UDC 512.5

J. K. Adashev (Inst. Math., Nat. Univ. Uzbekistan, Tashkent),
M. Ladra (Univ. Santiago de Compostela, Spain),
B. A. Omirov (Inst. Math., Nat. Univ. Uzbekistan, Tashkent)

THE CLASSIFICATION OF NATURALLY GRADED ZINBIEL ALGEBRAS WITH CHARACTERISTIC SEQUENCE EQUAL TO $(n-p,p)$ *

КЛАСИФІКАЦІЯ ПРИРОДНО ГРАДУЙОВАНИХ АЛГЕБР ЗІНБЄЛЯ З ХАРАКТЕРИСТИЧНОЮ ПОСЛІДОВНІСТЮ $(n-p,p)$

This work is a continuation of the description of some classes of nilpotent Zinbiel algebras. We focus on the study of Zinbiel algebras with restrictions imposed on gradation and characteristic sequence. Namely, we obtain the classification of naturally graded Zinbiel algebras with characteristic sequence equal to $(n-p,p)$.

Продовжено опис деяких класів нільпотентних алгебр Зінбєля. Основну увагу зосереджено на вивченні алгебр Зінбєля з обмеженнями на градацію та характеристичну послідовність, а саме, отримано класифікацію природно градуйованих алгебр Зінбєля з характеристичною послідовністю $(n-p,p)$.

1. Introduction. This paper is devoted to investigation of algebras, which are Koszul dual to Leibniz algebras. These algebras were introduced in the middle of 90th of the last century by the French mathematician J.-L. Loday [15] and they are called Zinbiel algebras (Leibniz written in reverse order).

A crucial fact of the theory of finite dimensional Zinbiel algebras is the nilpotency of such algebras over a field of zero characteristic [12]. Since the description of finite-dimensional complex Zinbiel algebras is a boundless problem (even if they are nilpotent), their study should be carried out by adding some additional restrictions (on index of nilpotency, gradation, characteristic sequence, etc.).

In general, investigation of Zinbiel algebras goes parallel to the study of nilpotent Leibniz algebras. For instance, n-dimensional Leibniz algebras of nilindices $n+1$ and n (which is equivalent to admit characteristic sequences equal to (n) and $(n-1,1)$, respectively) were described in papers [5] and [14]. Similar description for Zinbiel algebras were obtained in the paper [3].

In the study of n-dimensional Leibniz algebras of nilindex $n-1$ (see [10]) it was noted that characteristic sequences of such algebras are equal to either $(n-2,1,1)$ or $(n-2,2)$. Description of Leibniz (Zinbiel) algebras with such characteristic sequence were obtained in [9] and [10] (respectively, [4]).

Later on, naturally graded Leibniz algebras of nilindex $n-2$ that admit the following characteristic sequences:

$$(n-3,3), \quad (n-3,2,1), \quad (n-3,1,1,1)$$

* The work was partially supported by Agencia Estatal de Investigación (Spain), grant MTM2016-79661-P (European FEDER support included, UE), and by Xunta de Galicia, grant ED431C2019/10 (European FEDER support included).

were investigated in a series of papers [6, 7, 11], respectively. Description of naturally graded Zinbiel algebras with these properties was given in [1] and [2].

Finally, the latest progress in the description of the structure of nilpotent Leibniz algebras was obtained in papers [11] and [16]. In particular, naturally graded nilpotent n-dimensional Leibniz algebras with characteristic sequences equal to $(n-p,p)$ and $(n-p,1,\ldots,1)$ were described. Since the description of p-filiform Zinbiel algebras (that are Zinbiel algebras with characteristic sequence equal to $(n-p,1,\ldots,1)$) was obtained in [8], in order to complete the description similar to [11], in this paper we present the description (up to isomorphism) of naturally graded Zinbiel algebras with characteristic sequence equal to $(n-p,p)$.

All considered algebras and vector spaces in this work are assumed to be finite dimensional and complex. In order to keep tables of multiplications of algebras short, we will omit zero products.

2. Preliminaries. In this section we give definitions and known results necessary to proceed further to the main part of the work.

Definition 2.1. *An algebra A over a field F is called a Zinbiel algebra if for any $x,y,z\in A$ the following identity holds:*

$$(x\circ y)\circ z=x\circ(y\circ z)+x\circ(z\circ y),$$

where $\circ$ is the multiplication of the algebra A.

For an arbitrary Zinbiel algebra we define the *lower series* as follows:

$$A^1=A,\quad A^{k+1}=A\circ A^k,\quad k\geq 1.$$

Definition 2.2. *A Zinbiel algebra A is called nilpotent if there exists $s\in\mathbb{N}$ such that $A^s=0$. The minimal such number is called the nilindex of A.*

Definition 2.3. *An n-dimensional Zinbiel algebra A is called null-filiform if* $\dim A^i=(n+1)-i$ *for* $1\leq i\leq n+1$.

It is clear by definition that an algebra A being null-filiform is equivalent to admitting the maximal possible nilindex.

Let x be an element of the set $A\setminus A^2$. For an operator of a left multiplication L_x (defined as $L_x(y)=x\circ y$) we define a descending sequence $C(x)=(n_1,n_2,\ldots,n_k)$, where $n=n_1+n_2+\ldots+n_k$, which consists of the sizes of Jordan blocks of the operator L_x. On the set of such sequences we consider the lexicographical order, that is, $(n_1,n_2,\ldots,n_k)\leq(m_1,m_2,\ldots,m_s)$ if there exists $i\in N$ such that $n_j=m_j$ for all $j<i$ and $n_i<m_i$.

Definition 2.4. *The sequence $C(A)=\max_{x\in A\setminus A^2}C(x)$ is called the characteristic sequence of the algebra A.*

***Example* 2.1.** Let $C(A)=(1,1,\ldots,1)$. Then the algebra A is Abelian.

***Example* 2.2.** An n-dimensional Zinbiel algebra A is null-filiform if and only if $C(A)=(n)$.

Let A be a finite-dimensional Zinbiel algebra of nilindex s. We set $A_i:=A^i/A^{i+1}$, $1\leq i\leq s-1$, and $\operatorname{gr}A:=A_1\oplus A_2\oplus\cdots\oplus A_{s-1}$. From the condition $A_i\circ A_j\subseteq A_{i+j}$ we derive a graded algebra $\operatorname{gr}A$. The graduation constructed in a such way is called the natural graduation. If a Zinbiel algebra A is isomorphic to the algebra $\operatorname{gr}A$, then the algebra A is called a naturally graded Zinbiel algebra.

Further we need the following lemmas.

Lemma 2.1 [13]. *For any $n, a \in \mathbb{N}$ the following equality holds:*

$$\sum_{k=0}^{n}(-1)^k C_a^k C_{a+n-k-1}^{n-k} = 0.$$

Lemma 2.2 [12]. *Let A be a Zinbiel algebra with the following products to be known:*

$$e_1 \circ e_i = e_{i+1}, \quad 1 \le i \le k-1.$$

Then

$$e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \le i+j \le k,$$

where $C_a^b = \binom{a}{b}$ denotes the binomial coefficient.

3. Main results. Let A be a Zinbiel algebra and $C(A) = (n_1, n_2, \dots, n_k)$ its characteristic sequence. Then there exists a basis $\{e_1, e_2, \dots, e_n\}$ such that the matrix of an operator of the left multiplication by the element e_1 has the form

$$L_{e_1,\sigma} = \begin{pmatrix} J_{n_{\sigma(1)}} & 0 & \dots & 0 \\ 0 & J_{n_{\sigma(2)}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & J_{n_{\sigma(s)}} \end{pmatrix},$$

where $\sigma(i)$ belongs to $\{1, 2, \dots, s\}$.

By a suitable permutation of basis elements we can assume that $n_{\sigma(2)} \ge n_{\sigma(3)} \ge \dots \ge n_{\sigma(s)}$.

Let A be a naturally graded Zinbiel algebra with characteristic sequence equal to $(n_1, n_2, \dots, n_k)$.

Proposition 3.1. *There is no naturally graded Zinbiel algebra with $n_{\sigma(1)} = 1$ and $n_{\sigma(2)} \ge 4$.*

Proof. From the condition of the proposition we have the products

$$e_1 \circ e_1 = 0, \quad e_1 \circ e_i = e_{i+1}, \quad 2 \le i \le 4,$$

$$e_1 \circ e_i = e_{i+1}, \quad \sum_{k=1}^{t} n_{\sigma(k)} + 1 \le i \le \sum_{k=1}^{t+1} n_{\sigma(k)} - 1, \quad 3 \le t \le s-1,$$

$$e_1 \circ e_i = 0, \quad i = \sum_{k=1}^{t} n_{\sigma(k)}, \quad 3 \le t \le s.$$

By using the property of Zinbiel algebras

$$(a \circ b) \circ c = (a \circ c) \circ b,$$

we obtain

$$e_3 \circ e_1 = (e_1 \circ e_2) \circ e_1 = (e_1 \circ e_1) \circ e_2 = 0 \Rightarrow e_3 \circ e_1 = 0.$$

The chain of equalities

$$0 = (e_1 \circ e_1) \circ e_3 = e_1 \circ (e_1 \circ e_3) + e_1 \circ (e_3 \circ e_1) = e_1 \circ e_4 = e_5$$

implies $e_5 = 0$, that is, we get a contradiction with the condition $n_{\sigma(2)} \ge 4$ which completes the proof of the proposition.

The next example shows that the condition $n_{\sigma(2)} \ge 4$ of Proposition 3.1 is essential.

***Example* 3.1.** Let A be a four-dimensional Zinbiel algebra with the multiplication table

$$e_1 \circ e_2 = e_3, \quad e_1 \circ e_3 = e_4, \quad e_2 \circ e_1 = -e_3.$$

Then $C(A) = (3,1)$ and the matrix of the operator of the left multiplication on e_1 has the form $\begin{pmatrix} J_1 & 0 \\ 0 & J_3 \end{pmatrix}$.

Let A be an arbitrary Zinbiel algebra with characteristic sequence equal to $(n-p,p)$. Then the matrix of the operator of the left multiplication by e_1 admits one of the following forms:

$$\text{I. } \begin{pmatrix} J_{n-p} & 0 \\ 0 & J_p \end{pmatrix}; \quad \text{II. } \begin{pmatrix} J_p & 0 \\ 0 & J_{n-p} \end{pmatrix}, \quad n \geq 2p.$$

Definition 3.1. *A Zinbiel algebra is called an algebra of the first type* (*of type* I) *if the operator* L_{e_1} *has the form* $\begin{pmatrix} J_{n-p} & 0 \\ 0 & J_p \end{pmatrix}$; *otherwise it is called an algebra of the second type* (*of type* II).

Taking into account results of papers [4] and [3], we will consider only n-dimensional naturally graded Zinbiel algebras with $C(A) = (n-p,p)$, $p \geq 3$.

***3.1. Classification of Zinbiel algebras of type* I.** Let A be a Zinbiel algebra of type I. Then we have the existence of a basis $\{e_1, e_2, \ldots, e_{n-p}, f_1, f_2, \ldots, f_p\}$ such that the products containing an element e_1 on the left are as follows:

$$e_1 \circ e_i = e_{i+1}, \quad 1 \leq i \leq n-p-1.$$

From Lemma 2.2 we obtain

$$\begin{gathered} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i+j \leq n-p, \quad e_1 \circ e_p = 0, \\ e_1 \circ f_i = f_{i+1}, \quad 1 \leq i \leq p-1, \quad e_1 \circ f_p = 0. \end{gathered} \tag{1}$$

It is easy to see that

$$A_1 = \langle e_1, f_1 \rangle, A_2 = \langle e_2, f_2 \rangle, \ldots, A_p = \langle e_p, f_p \rangle, A_{p+1} = \langle e_{p+1} \rangle, \ldots, A_{n-p} = \langle e_{n-p} \rangle.$$

Let

$$\begin{gathered} f_1 \circ e_i = \alpha_i e_{i+1} + \beta_i f_{i+1}, \quad 1 \leq i \leq p-1, \\ f_1 \circ e_i = \alpha_i e_{i+1}, \quad p \leq i \leq n-p-1, \\ f_1 \circ f_i = \gamma_i e_{i+1} + \delta_i f_{i+1}, \quad 1 \leq i \leq p-1, \\ f_1 \circ f_p = \gamma_p e_{p+1}. \end{gathered} \tag{2}$$

Proposition 3.2. *Let* A *be a Zinbiel algebra of type* I. *Then, for the structural constants* α_i, β_i, γ_i *and* δ_i, *we have the following restrictions:*

$$\alpha_{i+1} = 0, \quad 1 \leq i \leq n-p-2,$$

$$\beta_{i+1} = \prod_{k=0}^{i} \frac{k+\beta_1}{k+1}, \quad 1 \leq i \leq p-2,$$

$$(i+1)\gamma_i = \beta_1 \left(2\gamma_1 + \sum_{k=2}^{i} \gamma_k\right), \quad 1 \le i \le n-p-2,$$

$$(i+\beta_1)\delta_i = \beta_1 \left(2\delta_1 + \sum_{k=2}^{i} \delta_k\right), \quad 1 \le i \le p-2.$$

Proof. First, we calculate the products $f_i \circ e_1$ and $f_2 \circ e_i$.

Consider

$$f_2 \circ e_1 = e_1 \circ (f_1 \circ e_1) + e_1 \circ (e_1 \circ f_1) = \alpha_1 e_3 + (1+\beta_1) f_3.$$

Using the chain of equalities

$$f_i \circ e_1 = (e_1 \circ f_{i-1}) \circ e_1 = e_1 \circ (f_{i-1} \circ e_1) + e_1 \circ (e_1 \circ f_{i-1}),$$

we deduce $f_i \circ e_1 = \alpha_1 e_{i+1} + (i-1+\beta_1) f_{i+1}$ for $1 \le i \le p-1$ and $f_i \circ e_1 = \alpha_1 e_{i+1}$ for $p \le i \le n-p-1$.

From the equality

$$e_i \circ f_1 = (e_1 \circ e_{i-1}) \circ f_1 = e_1 \circ (e_{i-1} \circ f_1) + e_1 \circ (f_1 \circ e_{i-1}),$$

we obtain $e_i \circ f_1 = \sum_{k=1}^{i-1} \alpha_k e_{i+1} + \sum_{k=0}^{i-1} \beta_k f_{i+1}$ for $1 \le i \le p-1$ and $e_i \circ f_1 = \sum_{k=1}^{i-1} \alpha_k e_{i+1}$ for $p \le i \le n-p-1$.

$$f_2 \circ e_2 = e_1 \circ (f_1 \circ e_2) + e_1 \circ (e_2 \circ f_1) = (\alpha_1 + \alpha_2) e_4 + (1 + \beta_1 + \beta_2) f_4.$$

From $f_2 \circ e_i = (e_1 \circ f_1) \circ e_i = e_1 \circ (f_1 \circ e_i) + e_1 \circ (e_i \circ f_1)$, we have

$$f_2 \circ e_i = \sum_{k=1}^{i} \alpha_k e_{i+2} + \sum_{k=0}^{i} \beta_k f_{i+2}, \quad 1 \le i \le p-2,$$

$$f_2 \circ e_i = \sum_{k=1}^{i} \alpha_k e_{i+2}, \quad p-1 \le i \le n-p-2.$$

Now we calculate the products $f_i \circ f_1$ and $f_2 \circ f_i$. We get

$$f_2 \circ f_1 = (e_1 \circ f_1) \circ f_1 = 2\gamma_1 e_3 + 2\delta_1 f_3,$$

$$f_3 \circ f_1 = (e_1 \circ f_2) \circ f_1 = (2\gamma_1 + \gamma_2) e_4 + (2\delta_1 + \delta_2) f_4.$$

By induction we obtain

$$f_i \circ f_1 = \left(2\gamma_1 + \sum_{k=2}^{i-1} \gamma_k\right) e_{i+1} + \left(2\delta_1 + \sum_{k=2}^{i-1} \delta_k\right) f_{i+1}, \quad 2 \le i \le p-1,$$

$$f_p \circ f_1 = \left(2\gamma_1 + \sum_{k=2}^{p-1} \gamma_k\right) e_{p+1}.$$

Similarly, from $f_2 \circ f_i = (e_1 \circ f_1) \circ f_i = e_1 \circ (f_1 \circ f_i) + e_1 \circ (f_i \circ f_1)$ we derive

$$f_2 \circ f_i = \left(2\gamma_1 + \sum_{k=2}^{i} \gamma_k\right) e_{i+2} + \left(2\delta_1 + \sum_{k=2}^{i} \delta_k\right) f_{i+2}, \quad 2 \le i \le p-2,$$

$$f_2 \circ f_i = \left(2\gamma_1 + \sum_{k=2}^{i} \gamma_k\right) e_{i+2}, \quad p-1 \le i \le p.$$

If $\beta_1 = 1$, then from the equality $(f_1 \circ f_1) \circ e_1 = (f_1 \circ e_1) \circ f_1$ we get

$$2\gamma_1(1-\beta_1) = \alpha_1^2 - \delta_1\alpha_1, \quad (1-\beta_1)\delta_1 = \alpha_1(1+\beta_1).$$

Consequently, $\alpha_1 = 0$.

Let $\beta_1 \neq 1$. Then taking the following change:

$$e_1' = e_1, \quad f_1' = \frac{\alpha_1}{\beta_1 - 1} e_1 + f_1,$$

we obtain $\alpha_1' = 0$.

Form the equalities

$$f_1 \circ e_{i+1} = f_1 \circ (e_1 \circ e_i) = (f_1 \circ e_1) \circ e_i - i f_1 \circ e_{i+1} = \beta_1 f_2 \circ e_i - i f_1 \circ e_{i+1},$$

we derive $(i+1) f_1 \circ e_{i+1} = \beta_1 f_2 \circ e_i$. Therefore,

$$(i+1)\alpha_{i+1} e_{i+2} + (i+1)\beta_{i+1} f_{i+2} = \beta_1 \left(\sum_{k=1}^{i} \alpha_k e_{i+2} + \sum_{k=0}^{i} \beta_k f_{i+2}\right).$$

Comparing coefficients at the basis elements and applying induction, we deduce

$$\alpha_{i+1} = 0, \quad 1 \le i \le n-p-2,$$

$$\beta_{i+1} = \prod_{k=0}^{i} \frac{k+\beta_1}{k+1}, \quad 1 \le i \le p-2.$$

Considering the equality $(f_1 \circ f_i) \circ e_1 = (f_1 \circ e_1) \circ f_i$ leads to the rest of the restrictions of the proposition.

In the next proposition we calculate the products $e_i \circ f_j$ and $f_j \circ e_i$.

Proposition 3.3. *Let A be a Zinbiel algebra of type* I. *Then the following expressions are true:*

$$e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j} \quad \textit{for} \quad 2 \le i+j \le p, \tag{3}$$

$$f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j} \quad \textit{for} \quad 2 \le i+j \le p, \tag{4}$$

where $\beta_0 = 1$.

Proof. We shall prove (3), (4) by induction. From (1) and (3.1) we get the correctness of (3) and (4) for $i = 1$.

Consider (4) for $j = 1$. Since $f_1 \circ e_1 = \beta_1 f_2$ and $f_2 \circ e_1 = e_1 \circ (f_1 \circ e_1) + e_1 \circ (e_1 \circ f_1) = (1+\beta_1) f_3$, then using the equalities $f_i \circ e_1 = (e_1 \circ f_{i-1}) \circ e_1 = e_1 \circ (f_{i-1} \circ e_1) + e_1 \circ (e_1 \circ f_{i-1})$ and induction, we deduce $f_i \circ e_1 = (i - 1 + \beta_1) f_{i+1}$ for $1 \le i \le p - 1$. From

$$e_i \circ f_1 = (e_1 \circ e_{i-1}) \circ f_1 = e_1 \circ (e_{i-1} \circ f_1) + e_1 \circ (f_1 \circ e_{i-1}),$$

it implies

$$e_i \circ f_1 = \sum_{k=0}^{i-1} \beta_k f_{i+1} \quad \text{for} \quad 1 \le i \le p - 1.$$

Therefore, the equalities (3) are true for $j = 1$ and arbitrary i.

Let us suppose that expressions (3), (4) are true for i and any value of j. The proof of the expressions for $i + 1$ is obtained by the following chain of equalities:

$$e_{i+1} \circ f_j = e_1 \circ (e_i \circ f_j) + e_1 \circ (f_j \circ e_i) =$$

$$= e_1 \circ \left(\sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j} + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k f_{i+j} \right) =$$

$$= \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j+1} + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k f_{i+j+1} =$$

$$= \left(\sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k \right) f_{i+j+1} = \sum_{k=0}^{i} C_{i+j-1-k}^{j-1} \beta_k f_{i+j+1}.$$

Here we used the well-known formula $C_n^{m-1} + C_n^m = C_{n+1}^m$.

The proof of expressions (4) is analogous.

Below, we clarify the restrictions on structural constants of the algebra with relation to the dimension and the parameter β_1.

Proposition 3.4. *Let A be a Zinbiel algebra of type* I. *Then the following restrictions are true:*

(1) *Case* $\dim A \ge 2p + 1$. *If* $\beta_1 \ne 1$, *then*

$$\gamma_i = 0, \quad 1 \le i \le p - 1,$$

$$\delta_i = 0, \quad 1 \le i \le p - 2,$$

$$(p - 1 + \beta_1)\gamma_p = 0,$$

$$(p - 2 + \beta_1)\delta_{p-1} = 0.$$

If $\beta_1 = 1$, *then*

$$\beta_i = 1, \quad 1 \le i \le p - 1,$$

$$\gamma_i = \gamma_1, \quad 1 \le i \le p,$$

$$\delta_i = \delta_1, \quad 1 \le i \le p - 1.$$

(2) *Case* $\dim A = 2p$. *If* $\beta_1 \neq 1$, *then*

$$\gamma_i = 0, \quad 1 \leq i \leq p-2,$$
$$\delta_i = 0, \quad 1 \leq i \leq p-2,$$
$$(p-2+\beta_1)\gamma_{p-1} = 0,$$
$$(p-2+\beta_1)\delta_{p-1} = 0.$$

If $\beta_1 = 1$, *then*

$$\beta_i = 1, \quad 1 \leq i \leq p-1,$$
$$\gamma_i = \gamma_1, \quad 1 \leq i \leq p-1,$$
$$\delta_i = \delta_1, \quad 1 \leq i \leq p-1.$$

Proof. Let $\dim A \geq 2p+1$. Then from Proposition 3.2 we have

$$\begin{aligned} (i+1)\gamma_i &= \beta_1\left(2\gamma_1 + \sum_{k=2}^{k}\gamma_k\right), \quad 1 \leq i \leq p-1, \\ (i+\beta_1)\delta_i &= \beta_1\left(2\delta_1 + \sum_{k=2}^{k}\delta_k\right), \quad 1 \leq i \leq p-2. \end{aligned} \tag{5}$$

Consider

$$(f_1 \circ f_i) \circ e_1 = f_1 \circ (f_i \circ e_1) + f_1 \circ (e_1 \circ f_i) = (i+\beta_1)(\gamma_{i+1}e_{i+2} + \delta_{i+1}f_{i+2}).$$

On the other hand,

$$(f_1 \circ f_i) \circ e_1 = (\gamma_i e_{i+1} + \delta_i f_{i+1}) \circ e_1 = (i+1)\gamma_i e_{i+2} + (i+\beta_1)\delta_i f_{i+2}.$$

Hence,

$$\begin{aligned} (i+1)\gamma_i &= (i+\beta_1)\gamma_{i+1}, \quad 1 \leq i \leq p-1, \\ (i+\beta_1)\delta_i &= (i+\beta_1)\delta_{i+1}, \quad 1 \leq i \leq p-2. \end{aligned} \tag{6}$$

Considering the cases $\beta_1 \neq 1$ and $\beta_1 = 1$ together with the expressions (5) and (6) leads to the restrictions of the case $\dim A \geq 2p+1$. The proof of the remaining case is carried out in a similar fashion.

Consider a general change of basis of the algebra A. It is known that for naturally graded Zinbiel algebras it is sufficient to take the change of basis in the form

$$e_1' = Ae_1 + Bf_1, \quad f_1' = Ce_1 + Df_1,$$

where $AD - BC \neq 0$.

Proposition 3.5. *Let A be a Zinbiel algebra of type* I *and let $\beta_1 \neq 1$. Then*

$$e'_{i+1} = e'_1 \circ e'_i, \quad 1 \leq i \leq n-p-1,$$

$$f'_{i+1} = e'_1 \circ f'_i, \quad 1 \leq i \leq p-1,$$

$$e'_i = A^i e_i + A^{i-1} B \sum_{k=0}^{i-1} \beta_k f_i, \quad 1 \leq i \leq p-1,$$

$$f'_i = A^{i-1} D f_i, \quad 1 \leq i \leq p-1,$$

$$C = 0.$$

Proof. From $f'_1 \circ f'_1 = 0$ we get $C = 0$. The proof of the proposition is completed by considering products $e'_1 \circ e'_i = e'_{i+1}$ and $e'_1 \circ f'_i = f'_{i+1}$.

Theorem 3.1. *Let A be an n-dimensional $(n \geq 2p+2)$ Zinbiel algebra of type* I *and with characteristic sequence equal to $(n-p, p)$. Then it is isomorphic to one of the following non-isomorphic algebras:*

$$A_1: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i+j \leq n-p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, & 2 \leq i+j \leq p, \end{cases}$$

where $\beta_{i+1} = \prod_{k=0}^{i} \dfrac{k+\beta_1}{k+1}$ *for* $1 \leq i \leq p-2$ *and* $\beta_1 \in \mathbb{C}$;

$$A_2: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i+j \leq n-p, \quad f_1 \circ f_{p-1} = f_p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \quad 2 \leq i+j \leq p, \end{cases}$$

where $\beta_i = (-1)^i C^i_{p-2}$ *for* $1 \leq i \leq p-2$;

$$A_3: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i+j \leq n-p, \\ e_i \circ f_j = f_i \circ e_j = f_i \circ f_j = C^j_{i+j-1} f_{i+j}, & 2 \leq i+j \leq p. \end{cases}$$

Proof. From Proposition 3.4 for $\beta_1 \neq 1$ we obtain a multiplication table of the algebra:

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i+j \leq n-p,$$

$$f_1 \circ f_i = \delta_i f_{i+1}, \quad 1 \leq i \leq p-1, \quad f_i \circ f_j = \varphi(\delta_1, \delta_2, \ldots, \delta_s) f_{i+j}, \quad 2 \leq i+j \leq p,$$

$$e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \quad 2 \leq i+j \leq p,$$

where $(p-1+\beta_1)\gamma_p = 0$, $(p-2+\beta_1)\delta_{p-1} = 0$.

Consider

$$(f_1 \circ f_p) \circ e_1 = f_1 \circ (f_p \circ e_1) + f_1 \circ (e_1 \circ f_p) = 0.$$

On the other hand,
$$(f_1 \circ f_p) \circ e_1 = \gamma_p e_{p+1} \circ e_1 = (p+1)\gamma_p e_{p+2}.$$
Therefore, we deduce $\gamma_p = 0$.

Applying Proposition 3.5 to the general change of basis, from the equalities
$$f_1' \circ f_{p-1}' = (Df_1) \circ \left(A^{p-2} D f_{p-1}\right) = A^{p-2} D^2 \delta_{p-1} f_p =$$
$$= \delta_{p-1}' f_p' = \delta_{p-1}' \left(A^{p-1} D + A^{p-2} B D \delta_{p-1}\right) f_p,$$
we get $\delta_{p-1}' = \dfrac{D\delta_{p-1}}{A + B\delta_{p-1}}$.

If $\delta_{p-1} = 0$, then $\delta_{p-1}' = 0$, and we obtain the algebra A_1.

If $\delta_{p-1} \neq 0$, then, by choosing $D = \dfrac{A + B\delta_{p-1}}{\delta_{p-1}}$ and from $(p - 2 + \beta_1)\delta_{p-1} = 0$, we have $\delta_{p-1}' = 1$, $\beta_1 = 2 - p$, that is, we have the algebra A_2.

In case of $\beta_1 = 1$ we have $\delta_i = \delta_1$, $1 \leq i \leq p - 1$. Putting $D = \dfrac{A + B\delta_1}{\delta_1}$ we obtain $\delta_1' = 1$. Consequently, $f_1 \circ f_i = f_{i+1}$ for $1 \leq i \leq p-1$. From Lemma 2.2 we deduce $f_i \circ f_j = C_{i+j-1}^{j} f_{i+j}$, $2 \leq i + j \leq p$. Thus, we get the algebra A_3.

In the following theorem the classification for $n = 2p + 1$ is presented.

Theorem 3.2. *Let A be a Zinbiel algebra of type* I *and with characteristic sequence equal to $(p+1, p)$. Then it is isomorphic to one of the following non-isomorphic algebras:*
$$A_4: \begin{cases} e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq p+1, \quad f_1 \circ f_p = e_{p+1}, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, \quad 2 \leq i+j \leq p, \end{cases}$$
where $\beta_0 = 1$, $\beta_i = (-1)^i C_{p-1}^{i}$ *for* $1 \leq i \leq p-1$;
$$A_5: \begin{cases} e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq p+1, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, \quad 2 \leq i+j \leq p, \end{cases}$$
where $\beta_0 = 1$, $\beta_{i+1} = \prod_{k=0}^{i} \dfrac{k + \beta_1}{k+1}$ *for* $1 \leq i \leq p-2$ *and* $\beta_1 \in \mathbb{C}$;
$$A_6: \begin{cases} e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, \quad 2 \leq i+j \leq p+1, \quad f_1 \circ f_{p-1} = f_p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, \quad 2 \leq i+j \leq p, \end{cases}$$
where $\beta_0 = 1$, $\beta_i = (-1)^i C_{p-2}^{i}$ *for* $1 \leq i \leq p-2$ *and* $\beta_{p-1} = 0$;
$$A_7: \begin{cases} e_i \circ e_j = C_{i+j-1}^{j} e_{i+j}, & 2 \leq i+j \leq p+1, \\ e_i \circ f_j = f_i \circ e_j = C_{i+j-1}^{j} f_{i+j}, & 2 \leq i+j \leq p, \\ f_i \circ f_j = \gamma_1 C_{i+j-1}^{j} e_{i+j} + \delta_1 C_{i+j-1}^{j} f_{i+j}, & 2 \leq i+j \leq p, \\ f_i \circ f_j = \gamma_1 C_{i+j-1}^{j} e_{p+1}, & i+j = p+1, \end{cases}$$
where γ_1, $\delta_1 \in \mathbb{C}$.

Proof. From Proposition 3.4 for $\beta_1 \neq 1$ we obtain a multiplication table of A

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \le i+j \le p+1,$$

$$f_1 \circ f_{p-1} = \delta_{p-1} f_p, \quad f_1 \circ f_p = \gamma_p e_{p+1},$$

$$e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \quad 2 \le i+j \le p,$$

where $\beta_0 = 1$, $(p-1+\beta_1)\gamma_p = 0$ and $(p-2+\beta_1)\delta_{p-1} = 0$.

Consider the general change of basis as above. Then from Proposition 3.5 we have

$$e'_p = A^p e_p + \left(A^{p-1} B \sum_{k=0}^{p-1} \beta_k + A^{p-2} B^2 \sum_{k=0}^{p-2} \beta_k \delta_{p-1} \right) f_p,$$

$$e'_{p+1} = \left(A^{p+1} + \left(A^{p-1} B^2 \sum_{k=0}^{p-1} \beta_k + A^{p-2} B^3 \sum_{k=0}^{p-2} \beta_k \delta_{p-1} \right) \gamma_p \right) e_{p+1},$$

$$f'_p = \left(A^{p-1} D + A^{p-2} B D \delta_{p-1} \right) f_p.$$

The equality $e'_1 \circ f'_p = 0$ in the new basis implies $B\gamma_p = 0$.

Case 1. Let $\gamma_p \neq 0$. Then $B = 0$ and $\beta_1 = 1 - p$, $\delta_{p-1} = 0$.

Considering the equality $f'_1 \circ f'_p = \gamma'_p e'_{p+1}$, we derive $A^2 \gamma'_p = D^2 \gamma_p$.

Setting $D = \dfrac{A}{\sqrt{\gamma_p}}$, we obtain $\gamma'_p = 1$. Thus, we get the algebra A_4.

Case 2. Let $\gamma_p = 0$. Then, considering the equality $f'_1 \circ f'_{p-1} = \delta'_{p-1} f'_p$, we deduce $\delta'_{p-1} = \dfrac{D\delta_{p-1}}{A + B\delta_{p-1}}$.

If $\delta_{p-1} = 0$, then $\delta'_{p-1} = 0$, that is, we obtain the algebra A_5.

If $\delta_{p-1} \neq 0$, then $\beta_1 = 2 - p$ and putting $D = \dfrac{A + B\delta_{p-1}}{\delta_{p-1}}$, we get $\delta'_{p-1} = 1$ and the algebra A_6.

Now we consider case $\beta_1 = 1$. Using Proposition 3.4, we obtain the algebra A_7.

Below, we present the classification of Zinbiel algebras with characteristic sequence equal to $C(A) = (p, p)$.

Theorem 3.3. *Let A be a Zinbiel algebra with characteristic sequence (p, p). Then it is isomorphic to one of the following non-isomorphic algebras:*

$$A_8: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \le i+j \le p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, & 2 \le i+j \le p, \end{cases}$$

where $\beta_0 = 1$, $\beta_{i+1} = \prod_{k=0}^{i} \dfrac{k+\beta_1}{k+1}$ *for* $1 \le i \le p-2$ *and* $\beta_1 \in \mathbb{C}$;

$$A_9: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \le i+j \le p, \quad f_1 \circ f_{p-1} = f_p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \quad 2 \le i+j \le p, \end{cases}$$

where $\beta_0 = 1,\ \beta_i = (-1)^i C_{p-2}^i$ *for* $1 \le i \le p-2$ *and* $\beta_{p-1} = 0$;

$$A_{10}: \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \ \ 2 \le i+j \le p, \ \ f_1 \circ f_{p-1} = e_p + \delta_{p-1} f_p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j}, \ \ f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, \ \ 2 \le i+j \le p, \end{cases}$$

where $\beta_0 = 1,\ \beta_i = (-1)^i C_{p-2}^i$ *for* $1 \le i \le p-2,\ \beta_{p-1} = 0$ *and* $\delta_{p-1} \in \mathbb{C}$;

$$A_{11}: e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad e_i \circ f_j = f_i \circ e_j = C_{i+j-1}^j f_{i+j}, \quad 2 \le i+j \le p,$$

$$A_{12}: \begin{cases} e_i \circ e_j = C_{i+j-1}^j e_{i+j}, & 2 \le i+j \le p, \\ e_i \circ f_j = f_i \circ e_j = f_i \circ f_j = C_{i+j-1}^j f_{i+j}, & 2 \le i+j \le p. \end{cases}$$

Proof. The proof of this theorem is carried out by applying the methods and arguments as in the proof of Theorems 3.1 and 3.2.

***3.2. Classification of Zinbiel algebras of type* II.** Consider a Zinbiel algebra of type II. From the condition on the operator L_{e_1} we have the existence of a basis $\{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_{n-p}\}$ such that the products involving e_1 on the left-hand side have the form

$$e_1 \circ e_i = e_{i+1}, \quad 1 \le i \le p-1.$$

Applying Lemma 2.2, we get

$$e_i \circ e_j = C_{i+j-1}^j e_{i+j}, \quad 2 \le i+j \le p, \quad e_1 \circ e_p = 0,$$

$$e_1 \circ f_i = f_{i+1}, \quad 1 \le i \le n-p-1, \quad e_1 \circ f_{n-p} = 0.$$

It is easy to see that

$$A_1 = \langle e_1, f_1 \rangle, A_2 = \langle e_2, f_2 \rangle, \ldots, A_p = \langle e_p, f_p \rangle, A_{p+1} = \langle f_{p+1} \rangle, \ldots, A_{n-p} = \langle f_{n-p} \rangle.$$

Let us introduce notations:

$$f_1 \circ e_i = \alpha_i e_{i+1} + \beta_i f_{i+1}, \quad 1 \le i \le p-1, \quad f_1 \circ e_p = \beta_p f_{p+1},$$

$$f_1 \circ f_i = \gamma_i e_{i+1} + \delta_i f_{i+1}, \quad 1 \le i \le p-1,$$

$$f_1 \circ f_i = \delta_i f_{i+1}, \quad p \le i \le n-p-1, \quad f_1 \circ f_{n-p} = 0.$$

The following proposition can be proved similar to Proposition 3.2.

Proposition 3.6. *Let A be a Zinbiel algebra of type* II. *Then for structural constants $\alpha_i,\ \beta_i,\ \gamma_i$ and δ_i the following restrictions hold:*

$$\alpha_{i+1} = 0, \quad 1 \le i \le p-2,$$

$$\beta_{i+1} = \prod_{k=0}^{i} \frac{k+\beta_1}{k+1}, \quad 1 \le i \le p-1,$$

$$(i+1)\gamma_i = \beta_1 \left(2\gamma_1 + \sum_{k=2}^{i} \gamma_k\right), \quad 1 \le i \le p-2,$$

$$(i+\beta_1)\delta_i = \beta_1 \left(2\delta_1 + \sum_{k=2}^{i} \delta_k\right), \quad 1 \le i \le n-p-2.$$

Proposition 3.7. *Let A be a Zinbiel algebra of type* II. *Then the following expressions hold:*

$$e_i \circ f_j = \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j}, \quad 1 \le i \le p, \quad p+1 \le i+j \le n-p, \tag{7}$$

$$f_i \circ e_j = \sum_{k=0}^{j} C_{i+j-2-k}^{i-2} \beta_k f_{i+j}, \quad 1 \le j \le p, \quad p+1 \le i+j \le n-p, \tag{8}$$

where $\beta_0 = 1$.

Proof. We shall prove the assertion of the proposition be induction. Clearly, the relation (7) is true for $i = 1$.

We have

$$f_2 \circ e_1 = e_1 \circ (f_1 \circ e_1) + e_1 \circ (e_1 \circ f_1) = (1+\beta_1) f_3.$$

Using the chain of equalities

$$f_i \circ e_1 = (e_1 \circ f_{i-1}) \circ e_1 = e_1 \circ (f_{i-1} \circ e_1) + e_1 \circ (e_1 \circ f_{i-1}),$$

and induction, we derive $f_i \circ e_1 = (i-1+\beta_1) f_{i+1}$ for $1 \le i \le p-1$ and $f_i \circ e_1 = (i-1+\beta_1) f_{i+1}$ for $p \le i \le n-p-1$. Therefore, the relation (8) is true for $j = 1$.

Let us assume that the relations (7), (8) are true for i and any value of j. The proof of these relations for $i+1$ follows from the following chain of equalities:

$$e_{i+1} \circ f_j = e_1 \circ (e_i \circ f_j) + e_1 \circ (f_j \circ e_i) =$$

$$= e_1 \circ \left(\sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j} + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k f_{i+j}\right) =$$

$$= \sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k f_{i+j+1} + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k f_{i+j+1} =$$

$$= \left(\sum_{k=0}^{i-1} C_{i+j-2-k}^{j-1} \beta_k + \sum_{k=0}^{i} C_{i+j-2-k}^{j-2} \beta_k\right) f_{i+j+1} = \sum_{k=0}^{i} C_{i+j-1-k}^{j-1} \beta_k f_{i+j+1}.$$

Checking the correctness of the remaining relations of the proposition is analogous.

Similar to the case of Zinbiel algebra of type I for algebras of type II we obtain the restrictions on structure constants with relation to parameter β_1.

Proposition 3.8. *Let A be a Zinbiel algebra of type* II.

If $\beta_1 \neq 1$, then

$$\gamma_i = 0, \quad 1 \leq i \leq p-2,$$

$$\delta_i = 0, \quad 1 \leq i \leq n-p-2,$$

$$(p-2+\beta_1)\gamma_{p-1} = 0,$$

$$(n-p-2+\beta_1)\delta_{n-p-1} = 0;$$

if $\beta_1 = 1$, *then*

$$\beta_i = 1, \quad 1 \leq i \leq p,$$

$$\gamma_i = \gamma_1, \quad 1 \leq i \leq p-1,$$

$$\delta_i = \delta_1, \quad 1 \leq i \leq n-p-1.$$

In the next theorem we prove that there is no n-dimensional Zinbiel algebras of type II with $n \geq 3p+2$.

Theorem 3.4. *There is no Zinbiel algebras of type* II *with characteristic sequence equal to* $(n-p, p)$ *for* $n \geq 3p+2$.

Proof. Consider for $1 \leq i \leq p+1$ equalities

$$0 = (e_1 \circ e_p) \circ f_i = e_1 \circ (e_p \circ f_i) + e_1 \circ (f_i \circ e_p).$$

Applying the relations (7), (8) and arguments similar to the ones that are used in the proof of Proposition 3.1, we derive the relation

$$\sum_{k=0}^{p} C_{p+i-1-k}^{i-1} \beta_k = 0, \tag{9}$$

where $\beta_0 = 1$ and $1 \leq i \leq n-p-1$.

Now we consider the determinant of the matrix of order $p+1$:

$$M = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ C_{p+1}^1 & C_p^1 & C_{p-1}^1 & \dots & C_2^1 & 1 \\ C_{p+2}^2 & C_{p+1}^2 & C_p^2 & \dots & C_3^2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_{2p-1}^{p-1} & C_{2p-2}^{p-1} & C_{2p-3}^{p-1} & \dots & C_p^{p-1} & 1 \\ C_{2p}^p & C_{2p-1}^p & C_{2p-2}^p & \dots & C_{p+1}^p & 1 \end{vmatrix}.$$

Taking into account identity $C_n^{m-1} + C_n^m = C_{n+1}^m$ and subtracting from each row the previous one we obtain

$$M = \begin{vmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ C_p^1 & C_{p-1}^1 & C_{p-2}^1 & \dots & 1 & 0 \\ C_p^2 & C_{p-1}^2 & C_{p-2}^2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_p^{p-1} & 1 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{vmatrix} = -1.$$

Since $M = -1$, the system of equations (9) for $i = p+1$ has only trivial solution with respect to unknown variables β_i. In particular, $\beta_0 = 0$. However, $\beta_0 = 1$, that is, we get a contradiction to the condition $i = p+1 \leq n-p-1$, which implies the non existence of an algebra under the condition $n \geq 3p+2$.

Let A be an n-dimensional algebra with a basis $\{e_1, e_2, \ldots, e_p, f_1, f_2, \ldots, f_{n-p}\}$ and the multiplication table

$$e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \quad 2 \leq i+j \leq p, \quad e_1 \circ e_p = 0,$$

$$e_1 \circ f_{n-p} = 0, \quad f_i \circ f_j = 0, \quad 1 \leq i, \quad j \leq n-p,$$

$$e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \quad f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \quad 2 \leq i+j \leq n-p,$$

where $\beta_i = (-1)^i C^i_p$, $0 \leq i \leq p$.

It is easy to check that this algebra is a Zinbiel algebra.

The correctness of the relation (9) for parameters β_i for $n = 3p+1$ follows from Lemma 2.1. Thus, the condition $n \geq 3p+2$ is necessary.

We list the next theorems on the description of Zinbiel algebras of type II without proofs. It can be carried out by applying similar arguments that were used above.

Theorem 3.5. *A Zinbiel algebra of type* II *with characteristic sequence equal to* $(p+1, p)$ *is isomorphic to one of the following non-isomorphic algebras:*

$$\widetilde{A_1}: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \ \ 2 \leq i+j \leq p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \ \ f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \ \ 2 \leq i+j \leq p+1, \end{cases}$$

where $\beta_0 = 1$, $\beta_{i+1} = \prod_{k=0}^{i} \dfrac{k+\beta_1}{k+1}$ *for* $1 \leq i \leq p-1$, $\beta_1 \in \{-p, -(p-1), \ldots, -2, -1\}$;

$$\widetilde{A_2}: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \ \ 2 \leq i+j \leq p, \ \ f_1 \circ f_{p-1} = e_p, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \ \ f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \ \ 2 \leq i+j \leq p+1, \end{cases}$$

where $\beta_i = (-1)^i C^i_{p-2}$ *for* $0 \leq i \leq p-2$ $\beta_{p-1} = \beta_p = 0$;

$$\widetilde{A_3}: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, \ \ 2 \leq i+j \leq p, \ \ f_1 \circ f_p = f_{p+1}, \\ e_i \circ f_j = \sum_{k=0}^{i-1} C^{j-1}_{i+j-2-k} \beta_k f_{i+j}, \ \ f_i \circ e_j = \sum_{k=0}^{j} C^{i-2}_{i+j-2-k} \beta_k f_{i+j}, \ \ 2 \leq i+j \leq p+1, \end{cases}$$

where $\beta_i = (-1)^i C^i_{p-1}$ *for* $0 \leq i \leq p-1$ *and* $\beta_p = 0$;

$$\widetilde{A_4}: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i+j \leq p, \\ e_i \circ f_j = f_i \circ e_j = C^j_{i+j-1} f_{i+j}, & 2 \leq i+j \leq p+1, \end{cases}$$

$$\widetilde{A_5}: \begin{cases} e_i \circ e_j = C^j_{i+j-1} e_{i+j}, & 2 \leq i+j \leq p, \\ e_i \circ f_j = f_i \circ e_j = C^j_{i+j-1} f_{i+j}, \ \ f_i \circ f_j = C^j_{i+j-1} f_{i+j}, & 2 \leq i+j \leq p+1. \end{cases}$$

Theorem 3.6. *A Zinbiel algebra of type* II *with characteristic sequence equal to* $(p+2,p)$ *is isomorphic to one of the following non-isomorphic algebras:*

$$\widetilde{A_6}: \begin{cases} e_i\circ e_j = C^j_{i+j-1}e_{i+j}, & 2\le i+j\le p,\\ e_i\circ f_j=\sum_{k=0}^{i-1}C^{j-1}_{i+j-2-k}\beta_k f_{i+j}, & 1\le i\le p,\ \ 2\le i+j\le p+2,\\ f_i\circ e_j=\sum_{k=0}^{j}C^{i-2}_{i+j-2-k}\beta_k f_{i+j}, & 1\le j\le p,\ \ 2\le i+j\le p+2,\end{cases}$$

where $\beta_0=1,\ \beta_{i+1}=\prod_{k=0}^{i}\dfrac{k+\beta_1}{k+1}$ *for* $1\le i\le p-1,\ \beta_1\in\{-p,-(p-1),\ldots,-2,-1\}$;

$$\widetilde{A_7}: \begin{cases} e_i\circ e_j = C^j_{i+j-1}e_{i+j}, & 2\le i+j\le p,\quad f_1\circ f_{p-1}=e_p,\\ e_i\circ f_j=\sum_{k=0}^{i-1}C^{j-1}_{i+j-2-k}\beta_k f_{i+j}, & 1\le i\le p,\ \ 2\le i+j\le p+2,\\ f_i\circ e_j=\sum_{k=0}^{j}C^{i-2}_{i+j-2-k}\beta_k f_{i+j}, & 1\le j\le p,\ \ 2\le i+j\le p+2,\end{cases}$$

where $\beta_i=(-1)^iC^i_{p-2}$ *for* $0\le i\le p-2,\ \beta_{p-1}=\beta_p=0$;

$$\widetilde{A_8}: \begin{cases} e_i\circ e_j = C^j_{i+j-1}e_{i+j}, & 2\le i+j\le p,\quad f_1\circ f_{p+1}=f_{p+2},\\ e_i\circ f_j=\sum_{k=0}^{i-1}C^{j-1}_{i+j-2-k}\beta_k f_{i+j}, & 1\le i\le p,\ \ 2\le i+j\le p+2,\\ f_i\circ e_j=\sum_{k=0}^{j}C^{i-2}_{i+j-2-k}\beta_k f_{i+j}, & 1\le j\le p,\ \ 2\le i+j\le p+2,\end{cases}$$

where $\beta_i=(-1)^iC^i_p$ *for* $0\le i\le p$.

Theorem 3.7. *A Zinbiel algebra of type* II *with characteristic sequence equal to* $(p+t,p)$, *for* $3\le t\le p+1$, *is isomorphic to one of the following non-isomorphic algebras:*

$$\widetilde{A_9}: \begin{cases} e_i\circ e_j = C^j_{i+j-1}e_{i+j}, & 2\le i+j\le p,\\ e_i\circ f_j=\sum_{k=0}^{i-1}C^{j-1}_{i+j-2-k}\beta_k f_{i+j}, & 1\le i\le p,\ \ 2\le i+j\le p+t,\\ f_i\circ e_j=\sum_{k=0}^{j}C^{i-2}_{i+j-2-k}\beta_k f_{i+j}, & 1\le j\le p,\ \ 2\le i+j\le p+t,\end{cases}$$

where $\beta_0=1,\ \beta_{i+1}=\prod_{k=0}^{i}\dfrac{k+\beta_1}{k+1}$ *for* $1\le i\le p-1,\ \beta_1\in\{-p,-(p-1),\ldots,-(t-1)\}$;

$$\widetilde{A_{10}}: \begin{cases} e_i\circ e_j = C^j_{i+j-1}e_{i+j}, & 2\le i+j\le p,\quad f_1\circ f_{p-1}=e_p,\\ e_i\circ f_j=\sum_{k=0}^{i-1}C^{j-1}_{i+j-2-k}\beta_k f_{i+j}, & 1\le i\le p,\ \ 2\le i+j\le p+t,\\ f_i\circ e_j=\sum_{k=0}^{j}C^{i-2}_{i+j-2-k}\beta_k f_{i+j}, & 1\le j\le p,\ \ 2\le i+j\le p+t,\end{cases}$$

where $\beta_i=(-1)^iC^i_{p-2}$ *for* $0\le i\le p-2$ *and* $\beta_{p-1}=\beta_p=0$.

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Received 15.07.16