the second term of the right-hand side of (3.34), write as

$$\frac{L_1(k)}{\alpha}\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)\nabla T_k(u_n)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt =$$

$$= \frac{L_1(k)}{\alpha}\int\limits_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)\times$$

$$\times\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt +$$

$$+\frac{L_1(k)}{\alpha}\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt +$$

$$+\frac{L_1(k)}{\alpha}\int\limits_{Q_T} a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\nabla T_k(u)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt,$$

and, as above, by letting first n then finally μ go to infinity, we can easily seen, that each one of last two integrals is of the form $\varepsilon(n,\mu)$. This implies that

$$\left|\int\limits_{\{|u_n|\le k\}} g_n(x,t,u_n,\nabla u_n)\varphi\big(T_k(u_n)-w_\mu^i\big)h_m(u_n)\,dx\,dt\right| \le$$

$$\le \frac{L_1(k)}{\alpha}\int\limits_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)\times$$

$$\times\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\,dx\,dt+\varepsilon(n,\mu). \qquad (3.35)$$

Combining (3.26), (3.27), (3.32), (3.33), and (3.35), we get

$$\int\limits_{Q_T}\Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big)-a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)\times$$

$$\times\big(\nabla T_k(u_n)-\nabla T_k(u)\big)\left(\varphi'(T_k(u)-w_\mu^i)-\frac{L_1(k)}{\alpha}\big|\varphi\big(T_k(u_n)-w_\mu^i\big)\big|\right)dx\,dt \le$$

$$\le \varepsilon(n,\mu,i,m),$$

and so, thanks to (3.23), we have

$$\int\limits_{Q_T} \Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,t,T_k(u_n),\nabla T_k(u)\big)\Big)\times$$
$$\times\big(\nabla T_k(u_n) - \nabla T_k(u)\big)\,dx\,dt \le \varepsilon(n).$$

Hence by passing to the limit sup over n, we obtain

$$\limsup_{n\to\infty}\int\limits_{Q_T} \Big(a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) - a\big(x,T_k(u_n),\nabla T_k(u)\big)\Big)(\nabla T_k(u_n) - \nabla T_k(u))\,dx\,dt = 0.$$

This implies that

$$T_k(u_n) \to T_k(u) \quad \text{strongly in} \quad L^p\big(0,T;W_0^{1,p}(\Omega)\big) \quad \text{for all} \quad k. \tag{3.36}$$

Now, observe that, for every $\sigma > 0$,

$$\text{meas}\Big\{(x,t)\in Q_T\colon |\nabla u_n - \nabla u| > \sigma\Big\} \le$$
$$\le \text{meas}\Big\{(x,t)\in Q_T\colon |\nabla u_n| > k\Big\} + \text{meas}\Big\{(x,t)\in Q_T\colon |u| > k\Big\}+$$
$$+\,\text{meas}\Big\{(x,t)\in Q_T\colon \big|\nabla T_k(u_n) - \nabla T_k(u)\big| > \sigma\Big\},$$

then as a consequence of (3.36) we have that ∇u_n converges to ∇u in measure and, therefore, always reasoning for a subsequence,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad Q_T,$$

which implies

$$a\big(x,t,T_k(u_n),\nabla T_k(u_n)\big) \rightharpoonup a\big(x,t,T_k(u),\nabla T_k(u)\big) \quad \text{weakly in} \quad \big(L^{p'}(Q_T)\big)^N. \tag{3.37}$$

Step 3: Equi-integrability of $H_n(x,t,\nabla u_n)$ and $g_n(x,t,u_n,\nabla u_n)$. We shall now prove that $H_n(x,t,\nabla u_n)$ converges to $H(x,t,\nabla u)$ and $g_n(x,t,u_n,\nabla u_n)$ converges to $g(x,t,u,\nabla u)$ strongly in $L^1(Q_T)$ by using Vitali's theorem. Since $H_n(x,t,\nabla u_n) \to H(x,t,\nabla u)$ a.e. Q_T and $g_n(x,t,u_n,\nabla u_n) \to g(x,t,u,\nabla u)$ a.e. Q_T, thanks to (2.5) and (2.8), it suffices to prove that $H_n(x,t,\nabla u_n)$ and $g_n(x,t,u_n,\nabla u_n)$ are uniformly equi-integrable in Q_T. We will now prove that $H(x,\nabla u_n)$ is uniformly equi-integrable, we use Hölder's inequality and (3.3), we have, for any measurable subset $E \subset Q_T$,

$$\int\limits_E \big|H(x,\nabla u_n)\big|\,dx\,dt \le \left(\int\limits_E h^p(x,t)\,dx\,dt\right)^{\frac{1}{p}} \left(\int\limits_{Q_T} |\nabla u_n|^p\,dx\,dt\right)^{\frac{1}{p'}} \le$$
$$\le c_1\left(\int\limits_E h^p(x,t)\,dx\,dt\right)^{\frac{1}{p}},$$

which is small uniformly in n when the measure of E is small.

To prove the uniform equi-integrability of $g_n(x,t,u_n,\nabla u_n)$. For any measurable subset $E \subset Q_T$ and $m \geq 0$,

$$\int\limits_E \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt = \int\limits_{E\cap\{|u_n|\leq m\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt +$$

$$+ \int\limits_{E\cap\{|u_n|>m\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt \leq$$

$$\leq L_1(m) \int\limits_{E\cap\{|u_n|\leq m\}} \big[L_2(x,t) + |\nabla u_n|^p\big]\,dx\,dt +$$

$$+ \int\limits_{E\cap\{|u_n|>m\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt =$$

$$= K_1 + K_2. \tag{3.38}$$

For fixed m, we get

$$K_1 \leq L_1(m) \int\limits_E \big[L_2(x,t) + \big|\nabla T_m(u_n)\big|^p\big]\,dx\,dt,$$

which is thus small uniformly in n for m fixed when the measure of E is small $\big($recall that $T_m(u_n)$ tends to $T_m(u)$ strongly in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)\big)$. We now discuss the behavior of the second integral of the right-hand side of (3.38), let ψ_m be a function such that

$$\psi_m(s) = \begin{cases} 0 & \text{if} \quad |s| \leq m-1, \\ \operatorname{sign}(s) & \text{if} \quad |s| \geq m, \end{cases}$$

$$\psi_m'(s) = 1 \quad \text{if} \quad m-1 < |s| < m.$$

We choose for $m > 1$, $\psi_m(u_n)$ as a test function in (3.2), and we obtain

$$\left[\int\limits_\Omega B_m^n(x,u_n)dx\right]_0^T + \int\limits_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n \psi_m'(u_n)\,dx\,dt +$$

$$+ \int\limits_{Q_T} g_n(x,t,u_n,\nabla u_n)\psi_m(u_n)\,dx\,dt + \int\limits_{Q_T} H_n(x,t,\nabla u_n)\psi_m(u_n)\,dx\,dt =$$

$$= \int\limits_{Q_T} f_n \psi_m(u_n)\,dx\,dt,$$

where $B_m^n(x,r) = \displaystyle\int_0^r \frac{\partial b_n(x,s)}{\partial s}\psi_m(s)\,ds$, which implies, since $B_m^n(x,r) \geq 0$ and using (2.4), Hölder's inequality

$$\int\limits_{\{m-1\leq|u_n|\}} \big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt \leq \int\limits_E |H_n(x,t,\nabla u_n)|\,dx\,dt + \int\limits_{\{m-1\leq|u_n|\}} |f|\,dx\,dt,$$

and by (3.3), we have

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\int\limits_{\{|u_n|>m-1\}}\big|g_n(x,t,u_n,\nabla u_n)\big|\,dx\,dt=0.$$

Thus we proved that the second term of the right-hand side of (3.38) is also small, uniformly in n and in E when m is sufficiently large. Which shows that $g_n(x,t,u_n,\nabla u_n)$ and $H_n(x,t,\nabla u_n)$ are uniformly equi-integrable in Q_T as required, we conclude that

$$\begin{aligned} H_n(x,t,\nabla u_n)\to H(x,t,\nabla u)\quad &\text{strongly in}\quad L^1(Q_T),\\ g_n(x,t,u_n,\nabla u_n)\to g(x,t,u,\nabla u)\quad &\text{strongly in}\quad L^1(Q_T). \end{aligned}\tag{3.39}$$

Step 4: We prove that u satisfies (2.10).

Lemma 4. *The limit u of the approximate solution u_n of* (3.2) *satisfies*

$$\lim_{m\to+\infty}\int\limits_{\{m\le|u|\le m+1\}}a(x,t,u,\nabla u)\nabla u\,dx\,dt=0.$$

Proof. Note that for any fixed $m\ge 0$, one has

$$\int\limits_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt=$$

$$=\int\limits_{Q_T}a(x,t,u_n,\nabla u_n)\big(\nabla T_{m+1}(u_n)-\nabla T_m(u_n)\big)\,dx\,dt=$$

$$=\int\limits_{Q_T}a\big(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n)\big)\nabla T_{m+1}(u_n)\,dx\,dt-$$

$$-\int\limits_{Q_T}a\big(x,t,T_m(u_n),\nabla T_m(u_n)\big)\nabla T_m(u_n)\,dx\,dt.$$

According to (3.37) and (3.36), one can pass to the limit as $n\to+\infty$ for fixed $m\ge 0$, to obtain

$$\lim_{n\to+\infty}\int\limits_{\{m\le|u_n|\le m+1\}}a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt=$$

$$=\int\limits_{Q_T}a\big(x,t,T_{m+1}(u),\nabla T_{m+1}(u)\big)\nabla T_{m+1}(u)\,dx\,dt\;-$$

$$-\int\limits_{Q_T}a\big(x,t,T_m(u),\nabla T_m(u)\big)\nabla T_m(u_n)\,dx\,dt=$$

$$=\int\limits_{\{m\le|u_n|\le m+1\}}a(x,t,u,\nabla u)\nabla u\,dx\,dt.\tag{3.40}$$

Taking the limit as $m \to +\infty$ in (3.40) and using the estimate (3.20) show that u satisfies (2.10) and the proof is complete.

Step 5: We prove that u satisfies (2.11) and (2.12).

Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let M be a positive real number such that support of S' is a subset of $[-M, M]$. Pointwise multiplication of the approximate equation (3.2) by $S'(u_n)$ leads to

$$\frac{\partial B_S^n(x,u_n)}{\partial t} - \operatorname{div}\Big(S'(u_n)a(x,t,u_n,\nabla u_n)\Big) + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n + \\ + S'(u_n)\Big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\Big) = fS'(u_n) \quad \text{in} \quad \mathcal{D}'(Q_T), \tag{3.41}$$

where

$$B_S^n(x,z) = \int\limits_0^z \frac{\partial b_n(x,r)}{\partial r} S'(r)\,dr.$$

In what follows we pass to the limit in (3.41) as n tends to $+\infty$:

Limit of $\dfrac{\partial B_S^n(x,u_n)}{\partial t}$. Since S is bounded and continuous, $u_n \to u$ a.e. in Q_T, implies that $B_S^n(x,u_n)$ converges to $B_S(x,u)$ a.e. in Q_T and $L^\infty(Q_T)$-weak*. Then $\dfrac{\partial B_S^n(x,u_n)}{\partial t}$ converges to $\dfrac{\partial B_S(x,u)}{\partial t}$ in $\mathcal{D}'(Q_T)$ as n tends to $+\infty$.

The limit of $-\operatorname{div}\big(S'(u_n)a(x,t,u_n,\nabla u_n)\big)$. Since $\operatorname{supp}(S') \subset [-M, M]$, we have for $n \geq M$: $S'(u_n)a_n(x,t,u_n,\nabla u_n) = S'(u_n)a\big(x,t,T_M(u_n),\nabla T_M(u_n)\big)$ a.e. in Q_T. The pointwise convergence of u_n to u, (3.37) and the bounded character of S' yield, as n tends to $+\infty$: $S'(u_n)a_n\big(x,t,u_n,\nabla u_n\big)$ converges to $S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)$ in $\big(L^{p'}(Q_T)\big)^N$, and $S'(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)$ has been denoted by $S'(u)a(x,t,u,\nabla u)$ in equation (2.11).

The limit of $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n$. Consider the "energy" term, $S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n = S''(u_n)a\big(x,t,T_M(u_n),\nabla T_M(u_n)\big)\nabla T_M(u_n)$ a.e. in Q_T.

The pointwise convergence of $S'(u_n)$ to $S'(u)$ and (3.37) as n tends to $+\infty$ and the bounded character of S'' permit us to conclude that $S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n$ converges to $S''(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)\nabla T_M(u)$ weakly in $L^1(Q_T)$. Recall that

$$S''(u)a\big(x,t,T_M(u),\nabla T_M(u)\big)\nabla T_M(u) = S''(u)a(x,t,u,\nabla u)\nabla u \quad \text{a.e. in} \quad Q_T.$$

The limit of $S'(u_n)\big(g_n(x,t,u_n,\nabla u_n) + H_n(x,t,\nabla u_n)\big)$. From $\operatorname{supp}(S') \subset [-M, M]$, by (3.39), we have $S'(u_n)g_n(x,t,u_n,\nabla u_n)$ converges to $S'(u)g(x,t,u,\nabla u)$ strongly in $L^1(Q_T)$ and $S'(u_n)H_n(x,t,\nabla u_n)$ converge to $S'(u)H(x,t,\nabla u)$ strongly in $L^1(Q_T)$, as n tends to $+\infty$.

The limit of $S'(u_n)f_n$. Since $u_n \to u$ a.e. in Q_T, we have $S'(u_n)f_n$ converges to $S'(u)f$ strongly in $L^1(Q_T)$, as n tends to $+\infty$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (3.41) and to conclude that u satisfies (2.11).

It remains to show that $B_S(x,u)$ satisfies the initial condition (2.12). To this end, firstly remark that, S being bounded and in view of (2.14), (3.15), we have $B_S^n(x,u_n)$ is bounded in $L^p\big(0,T;W_0^{1,p}(\Omega)\big)$. Secondly, (3.41) and the above considerations on the behavior of the terms of this equation show that $\dfrac{\partial B_S^n(x,u_n)}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}\big(0,T;W^{-1,p'}(\Omega)\big)$. As a consequence (see [21]), $B_S^n(x,u_n)(t=0) = B_S^n(x,u_{0n})$ converges to $B_S(x,u)(t=0)$ strongly in $L^1(\Omega)$. On the other hand, the smoothness of S and in view of (3.1) imply that $B_S(x,u)(t=0) = B_S(x,u_0)$ in Ω. As a conclusion, steps 1 – 5 complete the proof of Theorem 1.

References

1. *Akdim Y., Benkirane A., El Moumni M., Redwane H.* Existence of renormalized solutions for nonlinear parabolic equations // J. Part. Different. Equat. – 2014. – **27**, № 1. – P. 28 – 49.
2. *Alvino A., Trombetti G.* Sulle migliori costanti di maggiorazione per una classe di equazioni ellittiche degeneri // Ric. Mat. – 1978. – **27**. – P. 413 – 428.
3. *Beckenbach E.-F., Bellman R.* Inequalities. – New York: Springer-Verlag, 1965.
4. *Blanchard D., Murat F.* Renormalized solutions of nonlinear parabolic problems with L^1 data: existence and uniqueness // Proc. Roy. Soc. Edinburgh Sect. A. – 1997. – **127**. – P. 1137 – 1152.
5. *Blanchard D., Murat F., Redwane H.* Existence and uniqueness of renormalized solution for a fairly general class of nonlinear parabolic problems // J. Diferent. Equat. – 2001. – **177**. – P. 331 – 374.
6. *Blanchard D., Redwane H.* Existence of a solution for a class of parabolic equations with three unbounded nonlinearities, natural growth terms and L^1 data // Arab. J. Math. Sci. – 2014. – **20**, № 2. – P. 157 – 176.
7. *Boccardo L., Dall'Aglio A., Gallouët T., Orsina L.* Nonlinear parabolic equations with measure data // J. Funct. Anal. – 1997. – **147**, № 1. – P. 237 – 258.
8. *Boccardo L., Gallouët T.* On some nonlinear elliptic and parabolic equations involving measure data // J. Funct. Anal. – 1989. – **87**. – P. 149 – 169.
9. *Boccardo L., Giachetti D., Diaz J. I., Murat F.* Existence and regularity of renormalized solutions of some elliptic problems involving derivatives of nonlinear terms // J. Different. Equat. – 1993. – **106**. – P. 215 – 237.
10. *Dal Maso G., Murat F., Orsina L., Prignet A.* Definition and existence of renormalized solutions of elliptic equations with general measure data // C. R. Acad. Sci. Paris. – 1997. – **325**. – P. 481 – 486.
11. *Dall'Aglio A., Orsina L.* Nonlinear parabolic equations with natural growth conditions and L^1 data // Nonlinear Anal. – 1996. – **27**. – P. 59 – 73.
12. *Diperna R. J., Lions P.-L.* On the Cauchy problem for Boltzman equations: global existence and weak stability // Ann. Math. – 1989. – **130**, № 2. – P. 321 – 366.
13. *Droniou J., Porretta A., Prignet A.* Parabolic capacity and soft measures for nonlinear equations // Potential Anal. – 2003. – **19**, № 2. – P. 99 – 161.
14. *Kardar M., Parisi G., Zhang Y. C.* Dynamic scaling of growing interfaces // Phys. Rev. Lett. – 1986. – **56**. – P. 889 – 892.
15. *Landes R.* On the existence of weak solutions for quasilinear parabolic initial-boundary value problems // Proc. Roy. Soc. Edinburgh Sect. A. – 1981. – **89**. – P. 321 – 366.
16. *Lions J.-L.* Quelques méthodes de résolution des problème aux limites non lineaires. – Paris: Dundo, 1969.
17. *Lions P.-L.* Mathematical topics in fluid mechanics // Oxford Lect. Ser. Math. and Appl. – 1996. – Vol. 1.
18. *Liu W. J.* Extinction properties of solutions for a class of fast diffusive p-Laplacian equations // Nonlinear Anal. – 2011. – **74**. – P. 4520 – 4532.
19. *Murat F.* Soluciones renormalizadas de EDP elipticas no lineales // Lab. Anal. Numer. Paris. – 1993. – **6**.
20. *Murat F.* Equations elliptiques non lineaires avec second membre L^1 ou mesure // Compt. Rend. 26ème Congr. Nat. Anal. Numér. (Les Karellis). – 1994. – P. A12 – A24.
21. *Porretta A.* Existence results for nonlinear parabolic equations via strong convergence of truncations // Ann. Mat. Pura ed Appl. (IV). – 1999. – **177**. – P. 143 – 172.
22. *Porretta A.* Nonlinear equations with natural growth terms and measure data // Electron J. Different. Equat. – 2002. – P. 183 – 202.
23. *Porzio M. M.* Existence of solutions for some noncoercive parabolic equations // Discrete Contin. Dyn. Syst. – 1999. – **5**, № 3. – P. 553 – 568.
24. *Rakotoson J. M.* Uniqueness of renormalized solutions in a T-set for L^1 data problems and the link between various formulations // Indiana Univ. Math. J. – 1994. – **43**. – P. 685 – 702.
25. *Redwane H.* Solution renormalises de problèmes paraboliques et elleptique non linéaires: Ph. D. Thesis. – Rouen, 1997.

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