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APPLICATION OF THE INFINITE MATRIX THEORY TO THE SOLVABILITY OF SEQUENCE SPACES INCLUSION EQUATIONS WITH OPERATORS

ЗАСТОСУВАННЯ ТЕОРІЇ НЕСКІНЧЕННИХ МАТРИЦЬ ДО РОЗВ'ЯЗАННЯ ВІДНОШЕНЬ ВКЛЮЧЕННЯ ДЛЯ ПРОСТОРІВ ПОСЛІДОВНОСТЕЙ З ОПЕРАТОРАМИ

Given any sequence $a=(a_n)_{n\geq 1}$ of positive real numbers and any set E of complex sequences, we write E_a for the set of all sequences $y=(y_n)_{n\geq 1}$ such that $y/a=(y_n/a_n)_{n\geq 1}\in E$. In particular, c_a denotes the set of all sequences y such that y/a converges. We deal with sequence spaces inclusion equations (SSIE) of the form $F\subset E_a+F'_x$ with $e\in F$ and explicitly find the solutions of these SSIE when $a=(r^n)_{n\geq 1}$, F is either c or s_1, and E, F' are any sets c_0, c, s_1, ℓ_p, w_0, and w_∞. Then we determine the sets of all positive sequences satisfying each of the SSIE $c\subset D_r*(c_0)_\Delta+c_x$ and $c\subset D_r*(s_1)_\Delta+c_x$, where Δ is the operator of the first difference defined by $\Delta_n y=y_n-y_{n-1}$ for all $n\geq 1$ with $y_0=0$. Then we solve the SSIE $c\subset D_r*E_{C_1}+s_x^{(c)}$ with $E\in\{c,s_1\}$ and $s_1\subset D_r*(s_1)_{C_1}+s_x$, where C_1 is the Cesàro operator defined by $(C_1)_n y=n^{-1}\sum_{k=1}^{n} y_k$ for all y. We also deal with the solvability of the sequence spaces equations (SSE) associated with the previous SSIE and defined as $D_r*E_{C_1}+s_x^{(c)}=c$ with $E\in\{c_0,c,s_1\}$ and $D_r*E_{C_1}+s_x=s_1$ with $E\in\{c,s_1\}$.

Для заданої послідовності додатних дійсних чисел $a=(a_n)_{n\geq 1}$ і будь-якої множини комплексних послідовностей E вираз E_a позначає множину всіх послідовностей $y=(y_n)_{n\geq 1}$ таких, що $y/a=(y_n/a_n)_{n\geq 1}\in E$. Зокрема, c_a позначає множину всіх послідовностей y таких, що y/a збігається. Розглянуто відношення включення для просторів послідовностей (ВВПП) вигляду $F\subset E_a+F'_x$ з $e\in F$, а також знайдено явні розв'язки цих ВВПП у випадку, коли $a=(r^n)_{n\geq 1}$, F — це c або s_1, а E і F' — будь-які з множин c_0, c, s_1, ℓ_p, w_0 і w_∞. Крім того, визначено множини всіх додатних послідовностей, що задовольняють кожне з ВВПП $c\subset D_r*(c_0)_\Delta+c_x$ і $c\subset D_r*(s_1)_\Delta+c_x$, де Δ — оператор першої різниці, визначений як $\Delta_n y=y_n-y_{n-1}$ для всіх $n\geq 1$ з $y_0=0$. Також розв'язано ВВПП $c\subset D_r*E_{C_1}+s_x^{(c)}$, де $E\in\{c,s_1\}$, $s_1\subset D_r*(s_1)_{C_1}+s_x$, а C_1 — оператор Чезаро, визначений як сума $(C_1)_n y=n^{-1}\sum_{k=1}^{n} y_k$ для всіх y. Крім того, розглянуто питання про існування розв'язків рівнянь для просторів послідовностей (РПП), пов'язаних із попередніми ВВПП і визначених таким чином: $D_r*E_{C_1}+s_x^{(c)}=c$ з $E\in\{c_0,c,s_1\}$ і $D_r*E_{C_1}+s_x=s_1$ з $E\in\{c,s_1\}$.

1. Introduction. We write ω for the set of all complex sequences $y=(y_n)_{n\geq 1}$, ℓ_∞, c and c_0 for the sets of all bounded, convergent and null sequences, respectively, also $\ell_p=\Big\{y\in\omega\colon \sum_{n=1}^{\infty}|y_n|^p<\infty\Big\}$ for $1\leq p<\infty$. If $y,\ z\in\omega$, then we write $yz=(y_nz_n)_{n\geq 1}$. Let $U=\big\{y\in\omega\colon y_n\neq 0\big\}$ and $U^+=\big\{y\in\omega\colon y_n>0\big\}$. We write $z/u=(z_n/u_n)_{n\geq 1}$ for all $z\in\omega$ and all $u\in U$, in particular $1/u=e/u$, where e is the sequence with $e_n=1$ for all n. Finally, if $a\in U^+$ and E is any subset of ω, then we put $E_a=(1/a)^{-1}*E=\big\{y\in\omega\colon y/a\in E\big\}$. Let E and F be subsets of ω. In [1], the sets s_a, s_a^0 and $s_a^{(c)}$ were defined for positive sequences a by $(1/a)^{-1}*E$ and $E=\ell_\infty$, c_0, c, respectively. In [2], the sum E_a+F_b and the product E_a*F_b were defined, where E, F are any of the symbols s, s^0 or $s^{(c)}$. Then in [5] the solvability was determined of sequences spaces inclusion equations $G_b\subset E_a+F_x$, where $E,\ F,\ G\in\big\{s^0,s^{(c)},s\big\}$ and some applications were given to sequence spaces inclusions with operators. Recall that the spaces w_∞ and w_0 of strongly bounded and summable sequences are the sets of all y such that $\Big(n^{-1}\sum_{k=1}^{n}|y_k|\Big)_n$ is bounded and tends to zero, respectively. These spaces were studied by Maddox [21] and Malkowsky, Rakočević [20]. In

[9, 12], were given some properties of well known operators definedby the sets $W_a = (1/a)^{-1} * w_\infty$ and $W_a^0 = (1/a)^{-1} * w_0$. In this paper, we deal with special *sequence spaces inclusion equations* (*SSIE*) (*resp. sequence spaces equations* (*SSE*)), which are determined by an inclusion (resp. identity), for which each term is a *sum* or *a sum of products of sets of the form* $(E_a)_T$ *and* $\left(E_{f(x)}\right)_T$, where f maps U^+ to itself, E is any linear space of sequences and T is a triangle. Some results on SSE and SSIE were stated in [3–6, 8, 13, 14, 16, 17]. In [5], we dealt with the SSIE with operators $E_a + (F_x)_\Delta \subset s_x^{(c)}$, where E and F are any of the sets c_0, c or s_1. Then we gave a resolution of the next inclusion equations with operator $s_x^{(c)} + \left(s_b^0\right)_\Delta \subset s_b$ and $s_x^0 + \left(s_b^0\right)_\Delta \subset s_b^{(c)}$. Note that the SSIE $s_x^{(c)} + (s_b^0)_\Delta \subset s_b$ means $y_n/x_n \to l$ and $(z_n - z_{n-1})/b_n \to 0$ $n \to \infty$, together imply $|y_n + z_n| \le K b_n$ for all y, $z \in \omega$ and for some scalars l and K with $K > 0$. In [13], we determined the set of all positive sequences x for which the *SSIE* $\left(s_x^{(c)}\right)_{B(r,s)} \subset \left(s_x^{(c)}\right)_{B(r',s')}$ holds, where r, r', s', and s are real numbers, and $B(r,s)$ is the generalized operator of the first difference defined by $(B(r,s)y)_n = r y_n + s y_{n-1}$ for all $n \ge 2$ and $(B(r,s)y)_1 = r y_1$. In this way we determined the set of all positive sequences x for which $\big(r y_n + s y_{n-1}\big)/x_n \to l$ implies $\big(r' y_n + s' y_{n-1}\big)/x_n \to l$ $(n \to \infty)$ for all y and for some scalar l. In the paper [8], we used the sets of analytic and entire sequences denoted by $\mathbf{\Lambda}$ and $\mathbf{\Gamma}$ and defined by $\sup_{n\ge1}\left(|y_n|^{1/n}\right) < \infty$ and $\lim_{n\to\infty}\left(|y_n|^{1/n}\right) = 0$, respectively. Then we dealt with a class of SSE with operators of the form $E_T + F_x = F_b$, where T is either Δ or Σ and E is any of the sets c_0, c, ℓ_∞, ℓ_p $(p \ge 1)$, w_0, $\mathbf{\Gamma}$ or $\mathbf{\Lambda}$ and $F = c$, ℓ_∞ or $\mathbf{\Lambda}$. In [11], we solved the SSE defined by $(E_a)_\Delta + s_x^{(c)} = s_b^{(c)}$, where E is either c_0 or ℓ_p, and the SSE $(E_a)_\Delta + s_x^0 = s_b^0$, where E is either c or ℓ_∞. Finally, in [10], we dealt with the SSIE defined by $F \subset E_a + F'_x$, where a is positive sequence and E, F, and F' are linear subspaces of ω and we solved the SSE $E_r + (\ell_p)_x = (\ell_p)_b$.

In this paper, we deal with the SSIE of the form $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences a is a positive sequence with $e \in F$. We obtain a solvability of these SSIE for $a = (r^n)_{n\ge1}$. Throughout this paper we consider the SSIE $F \subset E_a + F'_x$ as a perturbed inclusion equation of the elementary inclusion equation $F \subset F'_x$. In this way it is interesting to determine the set of all positive sequences a for which the elementary and the perturbed inclusions equations have the same solutions. Then writing D_r for the diagonal matrix with $(D_r)_{nn} = r^n$, we study the solvability of the SSIE using the operator of the first difference Δ, defined by $c \subset D_r * E_\Delta + c_x$ with $E = c_0$ or s_1. Then we consider the SSIE $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E = c_0$, c or s_1 and $s_1 \subset D_r * (s_1)_{C_1} + s_x$ with $E = c$ or s_1, where C_1 is the Cesàro operator defined by $(C_1)_n y = \left(\sum_{k=1}^n y_k\right)/n$.

This paper is organized as follows. In Section 2, we recall some well-known results on sequence spaces and matrix transformations. In Section 3, we recall some results on the multipliers and on some characterizations of matrix transformations. In Section 4, we give some general results on the SSIE $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences and $e \in F$. In Section 5, we study the solvability of the SSIE of the form $F \subset E_a + F'_x$, where $a = (r^n)_{n\ge1}$. In Section 6, we deal with the SSIE of the form $F \subset E_a + F_x$ and we explicitly calculate the solutions of the SSIE of the form $F \subset E_a + F_x$, where $a = (r^n)_{n\ge1}$. Finally, in Section 7, we study some SSIE with operators of the form $F \subset (E_T)_r + F_x$, where T is a either Δ or C_1, and we solve the SSE of the form $(E_{C_1})_r + F_x = F$.

2. Preliminaries and notations. An FK space is a *complete linear metric space*, for which convergence implies *coordinatewise convergence*. A *BK space* is a Banach space of sequences that is an *FK space*. A BK space E *is said to have AK if for every sequence* $y=(y_k)_{k\geq 1}\in E$, *then* $y=\lim_{p\to\infty}\sum_{k=1}^{p}y_k e^{(k)}$, *where* $e^{(k)}=(0,\ldots,0,1,0,\ldots)$, 1 *being in the* k th *position*.

Let $\mathbb{R}$ be the set of all real numbers. For any given infinite matrix $A=(\mathbf{a}_{nk})_{n,k\geq 1}$ we define the operators $A_n=(\mathbf{a}_{nk})_{k\geq 1}$ for any integer $n\geq 1$, by $A_n y=\sum_{k=1}^{\infty}\mathbf{a}_{nk}y_k$, where $y=(y_k)_{k\geq 1}$, and the series are assumed convergent for all n. So we are led to the study of the operator A defined by $Ay=(A_n y)_{n\geq 1}$ mapping between sequence spaces. When A *maps* E *into* F, *where* E *and* F *are subsets of* ω, *we write* $A\in(E,F)$ (cf. [21, 22]). It is well known that if E has AK, then the set $\mathcal{B}(E)$ of all *bounded linear operators* L mapping in E, with norm $\|L\|=\sup_{y\neq 0}\left(\|L(y)\|_E/\|y\|_E\right)$ satisfies the identity $\mathcal{B}(E)=(E,E)$. We write ℓ_p for the set of all p-absolutely convergent series with $p\geq 1$, that is, $\ell_p=\left\{y\in\omega\colon\ \sum_{k=1}^{\infty}|y_k|^p<\infty\right\}$. For any subset F of ω, we write $F_A=\big\{y\in\omega\colon Ay\in F\big\}$ for the matrix domain of A in F. Then for any given sequence $u=(u_n)_{n\geq 1}\in\omega$ we define the diagonal matrix D_u by $[D_u]_{nn}=u_n$ for all n. It is interesting to rewrite the set E_u using a diagonal matrix. Let E be any subset of ω and $u\in U^+$, then we have $E_u=D_u*E=\big\{y=(y_n)_n\in\omega\colon y/u\in E\big\}$. We use the sets s_a^0, $s_a^{(c)}$, s_a, and $(\ell_p)_a$ defined as follows (cf. [1]). For given $a\in U^+$ and $p\geq 1$ we put $D_a*c_0=s_a^0$, $D_a*c=s_a^{(c)}$, $D_a*\ell_\infty=s_a$, and $D_a*\ell_p=(\ell_p)_a$. We will frequently write c_a instead of $s_a^{(c)}$ to simplify. Each of the spaces D_a*E, where $E\in\{c_0,c,\ell_\infty\}$ is a *BK space normed* by $\|y\|_{s_a}=\sup_n\left(|y_n|/a_n\right)$ and s_a^0 *has AK*. The set ℓ_p, $p\geq 1$, normed by $\|y\|_{\ell_p}=\left(\sum_{k=1}^{\infty}|y_k|^p\right)^{1/p}$ is a BK space with AK. If $a=(R^n)_{n\geq 1}$ with $R>0$, we write s_R, s_R^0, $s_R^{(c)}$ (or c_R), and $(\ell_p)_R$ for the sets s_a, s_a^0, $s_a^{(c)}$, and $(\ell_p)_a$, respectively. We also write D_R for $D_{(r^n)_{n\geq 1}}$. When $R=1$, we obtain $s_1=\ell_\infty$, $s_1^0=c_0$, and $s_1^{(c)}=c$. Recall that $S_1=(s_1,s_1)$ is a Banach algebra and $(c_0,s_1)=(c,s_1)=(s_1,s_1)=S_1$. We have $A\in S_1$ if and only if

$$\sup_n\left(\sum_{k=1}^{\infty}|\mathbf{a}_{nk}|\right)<\infty. \tag{1}$$

We also use the characterizations of the classes (c_0,c_0), (c_0,c), (c,c_0), (c,c), (s_1,c), and (ℓ_p,F), where $F=c_0$, c or ℓ_∞. In this way we state the next well-known results.

Lemma 1 ([20, p. 160], Theorem 1.36, [21])**.** (i) $A\in(c_0,c_0)$ *if and only if* (1) *holds and* $\lim_{n\to\infty}\mathbf{a}_{nk}=0$ *for all* k.

(ii) $A\in(c_0,c)$ *if and only if* (1) *holds and*

$$\lim_{n\to\infty}\mathbf{a}_{nk}=l_k \quad \textit{for all} \quad k \quad \textit{and for some scalar} \quad l_k. \tag{2}$$

(iii) $A\in(c,c_0)$ *if and only if* (1) *holds and* $\lim_{n\to\infty}\sum_{k=1}^{\infty}\mathbf{a}_{nk}=0$ *and* $\lim_{n\to\infty}\mathbf{a}_{nk}=0$ *for all* k.

(iv) $A\in(c,c)$ *if and only if* (1) *holds and* $\lim_{n\to\infty}\sum_{k=1}^{\infty}\mathbf{a}_{nk}=l$ *for some scalar* l.

(v) $A\in(s_1,c)$ *if and only if* (2) *holds and* $\lim_{n\to\infty}\sum_{k=1}^{\infty}|\mathbf{a}_{nk}|=\sum_{k=1}^{\infty}|l_k|$.

Characterization of (ℓ_p,F), where $F=c_0$, c or ℓ_∞. For this, we let $q=p/(p-1)$ for $p>1$. By using the notations of [20], we define $\mathcal{M}(\ell_p,\ell_\infty)=\sup_n\big(|\mathbf{a}_{nk}|\big)$ if $p=1$ and $\mathcal{M}(\ell_p,\ell_\infty)=\sup_n\left(\sum_{k=1}^{\infty}|\mathbf{a}_{nk}|^q\right)$ if $p>1$.

Lemma 2 ([20, p. 161], Theorem 1.37)**.** *Let* $p \geq 1$. *Then we have*:

(i) $A \in (\ell_p, \ell_\infty)$ *if and only if*

$$\mathcal{M}(\ell_p, \ell_\infty) < \infty. \tag{3}$$

(ii) $A \in (\ell_p, c_0)$ *if and only if the condition in* (3) *holds and* $\lim_{n\to\infty} \mathbf{a}_{nk} = 0$ *for all* k.

(iii) $A \in (\ell_p, c)$ *if and only if the conditions in* (3) *and* (2) *hold.*

We also use the well known properties, stated as follows.

Lemma 3. *Let* $a,\ b \in U^+$ *and let* $E,\ F \subset \omega$ *be any linear spaces. We have* $A \in (E_a, F_b)$ *if and only if* $D_{1/b} A D_a \in (E, F)$.

Lemma 4 ([3, p. 45], Lemma 9)**.** *Let* T' *and* T'' *be any given triangles and let* $E,\ F \subset \omega$. *Then for any given operator* T *represented by a triangle we have* $T \in (E_{T'}, F_{T''})$ *if and only if* $T'' T T'^{-1} \in (E, F)$.

3. Some results on matrix transformations and on the multipliers of special sets. ***3.1. On the triangles*** $\boldsymbol{C(\lambda)}$ ***and*** $\boldsymbol{\Delta(\lambda)}$ ***and the sets*** $\boldsymbol{W_a}$ ***and*** $\boldsymbol{W_a^0}$***.*** The infinite matrix $T = (t_{nk})_{n,k\geq 1}$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \neq 0$ for all n. For $\lambda \in U$ the infinite matrices $C(\lambda)$ and $\Delta(\lambda)$ are triangles. We have $[C(\lambda)]_{nk} = 1/\lambda_n$ for $k \leq n$, and the nonzero entries of $\Delta(\lambda)$ are determined by $\left[\Delta(\lambda)\right]_{nn} = \lambda_n$ for all n, and $\left[\Delta(\lambda)\right]_{n,n-1} = -\lambda_{n-1}$ for all $n \geq 2$. It can be shown that the matrix $\Delta(\lambda)$ is the inverse of $C(\lambda)$, that is, $C(\lambda)\big(\Delta(\lambda)y\big) = \Delta(\lambda)\big(C(\lambda)y\big) = y$ for all $y \in \omega$. If $\lambda = e$ we obtain the well known operator of the first difference represented by $\Delta(E) = \Delta$. Then we have $\Delta_n y = y_n - y_{n-1}$ for all $n \geq 1$ with the convention $y_0 = 0$. It is usually written $\Sigma = C(E)$, and then we may write $C(\lambda) = D_{1/\lambda}\Sigma$. Note that $\Delta = \Sigma^{-1}$. The Cesàro operator is defined by $C_1 = C\big((n)_{n\geq 1}\big)$. We use the sets of sequences that are *a-strongly bounded and a-strongly convergent to zero sequences* defined for $a \in U^+$ by

$$W_a = \left\{ y \in \omega \colon \|y\|_{W_a} = \sup_n \left(n^{-1} \sum_{k=1}^{n} |y_k|/a_k \right) < \infty \right\}$$

and

$$W_a^0 = \left\{ y \in \omega \colon \lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} |y_k|/a_k \right) = 0 \right\}$$

(cf. [7, 9, 12, 15]). It can easily be seen that $W_a = \left\{ y \in \omega \colon C_1 D_{1/a}|y| \in s_1 \right\}$. If $a = (r^n)_{n\geq 1}$ the sets W_a and W_a^0 are denoted by W_r and W_r^0. For $r = 1$, we obtain the well-known sets

$$w_\infty = \left\{ y \in \omega \colon \|y\|_{w_\infty} = \sup_n \left(n^{-1} \sum_{k=1}^{n} |y_k| \right) < \infty \right\}$$

and

$$w_0 = \left\{ y \in \omega \colon \lim_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} |y_k| \right) = 0 \right\}$$

called the *spaces of sequences that are strongly bounded and strongly summable to zero sequences by the Cesàro method* (cf. [18]).

3.2. On the multipliers of some sets. First we need to recall some well known results. Let y and z be sequences and let E and F be two subsets of ω, then we write $M(E, F) = \big\{ y \in \omega \colon yz \in F \text{ for all } z \in E \big\}$, the set $M(E, F)$ is called the *multiplierspace of* E *and* F. In the following we will use the next well known results.

Lemma 5. *Let E, $\widetilde{E}$, F and $\widetilde{F}$ be arbitrary subsets of ω. Then*: (i) $M(E,F) \subset M(\widetilde{E},F)$ *for all* $\widetilde{E} \subset E$; (ii) $M(E,F) \subset M(E,\widetilde{F})$ *for all* $F \subset \widetilde{F}$.

The α-dual of a set of sequences E is defined as $E^{\alpha} = M\left(E, \ell_1\right)$ and the β-dual of E is defined as $E^{\beta} = M(E, cs)$, where $cs = c_{\Sigma}$ is the set of all convergent series.

Lemma 6. *Let a, $b \in U^{+}$ and let E and F be two subsets of ω. Then $D_a E \subset D_b F$ if and only if $a/b \in M(E,F)$.*

In the following we use the results stated below.

Lemma 7. *Let $p \geq 1$. Then we have*:

(i) a) $M(c,c_0) = M(\ell_\infty, c) = M(\ell_\infty, c_0) = c_0$ *and* $M(c,c) = c$,

b) $M(E,\ell_\infty) = M(c_0,F) = \ell_\infty$ *for* $E,\ F = c_0,\ c$ *or* ℓ_∞,

c) $M(c_0,\ell_p) = M(c,\ell_p) = M(\ell_\infty,\ell_p) = \ell_p$,

d) $M(\ell_p,F) = \ell_\infty$ *for* $F \in \left\{c_0, c, s_1, \ell_p\right\}$;

(ii) a) $M(w_0,F) = M(w_\infty,\ell_\infty) = s_{(1/n)_{n\geq 1}}$ *for* $F = c_0,\ c$ *or* ℓ_∞,

b) $M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$,

c) $M(\ell_1, w_\infty) = s_{(n)_{n\geq 1}}$ *and* $M(\ell_1, w_0) = s^0_{(n)_{n\geq 1}}$,

d) $M(E, w_0) = w_0$ *for* $E = s_1$ *or* c,

e) $M(E, w_\infty) = w_\infty$ *for* $E = c_0,\ s_1$ *or* c.

Proof. Statements (i) a), (i) b) and (i) d) with $F \in \left\{c_0, c, \ell_\infty\right\}$ follow from [19, p. 648], (Lemma 3.1), Lemma 2 and [20, p. 157] (Example 1.28). The case $M(\ell_p,\ell_p) = \ell_\infty$ is immediate. Then statements (ii) a) with $F = c_0$ or c, (ii) b), (ii) c), (ii) d) follow from [14, p. 598] (Lemma 4.2). It remains to successively show the identity $M(c_0,\ell_p) = \ell_p$ in (i) c), statement (ii) a) with $F = \ell_\infty$ and the identity $M(c_0, w_\infty) = w_\infty$ in (ii) e). We show $M(c_0,\ell_p) = \ell_p$.

Case $p = 1$. We have $M(c_0,\ell_1) = c_0^{\alpha} = \ell_1$ (cf. [20], Theorem 1.29).

Case $p > 1$. By [22, p. 124] (Theorem 8.3.9) with $X = c_0$ and $Z = \ell_q$, we have $M(c_0,\ell_p) = M(c_0,\ell_q^{\beta}) = M(\ell_q, c_0^{\beta}) = M(\ell_q,\ell_1) = \ell_q^{\alpha}$. Then by [20] (Theorem 1.29) we have $\ell_q^{\alpha} = \ell_q^{\beta} = \ell_p$. We conclude $M(c_0,\ell_p) = \ell_p$. The identity $M(\ell_\infty,\ell_p) = \ell_p$ follows from [21], since $(c_0,\ell_p) = (\ell_\infty,\ell_p)$. We conclude by Lemma 5 that $\ell_p = M(\ell_\infty,\ell_p) \subset M(c,\ell_p) \subset M(c_0,\ell_p) = \ell_p$, which shows (i) c). For (ii) a) it is enough to notice that by [20, p. 219] (Theorem 3.58), we have $(w_0,s_1) = (w_\infty,s_1)$. Then, by [14, p. 598] (Lemma 4.2), we obtain $M(w_0,F) = s_{(1/n)_{n\geq 1}}$ for $F \in \{c_0, c, s_1\}$. It remains to show $M(c_0,w_\infty) = w_\infty$ in the statement (ii) e). By [20, p. 218] (Lemma 3.56), the set $\mathcal{M} = w_\infty^{\beta}$ is a BK space with AK, and is β-perfect, that is, $w_\infty^{\beta\beta} = w_\infty$. Again by [22, p. 124] (Theorem 8.3.9) with $X = c_0$ and $Z = \mathcal{M}$, we obtain $M(c_0,w_\infty) = M(c_0,w_\infty^{\beta\beta}) = M\big(\mathcal{M}, c_0^{\beta}\big)$. But we have $c_0^{\beta} = \ell_1$. We conclude $M(c_0,w_\infty) = \mathcal{M}^{\beta} = w_\infty$.

3.3. The equivalence relation $R_{\mathcal{E}}$. We need to recall some results on the equivalence relation $R_{\mathcal{E}}$ which is defined using the multiplier of sequence spaces. For $b \in U^{+}$ and for any subset $\mathcal{E}$ of ω, we denote by $cl^{\mathcal{E}}(b)$ the equivalence class for the equivalence relation $R_{\mathcal{E}}$ defined by $xR_{\mathcal{E}}y$ if $\mathcal{E}_x = \mathcal{E}_y$ for $x,y \in U^{+}$. It can easily be seen that $cl^{\mathcal{E}}(b)$ is the set of all $x \in U^{+}$ such that $x/b \in M(\mathcal{E},\mathcal{E})$ and $b/x \in M(\mathcal{E},\mathcal{E})$ (cf. [16]). Then we have $cl^{\mathcal{E}}(b) = cl^{M(\mathcal{E},\mathcal{E})}(b)$. For instance, $cl^{c}(b)$ is the set of all $x \in U^{+}$ such that $s_x^{(c)} = s_b^{(c)}$. This is the set of allsequences $x \in U^{+}$ such that $x_n \sim Cb_n$ $(n \to \infty)$ for some $C > 0$. In [16], we denote by $cl^{\infty}(b)$ the class $cl^{\ell_\infty}(b)$. Recall that $cl^{\infty}(b)$ is the set of all $x \in U^{+}$ such that $K_1 \leq x_n/b_n \leq K_2$ for all n and for some $K_1,\ K_2 > 0$.

4. On the SSIE $F \subset E_a + F'_x$ with $e \in F$ and $F' \subset M(F, F')$. Here we are interested in the study of the set of all positive sequences x that satisfy the inclusion $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences and a is a positive sequence. We may consider this problem as a *perturbation problem*. If we know the set $M(F, F')$, then the solutions of the *elementary inclusion* $F'_x \supset F$ are determined by $1/x \in M(F, F')$. Now the question is: let $\mathcal{E}$ be a linear space of sequences. What are the solutions of the *perturbed* inclusion $F'_x + \mathcal{E} \supset F$? An additionnal question may be the following one: what are the conditions on $\mathcal{E}$ under which the solutions of the elementary and the perturbed inclusions are the same? The solutions of the perturbed inclusion $F \subset E_a + F'_x$, where E, F, and F' are linear spaces of sequences cannot be obtained in the general case. So are led to deal with the case when $a = (r^n)_{n \geq 1}$, $r > 0$, for which most of these SSIE can be totally solved. In the following we write $\mathcal{I}_a(E, F, F') = \{x \in U^+ : F \subset E_a + F'_x\}$, where E, F, and F' are linear spaces of sequences and $a \in U^+$. For any set χ of sequences we let $\overline{\chi} = \{x \in U^+ : 1/x \in \chi\}$.

4.1. General case. The next theorem is the main result and is used throuhgout this paper. We use the set $\Phi = \{c_0, c, s_1, \ell_p, w_0, w_\infty\}$ with $p \geq 1$. By $c(1)$ we define the set of all sequences $\alpha \in U^+$ that satisfy $\lim_{n\to\infty} \alpha_n = 1$. Then we consider the condition

$$G \subset G_{1/\alpha} \quad \text{for all} \quad \alpha \in c(1) \tag{4}$$

for any given linear space G of sequences. Notice that condition (4) is satisfied for all $G \in \Phi$.

Theorem 1. *Let $a \in U^+$ and let E, F, F' be linear spaces of sequences. Assume*: a) $e \in F$, b) $F' \subset M(F, F')$, c) F' *satisfies condition* (4). *Then we have*: (i) $a \in M(E, c_0)$ *implies* $\mathcal{I}_a(E, F, F') = \overline{F'}$, (ii) $1/a \in M(F, E)$ *implies* $\mathcal{I}_a(E, F, F') = U^+$.

Proof. (i) Let $x \in \mathcal{I}_a(E, F, F')$. Then there are $\xi \in E$ and $f' \in F'$ such that $1 = a_n\xi_n + x_n f'_n$, hence,

$$\frac{1 - a_n\xi_n}{x_n} = f'_n \quad \text{for all} \quad n.$$

Since $a \in M(E, c_0)$ we have $1 - a_n\xi_n \to 1$ $(n \to \infty)$ and

$$\frac{1}{x_n} = \frac{1}{1 - a_n\xi_n} f'_n \quad \text{for all} \quad n.$$

By the condition in c) we conclude $x \in \overline{F'}$. Conversely, the condition $x \in \overline{F'}$ implies $1/x \in F'$, and the condition in b) implies $1/x \in M(F, F')$. We conclude $F \subset F'_x$ and $x \in \mathcal{I}_a(E, F, F')$. So we have shown (i). Statement (ii) follows from the equivalence of $1/a \in M(F, E)$ and $F \subset E_a$. This concludes the proof.

We immediately deduce the following.

***Corollary* 1.** *Let E, F, F' be linear spaces of sequences. Assume*: a) $e \in F$, b) $F' \subset M(F, F')$ *and* c) $E \subset c_0$. *Then the next statements are equivalent*: (i) $F \subset E + F'_x$, (ii) $F \subset F'_x$, (iii) $x \in \overline{F'}$.

In some cases, where $E = cs$ or ℓ_1 and $F' = \ell_1$, we obtain the next results using the α- and β-duals.

***Corollary* 2.** *Let $a \in U^+$ and let F and F' be linear spaces of sequences. Assume* a), b), c) *in Theorem* 1 *hold. Then the set $\mathcal{I}_a(cs, F, F')$ of all positive sequences x such that $F \subset cs_a + F'_x$, satisfies the next properties*: (i) $a \in s_1$ *implies* $\mathcal{I}_a(cs, F, F') = \overline{F'}$, (ii) $1/a \in F^\beta$ *implies* $\mathcal{I}_a(cs, F, F') = U^+$.

Proof. It is enough to show $M(cs, c_0) = s_1$. We have $a \in M(cs, c_0)$ if and only if $D_a\Delta \in \in (c, c_0)$, and $D_a\Delta$ is the infinite matrix whose the nonzero entries are $[D_a\Delta]_{nn} = -[D_a\Delta]_{n,n-1} = = a_n$ for all $n \geq 2$, and $[D_a\Delta]_{1,1} = a_1$. By the characterization of (c, c_0) recalled in Lemma 1 we conclude $M(cs, c_0) = s_1$.

***Corollary* 3.** *Let $a \in U^+$ and let F and F' be linear spaces of sequences. Assume* a), b), c) *in Theorem* 1 *hold. Then the set $\mathcal{I}_a(\ell_1, F, F')$ of all positive sequences x such that $F \subset (\ell_1)_a + + F'_x$, satisfies the next properties*: (i) $a \in s_1$ *implies* $\mathcal{I}_a(\ell_1, F, F') = \overline{F'}$, (ii) $1/a \in F^\alpha$ *implies* $\mathcal{I}_a(\ell_1, F, F') = U^+$.

Proof. This result follows from the identity $M(\ell_1, c_0) = s_1$ which is a direct consequence of Lemma 2.

***Corollary* 4.** *Let $a \in U^+$ and let E and F be linear spaces of sequences. Assume $e \in F$ and $\ell_1 \subset F^\alpha$. Then the set $\mathcal{I}_a(E, F, \ell_1)$ of all positive sequences x such that $F \subset E_a + (\ell_1)_x$ satisfies*: (i) *and* (ii) *in Theorem* 1 *with $F' = \ell_1$.*

4.2. On SSIE of the form $F \subset E_a + F'_x$, where E, F, and F' are any of the sets c_0, c, s_1, ℓ_p, w_0, or w_∞. In this part we use the set $\Omega = \big(\{s_1\} \times (\Phi \setminus \{c\})\big) \cup \big(\{c\} \times \Phi\big)$ with $p \geq 1$, and we deal with the perturbed inclusions of the form $F \subset E_a + F'_x$, where $E = c_0,\ s_1, \ell_p, w_0$, or w_∞ and $(F, F') \in \Omega$. As a direct consequence of Lemma 7 we obtain.

Lemma 8. *We have $(F, F') \in \Omega \Rightarrow F' \subset M(F, F')$.*

As a direct consequence of Corollary 1 and Lemma 8 we get the following result.

Proposition 1. *Let $E \subset c_0$ be a linear space of sequences and let $(F, F') \in \Omega$. Then the next statements are equivalent, where*

(i) $F \subset E + F'_x$, (ii) $F \subset F'_x$, (iii) $x \in \overline{F'}$.

Proposition 2. *Let $a \in U^+$ and $(F, F') \in \Omega$. We have*:

(i) $\mathcal{I}_a(c_0, F, F') = \overline{F'}$ *if* $a \in s_1$, *and* $\mathcal{I}_a(c_0, F, F') = U^+$ *if* $1/a \in c_0$,

(ii) $\mathcal{I}_a(s_1, F, F') = \overline{F'}$ *if* $a \in c_0$, *and* $\mathcal{I}_a(s_1, F, F') = U^+$ *if* $1/a \in s_1$,

(iii) $\mathcal{I}_a(\ell_p, F, F') = \overline{F'}$ *if* $a \in s_1$, *and* $\mathcal{I}_a(\ell_p, F, F') = U^+$ *if* $1/a \in \ell_p$ *for* $p \geq 1$,

(iv) $\mathcal{I}_a(w_0, F, F') = \overline{F'}$ *if* $a \in s_{(1/n)_{n\geq 1}}$, *and* $\mathcal{I}_a(w_0, F, F') = U^+$ *if* $1/a \in w_0$,

(v) $\mathcal{I}_a(w_\infty, F, F') = \overline{F'}$ *if* $a \in s^0_{(1/n)_{n\geq 1}}$, *and* $\mathcal{I}_a(w_\infty, F, F') = U^+$ *if* $1/a \in w_\infty$.

Proof. The proof is a direct consequence of Theorem 1 and Lemma 7. Indeed, we successively have $M(E, c_0) = s_1$ for $E = c_0$ or ℓ_p, $M(E, c_0) = c_0$ for $E = c$ or s_1, $M(w_0, c_0) = s_{(1/n)_{n\geq 1}}$ and $M(w_\infty, c_0) = s^0_{(1/n)_{n\geq 1}}$. Then we have $M(F, E) = M(s_1, E) = M(c, E)$ for $E \in \Phi \setminus \{c\}$ and $M(s_1, c_0) = c_0$, $M(s_1, s_1) = s_1$, $M(s_1, \ell_p) = \ell_p$, $M(s_1, w_0) = w_0$, and $M(s_1, w_\infty) = w_\infty$.

In the case when $E = c$ we obtain the following result.

Proposition 3. *Let $a \in U^+$ and $F' \in \Phi$. We have*:

(i) $\mathcal{I}_a(c, c, F') = \overline{F'}$ *if* $a \in c_0$, *and* $\mathcal{I}_a(c, c, F') = U^+$ *if* $1/a \in c$,

(ii) $\mathcal{I}_a(c, s_1, F') = \overline{F'}$ *if* $a \in c_0$, *and* $\mathcal{I}_a(c, s_1, F') = U^+$ *if* $1/a \in c_0$.

Proof. The proof follows from Theorem 1 and Lemma 7. Here we have $M(E, c_0) = M(c, c_0) = = c_0$ and $M(F, E) = M(F, c) = c_0$ for $F = s_1$, and $M(F, c) = c$ for $F = c$.

5. Solvability of the SSIE of the form $F \subset E_r + F'_x$, where E and F' are any of the sets c_0, c, s_1, ℓ_p $(p \geq 1)$, w_0, or w_∞. For $a = (r^n)_{n\geq 1}$, we write $\mathcal{I}_r(E, F, F')$ for the set $\mathcal{I}_a(E, F, F')$. Then we solve the perturbed inclusions $F \subset E_r + F'_x$, where F is either c or s_1 and $E \in \Phi \setminus \{w_0\}$, $F' \in \Phi$. It can easily beseen that in most of the cases the set $\mathcal{I}_r(E, F, F')$ may be determined by

$$\mathcal{I}_r(E,F,F') = \begin{cases} \overline{F'} & \text{if} \quad r < 1, \\ U^+ & \text{if} \quad r \geq 1 \end{cases} \tag{5}$$

or

$$\mathcal{I}_r(E,F,F') = \begin{cases} \overline{F'} & \text{if} \quad r \leq 1, \\ U^+ & \text{if} \quad r > 1. \end{cases} \tag{6}$$

As a direct consequence of Propositions 2 and 3 we obtain the following result.

Proposition 4. *Let $a \in U^+$ and $(F,F') \in \Omega$. We have*:

(i) *The sets $\mathcal{I}_r(s_1,F,F')$, $\mathcal{I}_r(c,c,F')$, and $\mathcal{I}_r(w_\infty,F,F')$ are determined by* (5).

(ii) *The sets $\mathcal{I}_r(c_0,F,F')$ and $\mathcal{I}_r(\ell_p,F,F')$ for $p \geq 1$ are determined by* (6).

Rewriting Proposition 4 we obtain.

Corollary **5.** *Let $r > 0$. Then we have*:

(i) *Let $F' \in \Phi$. Then*:

a) *The solutions of the SSIE $c \subset E_r + F'_x$ with $E = c$, s_1 or w_∞ are determined by* (5).

b) *The solutions of the SSIE $c \subset E_r + F'_x$ with $E = c_0$ or ℓ_p $(p \geq 1)$ are determined by* (6).

(ii) *Let $F' \in \Phi \diagdown \{c\}$. Then we have*:

a) *The solutions of the SSIE $s_1 \subset E_r + F'_x$ with $E = s_1$ or w_∞ are determined by* (5).

b) *The solutions of the SSIE $s_1 \subset E_r + F'_x$ with $E = c_0$ or ℓ_p $(p \geq 1)$ are determined by* (6).

Remark **1.** The set $\mathcal{I}_r(w_0,c,F')$ of all the solutions of the SSIE $c \subset W_r^0 + F'_x$, where $F' \in \Phi$, is determined for all $r \neq 1$. We obtain $\mathcal{I}_r(w_0,c,F') = \overline{F'}$ for $r < 1$, and $\mathcal{I}_r(w_0,c,F') = U^+$ if $r > 1$.

6. Application to the SSIE of the form $F \subset E_a + F_x$ with $e \in F$. In the following we write $\mathcal{I}_a(E,F) = \mathcal{I}_a(E,F,F) = \{x \in U^+ : F \subset E_a + F_x\}$. In this part we give results on the SSIE $F \subset E_a + F_x$ and we explicitly calculate the solutions of special SSIE of the form $F \subset E_r + F_x$.

6.1. Some general results on the SSIE of the form $F \subset E_a + F_x$. From Theorem 1 we obtain the next corollary.

Corollary **6.** *Let $a \in U^+$ and let E, F be two linear spaces of sequences. Assume*: a) $e \in F$, b) $F \subset M(F,F)$ *and* c) F *satisfies condition* (4). *Then we have*:

(i) *$a \in M(E,c_0)$ implies $\mathcal{I}_a(E,F) = \overline{F}$*,

(ii) *$1/a \in M(F,E)$ implies $\mathcal{I}_a(E,F) = U^+$.*

Now we deal with the SSIE $F \subset E_a + F_x$, where F is either c or s_1 and $E \in \Phi$. By Corollary 6 and Lemma 7 we obtain the following result.

Corollary **7.** *Let $a \in U^+$. We have*:

(i) a) *$\mathcal{I}_a(c_0,c) = \overline{c}$ if $a \in s_1$, and $\mathcal{I}_a(c_0,c) = U^+$ if $1/a \in c_0$*,

b) *$\mathcal{I}_a(c,c) = \overline{c}$ if $a \in c_0$, and $\mathcal{I}_a(c,c) = U^+$ if $1/a \in c$*,

c) *$\mathcal{I}_a(s_1,c) = \overline{c}$ if $a \in c_0$, and $\mathcal{I}_a(s_1,c) = U^+$ if $1/a \in s_1$*,

d) *$\mathcal{I}_a(\ell_p,c) = \overline{c}$ if $a \in s_1$, and $\mathcal{I}_a(\ell_p,c) = U^+$ if $1/a \in \ell_p$ for $p \geq 1$*,

e) *$\mathcal{I}_a(w_0,c) = \overline{c}$ if $a \in s_{(1/n)_{n\geq 1}}$, and $\mathcal{I}_a(w_0,c) = U^+$ if $1/a \in w_0$*,

f) *$\mathcal{I}_a(w_\infty,c) = \overline{c}$ if $a \in s^0_{(1/n)_{n\geq 1}}$, and $\mathcal{I}_a(w_\infty,c) = U^+$ if $1/a \in w_\infty$*;

(ii) a) *$\mathcal{I}_a(c_0,s_1) = \overline{s_1}$ if $a \in s_1$, and $\mathcal{I}_a(c_0,s_1) = U^+$ if $1/a \in c_0$*,

b) *$\mathcal{I}_a(c,s_1) = \overline{s_1}$ if $a \in c_0$, and $\mathcal{I}_a(c,s_1) = U^+$ if $1/a \in c_0$*,

c) *$\mathcal{I}_a(s_1,s_1) = \overline{s_1}$ if $a \in c_0$, and $\mathcal{I}_a(s_1,s_1) = U^+$ if $1/a \in s_1$*,

d) *$\mathcal{I}_a(\ell_p,s_1) = \overline{s_1}$ if $a \in s_1$, and $\mathcal{I}_a(\ell_p,s_1) = U^+$ if $1/a \in \ell_p$ for $p \geq 1$*,

e) *$\mathcal{I}_a(w_0,s_1) = \overline{s_1}$ if $a \in s_{(1/n)_{n\geq 1}}$, and $\mathcal{I}_a(w_0,s_1) = U^+$ if $1/a \in w_0$*,

f) $\mathcal{I}_a(w_\infty, s_1) = \overline{s_1}$ *if* $a \in s^0_{(1/n)_{n\geq 1}}$, *and* $\mathcal{I}_a(w_\infty, s_1) = U^+$ *if* $1/a \in w_\infty$.

6.2. Solvability of SSIE of the form $F \subset E_r + F_x$. In this part we consider the case when $a = (r^n)_{n\geq 1}$ for $r > 0$. We write E_r for $E_{(r^n)_{n\geq 1}}$, and the set of all positive sequences x that satisfy

$$F \subset E_r + F_x \tag{7}$$

is denoted by $\mathcal{I}_r(E, F)$. Here we explicitly calculate the solutions of the SSIE defined by (7), where F is either c or s_1 and $E \in \Phi$. We consider the conditions

$$(r^n)_{n\geq 1} \in M(E, c_0) \quad \text{for all} \quad 0 < r < 1, \tag{8}$$

$$(r^n)_{n\geq 1} \in M(E, c_0) \quad \text{for all} \quad 0 < r \leq 1, \tag{9}$$

$$(r^{-n})_{n\geq 1} \in M(F, E) \quad \text{for all} \quad r \geq 1, \tag{10}$$

$$(r^{-n})_{n\geq 1} \in M(F, E) \quad \text{for all} \quad r > 1. \tag{11}$$

We will use the next result which is a direct consequence of Corollary 6.

***Corollary* 8.** *Let $r > 0$ and let E and F be linear spaces of sequences. We assume that F satisfies the conditions* a), b) *and* c) *in Corollary* 6. *Then we have*:

(i) *Assume that the conditions in* (8) *and* (10) *hold. Then the solutions of the SSIE defined by* (7) *are determined by* (5) *with $F' = F$.*

(ii) *Assume that the conditions in* (9) *and* (11) *hold. Then the solutions of the SSIE defined by* (7) *are determined by* (6) *with $F' = F$.*

As a direct consequence of the preceding we obtain the following result.

***Corollary* 9.** *Let $r > 0$. Then we have*:

(i) a) *The solutions of the SSIE $c \subset E_r + c_x$ for $E = c,\ s_1$ or w_∞ are determined by* (5) *with* $F' = c$.

b) *The solutions of the SSIE $c \subset E_r + c_x$ with $E = c_0$ or ℓ_p $(p \geq 1)$ are determined by* (6) *with* $F' = c$.

(ii) a) *The solutions of the SSIE $s_1 \subset E_r + s_x$ with $E = s_1$ or w_∞ are determined by* (5) *with* $F' = s_1$.

b) *The solutions of the SSIE $s_1 \subset E_r + s_x$ with $E = c_0$ or ℓ_p $(p \geq 1)$ are determined by* (6) *with* $F' = s_1$.

7. On some SSIE and SSE with operators. In this part we consider the SSIE associated with the operator Δ, defined by $c \subset D_r * (c_0)_\Delta + c_x,\ c \subset D_r * c_\Delta + c_x,\ c \subset D_r * (s_1)_\Delta + c_x$, $s_1 \subset s_r + s_x$ and $s_1 \subset D_r * (s_1)_\Delta + s_x$ for $r > 0$. Then we consider the SSIE $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E \in \{c, s_1\}$ and $s_1 \subset D_r * (s_1)_{C_1} + s_x$, where C_1 is the Cesàro operator. Then we solve the SSE $D_r * E_{C_1} + s_x^{(c)} = c$ with $E \in \{c_0, c, s_1\}$, and $D_r * (s_1)_{C_1} + s_x = s_1$. Notice that since $D_a * E_T = E_{TD_{1/a}}$, where $TD_{1/a}$ is a triangle, for any linear space E of sequences and any triangle T the previous inclusions and identities can be considered as SSIE and SSE. More precisely, the previous SSE can be considered as the perturbed equations of the equations $F_x = F$ with $F = c$, or s_1.

7.1. On the SSIE of the form $F \subset D_r * E_\Delta + F_x$. In the next result among other things we deal with the SSIE $c \subset D_r * (c_0)_\Delta + c_x$ which is associated with the next statement. The condition $y_n \to l$ $(n \to \infty)$ implies that there are $u,\ v \in \omega$ such that $y = u + v$ and $u_n r^{-n} - u_{n-1} r^{-(n-1)} \to 0$ and $v_n / x_n \to l'$ $(n \to \infty)$ for some scalars $l,\ l'$ and for all y. The corresponding set of sequences is denoted by $\mathcal{I}_r((c_0)_\Delta, c)$. In a similar way $\mathcal{I}_r\big((s_1)_\Delta, c\big)$ is the set of all sequences that satisfy the SSIE $c \subset D_r * (s_1)_\Delta + c_x$. We obtain the following result.

***Corollary* 10.** *Let* $r > 0$. *We have*:

(i) $\mathcal{I}_r(E, c) = \mathcal{I}_r(s_1, c)$ *for* $E = c$, $(c_0)_\Delta$, c_Δ *or* $(s_1)_\Delta$, *and* $\mathcal{I}_r(s_1, c)$ *is determined by* (5) *with* $F' = c$.

(ii) $\mathcal{I}_r((s_1)_\Delta, s_1) = \mathcal{I}_r(s_1, s_1)$ *is determined by* (5) *with* $F' = s_1$.

Proof. (i) The identity $\mathcal{I}_r(s_1, c) = \mathcal{I}_r(c, c)$ is a direct consequence of Corollary 8, and $\mathcal{I}_r(s_1, c)$ is determined by (5) with $F = c$. Indeed, we have $M(E, c_0) = M(c, c_0) = M(s_1, c_0) = c_0$. Then we get $M(E, c) = c$ for $E = c$, and $M(E, s_1) = s_1$ for $E = s_1$. Now we deal with $\mathcal{I}((c_0)_\Delta, c)$. We have $(R^n)_{n\geq 1} \in M((c_0)_\Delta, c_0)$ if and only if $D_R\Sigma \in (c_0, c_0)$. The operator $D_R\Sigma$ is the triangle defined by $(D_R\Sigma)_{nk} = R^n$ for $k \leq n$, for all n. Then from the characterization of (c_0, c_0) the condition $D_R\Sigma \in (c_0, c_0)$ is equivalent to $nR^n = O(1)$ $(n \to \infty)$, and $R < 1$. Then the condition $(R^{-n})_{n\geq 1} \in M(c, (c_0)_\Delta)$ implies $\Delta D_{1/R} \in (c, c_0)$. The nonzero entries of $\Delta D_{1/R}$ are given by $[\Delta D_{1/R}]_{nn} = R^{-n}$ and $[\Delta D_{1/R}]_{n,n-1} = -R^{-n+1}$ for all $n \geq 1$, and from the characterization of (c, c_0) we conclude $R \geq 1$. By similar arguments we obtain $\mathcal{I}_r((s_1)_\Delta, c) = \mathcal{I}_r(c_\Delta, c) = \mathcal{I}_r(s_1, c)$. The proof of (ii) is similar and is left to the reader.

7.2. On the SSIE of the form $F \subset D_r * E_{C_1} + F_x$, where C_1 is the Cesàro operator. In this part we consider the SSIE with the Cesàro operator C_1 of the form $c \subset D_r * E_{C_1} + s_x^{(c)}$ with $E \in \{c, s_1\}$ and of the form $s_1 \subset D_r * E_{C_1} + s_x$ with $E \in \{c, s_1\}$. We obtain the following result.

Proposition 5. *Let* $r > 0$. *Then we have*:

(i) *The solutions of the SSIE defined by* $c \subset D_r * E_{C_1} + s_x^{(c)}$ *with* $E \in \{c, s_1\}$, *are determined by* (5) *with* $F' = c$.

(ii) *The solutions of the SSIE* $s_1 \subset D_r * (s_1)_{C_1} + s_x$ *are determined by* (5) *with* $F' = s_1$.

Proof. (i). *Case* $E = c$. Let $R > 0$. We have $(r^n)_{n\geq 1} \in M(c_{C_1}, c_0)$ if and only if $D_R C_1^{-1} \in (c, c_0)$. It can easily be seen that the entries of the matrix C_1^{-1} are defined by $\big[C_1^{-1}\big]_{nn} = n$, $[C_1^{-1}]_{n,n-1} = -(n-1)$ for all $n \geq 2$ and $[C_1^{-1}]_{1,1} = 1$. Then $D_R C_1^{-1}$ is the triangle whose the nonzero entries are given by $\big[D_R C_1^{-1}\big]_{nn} = nR^n$, $\big[D_R C_1^{-1}\big]_{n,n-1} = -(n-1)R^n$ for all $n \geq 2$ and $\big[D_R C_1^{-1}\big]_{1,1} = R$. From the characterization of (c, c_0) this means $\lim_{n\to\infty}\Big\{R^n\big[n - (n - 1)\big]\Big\} = \lim_{n\to\infty} R^n = 0$, and $(2n-1)R^n \leq K$ for some $K > 0$ and for all n. We conclude $(R^n)_{n\geq 1} \in M(c_{C_1}, c_0)$ if and only if $R < 1$. Then we have $(r^{-n})_{n\geq 1} \in M(c, c_{C_1})$ if and only if $C_1 D_{1/R} \in (c, c)$. But $C_1 D_{1/R}$ is the triangle defined by $[C_1 D_{1/R}]_{nk} = n^{-1}R^{-k}$ for $k \leq n$, for all n. So the condition $C_1 D_{1/R} \in (c, c)$ is equivalent to $n^{-1}\sum_{k=1}^{n} R^{-k} \to L$ $(n \to \infty)$ for some scalar L, and $R \geq 1$. We conclude by Corollary 8.

Case $E = s_1$. We have $(r^n)_{n\geq 1} \in M\big((s_1)_{C_1}, c_0\big)$ if and only if $D_R C_1^{-1} \in (s_1, c_0)$, and from the characterization of (s_1, c_0), that is, $\lim_{n\to\infty}\big[(2n-1)R^n\big] = 0$ and $R < 1$. Then we have $(r^{-n})_{n\geq 1} \in M\big(c, (s_1)_{C_1}\big)$ if and only if $C_1 D_{1/R} \in (c, s_1)$, that is, $\sup_n\Big(n^{-1}\sum_{k=1}^{n} R^{-k}\Big) < \infty$ and $R \geq 1$. Again we conclude by Corollary 8.

(ii). As we have just seen above we have $(r^n)_{n\geq 1} \in M\big((s_1)_{C_1}, c_0\big)$ if and only if $R < 1$. Then we have $(r^{-n})_{n\geq 1} \in M\big(s_1, (s_1)_{C_1}\big)$ if and only if $C_1 D_{1/R} \in (s_1, s_1)$, that is,

$$\sup_n\left(n^{-1}\sum_{k=1}^{n} R^{-k}\right) < \infty$$

and we conclude $R \geq 1$.

***Corollary* 11.** *The solutions of the SSIE $c \subset D_r * (c_0)_{C_1} + s_x^{(c)}$ are determined by*

$$\mathcal{I}_r\big((c_0)_{C_1}, c\big) = \begin{cases} \overline{c} & \textit{if} \quad r < 1, \\ U^+ & \textit{if} \quad r > 1. \end{cases}$$

Proof. We apply Theorem 1. We have $(r^n)_{n\geq 1} \in M\left((c_0)_{C_1}, c_0\right)$ if and only if $D_R C_1^{-1} \in \in (c_0, c_0)$. We conclude $D_R C_1^{-1} \in (c_0, c_0)$ if and only if $\big((2n-1)R^n\big)_{n\geq 1} \in \ell_\infty$ and $R < 1$. Then we get $(r^{-n})_{n\geq 1} \in M\big(c, (c_0)_{C_1}\big)$ if and only if $C_1 D_{1/R} \in (c, c_0)$, that is, $n^{-1}\sum_{k=1}^n R^{-k} \to 0$ $(n \to \infty)$ and $R > 1$.

7.3. Application to the solvability of the SSE with operator of the form $D_r * E_{C_1} + F_x = F$. In this subsection the deal with the solvability of each of the SSE $D_r * E_{C_1} + s_x^{(c)} = c$, where $E \in \{c_0, c, s_1\}$ and $D_r * E_{C_1} + s_x = s_1$ with $E \in \{c, s_1\}$. For instance, the SSE defined by $D_r * (s_1)_{C_1} + c_x = c$ is associated with the next statement. The condition $y_n \to l$ $(n \to \infty)$ holds if and only if there are $u,\ v \in \omega$ such that $y = u + v$ and $\sup_n \left\{ n^{-1} \left| \sum_{k=1}^n u_k/r^k \right| \right\} < < \infty$ and $v_n/x_n \to l'$ $(n \to \infty)$ for some scalars $l,\ l'$ and for all y. Here we also use the SSIE defined by $D_r * (c_0)_{C_1} + c_x \subset c$ which is associated with the next statement. The conditions $n^{-1}\left(\sum_{k=1}^n u_k/r^k\right) \to 0$ and $v_n/x_n \to l$ together imply $u_n + v_n \to l'$ $(n \to \infty)$ for all $u,\ v \in \omega$ and for some scalars $l,\ l'$. Let E and F be two linear spaces of sequences. We write $\mathcal{I}'_a(E,F) = = \{x \in U^+ : E_a + F_x \subset F\}$. Notice that since E and F are linear spaces of sequences, we have $x \in \mathcal{I}'_a(E,F)$ if and only if and $E_a \subset F$ and $F_x \subset F$. This means that $x \in \mathcal{I}'_a(E,F)$ if and only if $a \in M(E,F)$ and $x \in M(F,F)$. Then we have $\mathcal{S}(E,F) = \mathcal{I}_a(E,F) \cap \mathcal{I}'_a(E,F) = \{x \in U^+ : E_a + F_x = F\}$, see [16].

From Proposition 5 and Corollary 11 we obtain the next results on the SSE $D_r * E_{C_1} + s_x^{(c)} = c$ with $E \in \{c_0, c, s_1\}$, and $D_r * E_{C_1} + s_x = s_1$ with $E \in \{c, s_1\}$, where we write $G^+ = G \cap U^+$ for any set G of sequences.

Proposition 6. *Let $r > 0$. Then we have*:

(i) *The solutions of the SSIE $D_r * E_{C_1} + s_x^{(c)} \subset c$ with $E \in \{c_0, c, s_1\}$, and $D_r * E_{C_1} + s_x \subset s_1$ with $E \in \{c, s_1\}$ are determined by*

$$\mathcal{I}'_r(E_{C_1}, c) = \begin{cases} c^+ & \textit{if} \quad r < 1, \\ \varnothing & \textit{if} \quad r \geq 1, \end{cases} \quad \textit{for} \quad E \in \{c_0, c, s_1\}, \tag{12}$$

and

$$\mathcal{I}'_r(E_{C_1}, s_1) = \begin{cases} s_1^+ & \textit{if} \quad r < 1, \\ \varnothing & \textit{if} \quad r \geq 1, \end{cases} \quad \textit{for} \quad E \in \{c, s_1\}.$$

(ii) *The solutions of the perturbed equations $D_r * E_{C_1} + s_x^{(c)} = c$ with $E \in \{c_0, c, s_1\}$, and $D_r * E_{C_1} + s_x = s_1$ with $E \in \{c, s_1\}$ are determined by*

$$\mathcal{S}_r(E_{C_1}, c) = \begin{cases} cl^c(E) & \textit{if} \quad r < 1, \\ \varnothing & \textit{if} \quad r \geq 1, \end{cases} \quad \textit{for} \quad E \in \{c_0, c, s_1\},$$

and

$$\mathcal{S}_r(E_{C_1}, s_1) = \begin{cases} cl^{\infty}(E) & \textit{if} \quad r < 1, \\ \varnothing & \textit{if} \quad r \geq 1, \end{cases} \quad \textit{for} \quad E \in \{c, s_1\}.$$

Proof. (i) The inclusion $D_r * (c_0)_{C_1} \subset c$ is equivalent to $D_r C_1^{-1} \in (c_0, c)$, that is, $D_r C_1^{-1} \in S_1$ and $(n + n - 1)r^n \leq K$ for all n and for some $K > 0$. So we have $D_r * (c_0)_{C_1} \subset c$ if and only if $r < 1$. Then the inclusion $s_x^{(c)} \subset c$ is equivalent to $x \in c$. So the SSIE $D_r * (c_0)_{C_1} + s_x^{(c)} \subset c$ is equivalent to $r < 1$ and $x \in c$ and the identity in (12) holds for $E = c_0$. Case $E = c$. The inclusion $D_r * c_{C_1} \subset c$ is equivalent to $D_r C_1^{-1} \in (c, c)$, that is, $\big[n - (n-1)\big] r^n = r^n \to L \ (n \to \infty)$ for some scalar L, and $\big[n + (n-1)\big] r^n = O(1) \ (n \to \infty)$. This implies $r < 1$. Using similar arguments as those above we conclude (12) holds for $E = c$. The proof of the case $E = s_1$ is similar and is left to the reader.

(ii) is obtained from (i) and Proposition 5 (i) for $\mathcal{S}_r(E_{C_1}, c)$ with $E = c$, or s_1, and is obtained from (i) and Corollary 11 for $\mathcal{S}_r\big((c_0)_{C_1}, c\big)$. Then the determination of the set $\mathcal{S}_r\left((s_1)_{C_1}, s_1\right)$ is obtained from (i) and Proposition 5 (ii). It remains to determine the set $\mathcal{S}_r(c_{C_1}, s_1)$. For this we deal with the solvability of the SSIE $s_1 \subset D_r * c_{C_1} + s_x$. As we have seen in the proof of Proposition 5 we have $(r^n)_{n \geq 1} \in M(c_{C_1}, c_0)$ if and only if $D_r C_1^{-1} \in (c, c_0)$, that is, $r < 1$. Then we have $(r^{-n})_{n \geq 1} \in M(s_1, c_{C_1})$ if and only if $C_1 D_{1/r} \in (s_1, c)$. Since $\lim_{n \to \infty} [C_1 D_{1/R}]_{nk} = 0$ for all $k \geq 1$, by v) in Lemma 1 we have $C_1 D_{1/r} \in (s_1, c)$ if and only if $n^{-1} \sum_{k=1}^{n} r^{-k} \to 0 \ (n \to \infty)$ and $r > 1$. So we have shown $\mathcal{I}_r(c_{C_1}, s_1) = \overline{s_1}$ for $r < 1$ and $\mathcal{I}_r(c_{C_1}, s_1) = \overline{s_1}$ for $r > 1$. We conclude for the set $\mathcal{S}_r(c_{C_1}, s_1)$ using the determination of $\mathcal{I}'_r(c_{C_1}, s_1)$ given in (i).

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