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INSTABILITY INTERVALS FOR HILL'S EQUATION WITH SYMMETRIC SINGLE-WELL POTENTIAL

ІНТЕРВАЛИ НЕСТАБІЛЬНОСТІ ДЛЯ РІВНЯННЯ ХІЛЛА З СИМЕТРИЧНИМ ОДНОЯМНИМ ПОТЕНЦІАЛОМ

We deduce some explicit estimates for the periodic and semiperiodic eigenvalues and the lengths of the instability intervals of Hill's equation with symmetric single-well potentials by using an auxiliary eigenvalue problem. We also give bounds for the gaps of the Dirichlet and Neumann eigenvalues.

За допомогою допоміжної задачі на власні значення отримано деякі явні оцінки для періодичних і напівперіодичних власних значень та довжин інтервалів нестабільності для рівняння Хілла з симетричним одноямним потенціалом. Також наведено оцінки для щілин у множинах власних значень Діріхле таНоймана.

1. Introduction. We consider the differential equation

$$y''(t) + (\lambda - q(t))\, y(t) = 0, \tag{1.1}$$

where λ is a real parameter and $q(t)$ is a real-valued, continuous and periodic function with period a. Our interest is with two eigenvalue problems associated with (1.1) on $[0, a]$. The periodic problem of (1.1) with the boundary conditions $y(0) = y(a),\ y'(0) = y'(a)$. This problem has a countable infinity of eigenvalues denoted by $\{\lambda_n\}$. We are also concerned with the semiperiodic problem of (1.1) with the boundary conditions $y(0) = -y(a),\ y'(0) = -y'(a)$ and the eigenvalues are denoted by $\{\mu_n\}$. It is known [4] that the two sets of eigenvalues satisfy the relation

$$-\infty < \lambda_0 < \mu_0 \le \mu_1 < \lambda_1 \le \lambda_2 < \mu_2 \le \mu_3 < \ldots .$$

We also denote the eigenvalues of (1.1) with the Dirichlet boundary conditions $y(0) = y(a) = 0$ by Λ_n and the Neumann boundary conditions $y'(0) = y'(a) = 0$ by ν_n. It is also known [4] that, for $n = 0, 1, 2, \ldots,$

$$\mu_{2n} \le \Lambda_{2n} \le \mu_{2n+1}, \quad \lambda_{2n+1} \le \Lambda_{2n+1} \le \lambda_{2n+2}, \tag{1.2}$$

and

$$\mu_{2n} \le \nu_{2n+1} \le \mu_{2n+1}, \quad \lambda_{2n+1} \le \nu_{2n+2} \le \lambda_{2n+2}. \tag{1.3}$$

The instability intervals of (1.1) are defined to be $(-\infty, \lambda_0)$, (μ_{2n}, μ_{2n+1}), $(\lambda_{2n+1}, \lambda_{2n+2})$ and called the zeroth, $(2n+1)$th and $(2n+2)$th instability interval, respectively. The length of the nth instability interval of (1.1), whether it is absent or not, will be denoted by l_n. We note that the absence of an instability interval means that there is a value of λ for which all solutions of (1.1) have either period a or semiperiod a. Instability intervals for Hill's equation with various types of restrictions on potential have been investigated by many authors over the years [1, 4, 9]. We refer in particular to [7, 8] in which $q(t)$ is a symmetric single-well potential. Some results about the first instability interval were obtained in [7] and the eigenvalue gap for Schrödinger operators on an interval with Dirichlet and Neumann boundary conditions was considered in [8].

In this paper we obtain estimates about the instability intervals of (1.1) with $q(t)$ being of a symmetric single-well potential with mean value zero. By a symmetric single-well potential on $[0,a]$, we mean a continuous function $q(t)$ on $[0,a]$ which is symmetric about $t=\dfrac{a}{2}$ and non-increasing on $\left[0,\dfrac{a}{2}\right]$. Our analysis is based on the following theorem of Hochstadt, which involves $\Lambda_n(\tau)$ the eigenvalues of (1.1) considered on the interval $[\tau,\tau+a]$ where $0\le\tau<a$ with Dirichlet boundary conditions

$$y(\tau)=y(\tau+a)=0.$$

We refer to this problem as "auxiliary eigenvalue problem". Here we note that this problem is equivalent to the following problem [6]:

$$y''(t)+(\lambda-q\,(t+\tau))\,y(t)=0,$$

$$y(0)=y(a)=0.$$

We note that $q'(t)$ exists since a monotone function on an interval I is differentiable almost everywhere on I [5].

We now state an asymptotic approximation previously obtained for the auxiliary eigenvalues [1 – 3] which will be used to prove our results. It was shown in [3, p. 1275] (for $N=2$) as $n\to\infty$:

$$\Lambda_n^{1/2}(\tau)=\frac{(n+1)\,\pi}{a}+\frac{a}{4\,(n+1)^2\,\pi^2}\times$$

$$\times\left[\cos\left(\frac{2\,(n+1)\,\pi}{a}\tau\right)\int\limits_{\tau}^{\tau+a}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt-\right.$$

$$\left.-\sin\left(\frac{2\,(n+1)\,\pi}{a}\tau\right)\int\limits_{\tau}^{\tau+a}q'(t)\cos\left(\frac{2(n+1)\pi}{a}t\right)dt\right]-$$

$$-\frac{a^2}{8\,(n+1)^3\,\pi^3}\int\limits_0^a q^2\,(t)\,dt+o\left(n^{-3}\right). \tag{1.4}$$

As an illustration of our results, we give the following theorem.

Theorem 1.1. *Let $q(t)$ be a symmetric single-well potential on $[0,a]$. Then, as $n\to\infty$,*

$$\begin{matrix}\Lambda_{2n+1}-\Lambda_{2n}\\ \nu_{2n+2}-\nu_{2n+1}\end{matrix}\ \ge\frac{(4n+3)\,\pi^2}{a^2}-\frac{1}{2\,(n+1)\,\pi}\left|\int\limits_0^{a/2}q'(t)\sin\left(\frac{4\,(n+1)\,\pi}{a}t\right)dt\right|-$$

$$-\frac{1}{(2n+1)\pi}\left|\int\limits_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right| + o\left(n^{-2}\right)$$

and

$$\begin{matrix}\Lambda_{2n+1}-\Lambda_{2n} \\ \nu_{2n+2}-\nu_{2n+1}\end{matrix} \le \frac{(4n+3)\pi^2}{a^2} + \frac{1}{2(n+1)\pi}\left|\int\limits_0^{a/2} q'(t)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right| +$$

$$+\frac{1}{(2n+1)\pi}\left|\int\limits_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right| + o\left(n^{-2}\right).$$

The following theorem [4] which involves the auxiliary eigenvalues $\Lambda_n(\tau)$ plays an important role to obtain periodic and semiperiodic eigenvalues.

Theorem 1.2 [4]. *The ranges of* $\Lambda_{2n}(\tau)$ *and* $\Lambda_{2n+1}(\tau)$ *as functions of* τ *are* $[\mu_{2n}, \mu_{2n+1}]$ *and* $[\lambda_{2n+1}, \lambda_{2n+2}]$, *respectively.*

By this theorem and the fact that $\Lambda_n(\tau)$ is a continuous function of τ, we observe that

$$\begin{gathered}\max_\tau \Lambda_{2n}(\tau) = \mu_{2n+1}, \quad \min_\tau \Lambda_{2n}(\tau) = \mu_{2n}, \\ \max_\tau \Lambda_{2n+1}(\tau) = \lambda_{2n+2}, \quad \min_\tau \Lambda_{2n+1}(\tau) = \lambda_{2n+1}.\end{gathered} \tag{1.5}$$

2. Proof of the result. Before we prove the results, we first state the following lemma.

Lemma 2.1. *If* $q(t)$ *is a symmetric single-well potential, then*

(i) $\displaystyle\int_\tau^{\tau+a} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt = 2\int_0^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt,$

(ii) $\displaystyle\int_\tau^{\tau+a} q'(t)\cos\left(\frac{2(n+1)\pi}{a}t\right)dt = 0,$

(iii) $\displaystyle\int_0^a q^2(t)dt = aq^2(a) + 2a\int_0^{a/2} q(t)q'(t)dt - 4\int_0^{a/2} tq(t)q'(t)dt.$

Proof. (i) Since $q'(t)\sin\left(\dfrac{2(n+1)\pi}{a}t\right)$ is a periodic function with period a, we get

$$\int\limits_\tau^{\tau+a} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt = \int\limits_0^a q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt =$$

$$= \int\limits_0^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt + \int\limits_{a/2}^a q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt =$$

$$= \int\limits_0^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt - \int\limits_{a/2}^a q'(a-t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt =$$

$$= 2\int_0^{a/2} q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt.$$

The last equality holds since $q(t)$ is symmetric and $q'(t)$ exists.

(ii) It can be proved similarly.

(iii) By using integration by parts, we have

$$\int_0^a q^2(t)dt = tq^2(t)\big|_0^a - 2\int_0^a tq(t)q'(t)\,dt =$$

$$= aq^2(a) - 2\left\{\int_0^{a/2} tq(t)q'(t)dt + \int_{a/2}^a tq(t)q'(t)\,dt\right\} =$$

$$= aq^2(a) - 2\left\{\int_0^{a/2} tq(t)q'(t)dt - \int_{a/2}^a tq(a-t)\,q'(a-t)\,dt\right\} =$$

$$= aq^2(a) - 2\left\{\int_0^{a/2} tq(t)q'(t)dt + \int_{a/2}^0 (a-t)\,q(t)q'(t)dt\right\} =$$

$$= aq^2(a) + 2a\int_0^{a/2} q(t)q'(t)\,dt - 4\int_0^{a/2} tq(t)q'(t)dt.$$

Theorem 2.1. *The periodic and semiperiodic eigenvalues of* (1.1) *satisfy, as* $n\to\infty$,

$$\begin{matrix}\lambda_{2n+1}^{1/2}\\ \lambda_{2n+2}^{1/2}\end{matrix} = \frac{2(n+1)\pi}{a} \mp \frac{a}{8(n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{4(n+1)\pi}{a}t\right)dt\right| -$$

$$-\frac{a^2}{64(n+1)^3\pi^3}\times$$

$$\times\left[aq^2(a) + 2a\int_0^{a/2} q(t)q'(t)dt - 4\int_0^{a/2} tq(t)q'(t)dt\right] + o\left(n^{-3}\right)$$

and

$$\begin{matrix}\mu_{2n}^{1/2}\\ \mu_{2n+1}^{1/2}\end{matrix} = \frac{(2n+1)\pi}{a} \mp \frac{a}{2(2n+1)^2\pi^2}\left|\int_0^{a/2} q'(t)\sin\left(\frac{2(2n+1)\pi}{a}t\right)dt\right| -$$

$$-\frac{a^2}{8\left(2n+1\right)^3\pi^3}\times$$

$$\times\left[aq^2(a)+2a\int\limits_0^{a/2}q(t)q'(t)dt-4\int\limits_0^{a/2}tq(t)q'(t)dt\right]+o\left(n^{-3}\right).$$

Proof. From (1.4) and Lemma 2.1, we observe that

$$\Lambda_n^{1/2}(\tau)=\frac{(n+1)\,\pi}{a}+$$

$$+\frac{a}{2\left(n+1\right)^2\pi^2}\cos\left(\frac{2\left(n+1\right)\pi}{a}\tau\right)\int\limits_0^{a/2}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt-$$

$$-\frac{a^2}{8\left(n+1\right)^3\pi^3}\times$$

$$\times\left[aq^2(a)+2a\int\limits_0^{a/2}q(t)q'(t)dt-4\int\limits_0^{a/2}tq(t)q'(t)dt\right]+o\left(n^{-3}\right).$$

If we minimize and maximize the last equation, we find

$$\min_{\tau}\Lambda_n^{1/2}(\tau)=\frac{(n+1)\,\pi}{a}-\frac{a}{2\left(n+1\right)^2\pi^2}\left|\int\limits_0^{a/2}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt\right|-$$

$$-\frac{a^2}{8\left(n+1\right)^3\pi^3}\times$$

$$\times\left[aq^2(a)+2a\int\limits_0^{a/2}q(t)q'(t)dt-4\int\limits_0^{a/2}tq(t)q'(t)dt\right]+o\left(n^{-3}\right)\qquad(2.1)$$

and

$$\max_{\tau}\Lambda_n^{1/2}(\tau)=\frac{(n+1)\,\pi}{a}+\frac{a}{2\left(n+1\right)^2\pi^2}\left|\int\limits_0^{a/2}q'(t)\sin\left(\frac{2(n+1)\pi}{a}t\right)dt\right|-$$

$$-\frac{a^2}{8\left(n+1\right)^3\pi^3}\times$$

$$\times\left[aq^2(a)+2a\int\limits_0^{a/2}q(t)q'(t)dt-4\int\limits_0^{a/2}tq(t)q'(t)dt\right]+o\left(n^{-3}\right).\qquad(2.2)$$

Now, (1.5), (2.1) and (2.2) prove the theorem.

***Corollary* 2.1.** *l_n satisfies, as $n \to \infty$,*

$$l_n = \frac{2}{n\pi}\left|\int\limits_0^{a/2} q'(t)\sin\left(\frac{2n\pi}{a}t\right)dt\right| + o\left(n^{-2}\right).$$

Proof. Follows from Theorem 2.1.

***Proof of Theorem* 1.1.** Theorem 2.1, (1.2) and (1.3) are used to prove the theorem.

***Corollary* 2.2.** *Let $q(t)$ be a constant. Then, as $n \to \infty$,*

$$\begin{matrix}\Lambda_{2n+1} - \Lambda_{2n}\\ \nu_{2n+2} - \nu_{2n+1}\end{matrix} = \frac{(4n+3)\,\pi^2}{a^2} + o\left(n^{-2}\right).$$

Proof. Follows from Theorem 1.1.

3. An example. To illustrate the foregoing results, we consider an eigenvalue problem

$$y''(t) + (\lambda - q(t))\,y(t) = 0, \quad t \in [0,\pi),$$

where $q(t) = \dfrac{1}{4}\left(t - \dfrac{\pi}{2}\right)^4 + \dfrac{1}{2}\left(t - \dfrac{\pi}{2}\right)^2$ and extended by periodicity. Since we assumed that $q(t)$ has mean value zero in our results, we take $q(t)$ as follows:

$$q(t) = \frac{1}{4}\left(t - \frac{\pi}{2}\right)^4 + \frac{1}{2}\left(t - \frac{\pi}{2}\right)^2 - \frac{\pi^2}{24} - \frac{\pi^4}{160}.$$

In this case, by evaluating integral terms in Theorem 1.1, Theorem 2.1 and Corollary 2.1, we obtain, as $n \to \infty$,

$$\begin{matrix}\lambda_{2n+1}^{1/2}\\ \lambda_{2n+2}^{1/2}\end{matrix} = 2(n+1) \mp \frac{\pi^2+4}{256\,(n+1)^3\,\pi} -$$

$$-\frac{1}{64\,(n+1)^3}\frac{1}{1290240}\pi^4\left[35\pi^4 + 384\pi^2 + 1792\right] + o\left(n^{-3}\right),$$

$$\begin{matrix}\mu_{2n}^{1/2}\\ \mu_{2n+1}^{1/2}\end{matrix} = 2n+1 \mp \frac{\pi^2+4}{32\,(2n+1)^3\,\pi} -$$

$$-\frac{1}{8\,(2n+1)^3}\frac{1}{1290240}\pi^4\left[35\pi^4 + 384\pi^2 + 1792\right] + o\left(n^{-3}\right),$$

$$\begin{matrix}\Lambda_{2n+1} - \Lambda_{2n}\\ \nu_{2n+2} - \nu_{2n+1}\end{matrix} \geq 4n+3 - \frac{\pi^2+4}{64\,(n+1)^2} - \frac{\pi^2+4}{16\,(2n+1)^2} + o\left(n^{-2}\right),$$

$$\begin{matrix}\Lambda_{2n+1} - \Lambda_{2n}\\ \nu_{2n+2} - \nu_{2n+1}\end{matrix} \leq 4n+3 + \frac{\pi^2+4}{64\,(n+1)^2} + \frac{\pi^2+4}{16\,(2n+1)^2} + o\left(n^{-2}\right),$$

and

$$l_{n+1} = \frac{\pi^2+4}{8\,(n+1)^2} + o\left(n^{-2}\right).$$

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