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THE $\Box_b$-HEAT EQUATION ON FINITE TYPE CR MANIFOLDS WITH COMPARABLE LEVI FORM*

РІВНЯННЯ $\Box_b$-ТЕПЛОПРОВІДНОСТІ НА CR МНОГОВИДАХ СКІНЧЕННОГО ТИПУ ЗІ СПІВВИМІРНИМИ ФОРМАМИ ЛЕВІ

The main purpose of this paper is to study the initial-value problems for the heat equations associated with the operator $\Box_b$ on compact CR manifolds of finite type. The critical component of our analysis is the condition called $D^\epsilon(q)$ and introduced by K. D. Koenig [Amer. J. Math. – 2002. – **124**. – P. 129 – 197]. Actually, it states that the $\min\{q, n-1-q\}$th smallest eigenvalue of the Levi form is comparable with the largest eigenvalue of the Levi form.

Основною метою цієї роботи є вивчення початкових задач для рівнянь теплопровідності, асоційованих з оператором $\Box_b$, на компактних CR многовидах скінченного типу. Критичним компонентом нашого аналізу є так звана умова $D^\epsilon(q)$, що була запропонована К. Д. Кьонігом [Amer. J. Math. – 2002. – **124**. – P. 129 – 197]. Фактично вона встановлює, що $\min\{q, n-1-q\}$-найменше власне значення форми Леві є співвимірним із найбільшим власним значенням форми Леві.

1. Introduction. In Riemannian geometry, the Laplace – Beltrami operator defined on a Riemannian manifold M is $\Delta = d^*d$. In order to study the relation between geometry and analysis on M, a well-known approach is to use the heat equation associated to the Laplace – Beltrami operator. Let u be defined on $(0,\infty)\times M$. We say that u solves the heat equation on M when

$$\frac{\partial u}{\partial s} + \Delta u = 0 \quad \text{on } (0,\infty)\times M.$$

Moreover, we are also interested in the initial-value problem for the heat equation. That is, finding a function $u(s,x)$ solving the heat equation on M and satisfying

$$\lim_{s\to 0^+} u(s,\cdot) = f$$

with convergence in an appropriate norm on M. It is well-known that there is a unique fundamental solution $H(s,x,y)$ of the initial-value problem so that

$$u(s,x) = \int_M H(s,x,y) f(y)\, dV(y),$$

where $dV(\cdot)$ is the volume form on M. The kernel $H(s,x,y)$ is a smooth function only for $s>0$. For $s=0$, it agrees with the delta distribution of the diagonal, and it is obviously not smooth. The smoothness is a consequence of the ellipticity of the Laplace – Beltrami operator.

The study of the heat equation and the heat kernel for operators of the Laplace-type has numerous applications, including heat-kernel proofs of the Atiyah – Singer index theorem and its various generalizations.

In the present work, we will consider one analogue of the heat equation in Cauchy – Riemann (CR) geometry. That is an equation associated to the $\Box_b$-heat operator. Here, the operator $\Box_b$

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is the Laplacian associated to the $\bar{\partial}_b$-complex on a given CR manifold. Unfortunately, none of these are elliptic on CR manifolds. Hence, the classical analysis approach in Riemannian geometry does not allow us to deal with the $\Box_b$-heat equation. For non degenerate CR manifolds, the study of the $\Box_b$-heat equation culminates with Beals – Greiner – Stanton [1]. They actually built a class of pseudodifferential operators with a full symbolic calculus that allows us to construct explicit parametrices for the $\Box_b$-heat equation on (p, q)-forms under the condition $Y(q)$. In the strictly pseudoconvex case, this was used to derive a full short-time asymptotic for the heat kernel in terms of local pseudo-Hermitian invariants (i.e., universal polynomial in the covariant derivatives of the curvature and torsion tensors of the Tanaka – Webster connection). These results are the complete analogues of the results of the heat equation associated with Laplace type operators in the Riemannian setting.

The condition $Y(q)$ cannot hold on weakly pseudoconvex CR manifolds that are not strongly pseudoconvex. For finite type CR manifolds there were various attempts to give estimates in terms of the associated Carnot – Carathéodory metric for the fundamental solution of the $\Box_b$-equation. In particular, Nagel – Stein [8] established such estimates for the $\Box_b$-heat equation on finite type domains in $\mathbb{C}^2$.

The main purpose of this paper is to give an attempt to extend Nagel – Stein's results to finite type, compact CR manifolds of real dimension ≥ 5 by using the $D^{\epsilon}(q)$-condition introduced by K. Koenig [6]. Using this condition, K. Koenig [6] established that the Kohn – Laplacian has an inverse that belongs to a class of operators called nonisotropic smoothing (NIS) operators. This implies that its Schwartz kernel (i.e., the Green function of $\Box_b$) satisfies suitable metric distance estimates. The present work is also motivated to the fourth level in Fefferman's hierarchy [4], deriving estimates directly from the singularities of the integral kernels.

The main result of this paper is as follows:

Theorem 1.1. *Let M be a pseudoconvex, finite type, compact CR manifold for which the range of $\bar{\partial}_b$ is closed in L^2 and which satisfies the $D^{\epsilon}(q_0)$ condition. Then, for every $s > 0$, the heat solution operators $e^{-s\Box_b}$ is a NIS operator of order zero on $(0, q)$-forms, with $q_0 \leq q \leq n - 1 - q_0$, and associated estimates are uniform in $s > 0$.*

The paper is organized as follows. We will recall the definition of the operator $\Box_b$ and its properties, see [10] for all notions. Section 3 includes a short review on the class of NIS operators. The last section contains the full proof of Theorem 1.1, which is divided into two main steps: Theorems 4.3 and 4.4.

2. The hypoellipticity of the $\Box_b$-heat operator. Throughout this paper M is a compact oriented CR manifold of dimension $(n-1)$ with $n \geq 3$. The existence of a CR structure means there is a rank $(n - 1)$ complex subbundle $T^{1,0}(M)$ of the complexified tangent bundle $\mathbb{C}T(M) = T(M) \otimes_{\mathbb{R}} \mathbb{C}$ such that

(1) $T^{1,0}(M) \cap T^{0,1}(M) = \{0\}$, where $T^{0,1}(M) = \overline{T^{1,0}(M)}$,

and

(2) if Z and W are smooth sections of $T^{1,0}(M)$, then $[Z, W]$ is also a smooth section of $T^{1,0}(M)$.

We always assume that the manifold M is equipped with a Hermitian metric on $\mathbb{C}T(M)$ so that $T^{1,0}(M)$ is orthogonal to $T^{0,1}(M)$. Denote by $\eta(M)$ the orthogonal complement of $T^{1,0}(M) \oplus \oplus T^{0,1}(M)$. Let $T^{*1,0}(M)$ and $T^{*0,1}(M)$ be the dual bundles of $T^{1,0}(M)$ and $T^{0,1}(M)$, respectively.

For $0 \leq p, q \leq n-1$, the vector bundle $\Lambda^{p,q}(M)$ is defined as

$$\Lambda^{p,q}(M) = \Lambda^p T^{*1,0}(M) \oplus \Lambda^q T^{*0,1}(M).$$

The tangential CR complex $\bar{\partial}_b \colon C^\infty(\Lambda^{p,q}(M)) \to \mathcal{C}^\infty(\Lambda^{p,q+1}(M))$ is defined by

$$\bar{\partial}_b := \pi_{p,q+1} \circ d,$$

where $\pi_{p,q+1}$ is the orthogonal projection of $\Lambda^{p+q+1}(M)$ onto $\Lambda^{p,q+1}(M)$ and d is the exterior differentiation. Let $\bar{\partial}_b^*$ be the formal adjoint of $\bar{\partial}_b$ in $L^2_{(p,q)}(M)$, where $L^2_{(p,q)}(M)$ is the closure of $C^\infty(M, \Lambda^{p,q}(M))$ with respect to the appropriate pre-Hermitian inner product. The operator $\Box_b$ is the Laplacian associated to the $\bar{\partial}_b$-complex

$$\Box_b = \bar{\partial}_b \bar{\partial}_b^* + \bar{\partial}_b^* \bar{\partial}_b.$$

The associated initial-value problem is to find a function u on $(0,\infty) \times M$ such that

$$\mathfrak{H}[u](s,x) := \left[\frac{\partial}{\partial s} + \Box_b\right] u(s,x) = 0 \qquad \text{for } s > 0 \text{ and } x \in M,$$

$$\lim_{s \to 0^+} u(s,\cdot) = \phi(\cdot) \quad \text{in } L^2_{(0,q)}(M).$$

Let U be a suitable small open subset of M. We pick an orthogonal basis $\{\omega_1, , \ldots, \omega_{n-1}, \overline{\omega}_1, \overline{\omega}_2, \ldots, \overline{\omega}_{n-1}, \omega_0\}$ of $T^*(U)$ such that $\{\omega_1, \ldots, \omega_{n-1}\}$ is a frame of $\Lambda^{1,0}(U)$, and ω_0 is a real annihilator of $T^{0,1}$. Next, let $\{L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}, T\}$ be the (local) basis dual to $\{\omega_1, \ldots, \omega_{n-1}, \overline{\omega}_1, \overline{\omega}_2, \ldots, \overline{\omega}_{n-1}, \omega_0\}$.

Definition 2.1. *The Levi matrix associated with the Levi form is a Hermitian matrix* $(c_{kj})_{k,j=1,\ldots,n-1}$ *presented by*

$$[L_k, \overline{L}_j] = i c_{kj} T, \qquad \operatorname{mod}(L_1, \ldots, L_{n-1}, \overline{L}_1, \ldots, \overline{L}_{n-1}).$$

Now, we will consider what is called the comparable Levi form condition on the CR manifolds. For $1 \leq q \leq n-1$, let σ_q denote any of the $\binom{n-1}{q}$ sums of q eigenvalues λ_j of the Levi matrix (c_{kj}) and $\tau = \sum_{j=1}^{n-1} \lambda_j$ be the trace.

Definition 2.2 [6]. *We say that the $D^\epsilon(q)$ condition holds on M if there exists $\epsilon > 0$ such that*

$$\epsilon\tau \leq \sigma_q \leq (1-\epsilon)\tau \quad \textit{on } M, \textit{for all possible } \sigma_q.$$

Moreover, for $1 \leq q_0 \leq n-2$, if the $D^\epsilon(q_0)$ condition holds in U, so does the $D^\epsilon(q)$ condition also holds for all $\min(q_0, n-1-q_0) \leq q \leq \max(q_0, n-1-q_0)$. For this reason, we will always assume that $1 \leq q_0 \leq \dfrac{n-1}{2}$, that means the range of $D^\epsilon(q_0)$ is for $q \in [q_0, n-1-q_0]$.

Theorem 2.1. *Assume $U \subset M$ (suitable small open subset) is of finite commutator type and satisfies the $D^\epsilon(q_0)$ condition defined as above, for $1 \leq q_0 \leq (n-1)/2$. Then the heat operator $\mathfrak{H}$ acting on $(0,q)$ forms is hypoelliptic in $(0,\infty) \times U$ with $q_0 \leq q \leq n-1-q_0$.*

Here and in what follows, $\lesssim$ and $\gtrsim$ denote inequality up to a positive constant. Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

Proof. It suffices to prove the theorem on $(0, q_0)$-forms.

Since the finite type and $D^\epsilon(q_0)$ conditions, in the distributional sense, we have

$$\|\mathfrak{H}[u]\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)} = \|\partial_s u\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)} + \|\Box_b u\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)} +$$

$$+\langle \partial_s u, \Box_b u\rangle_{L^2_{(0,q_0)}((0,\infty)\times M)} + \langle \Box_b u, \partial_s u\rangle_{L^2_{(0,q_0)}((0,\infty)\times M)}.$$

But $\Box_b$ is self-adjoint and $\partial_s^* = -\partial_s$, then

$$\|\mathfrak{H}[u]\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)} = \|\partial_s u\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)} + \|\Box_b u\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)}.$$

Let $u = u(s, x)$, for $s > 0$ and $x \in U$. The condition of finite commutator type for $L_1, \ldots, L_{n-1}$, $\bar{L}_1, \ldots, \bar{L}_{n-1}$ on U also implies the finite commutator type for $L_1, \ldots, L_{n-1}$, $\bar{L}_1, \ldots, \bar{L}_{n-1}, \partial_s$ on $(0,\infty) \times U$. Since the $D^\epsilon(q)$ condition holds, we have obtained the well-known subelliptic estimate for the heat operator and the maximal estimate for $\Box_b$ [6]. Then these imply that

$$\|u\|^2_{H^{\epsilon_0}} \lesssim \|\mathfrak{H}[u]\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)} + \|u\|^2_{L^2_{(0,q_0)}((0,\infty)\times M)}.$$

Let ζ, ζ_1 be smooth real-valued cutoff functions supported in U, with $\zeta \prec \zeta_1$ (i.e., $\zeta = 1$ on suppζ_1). For any $\delta \in \mathbb{R}$ and $N > 0$, by the same method to prove Theorem 8.2.9 in [3], the following estimate holds:

$$\|\zeta u\|_{H^{\delta+\epsilon_0}} \le C_{\delta,N}(\|\zeta_1 \mathfrak{H}[u]\|_{H^\delta} + \|\zeta_1 u\|_{H^{-N}}).$$

Therefore, $\mathfrak{H}$ is hypoelliptic on all $(0, q_0)$-forms defined on $(0,\infty) \times U$.

Theorem 2.1 is proved.

3. Spaces of homogeneous type. In this section, we also assume that the holomorphic vectors fields $L_1, \ldots, L_{n-1}, \bar{L}_1, \ldots, \bar{L}_{n-1}$ defined on $U \subset M$ satisfy the condition of finite commutator type and the condition $D^\epsilon(q)$. The real vector fields $X_1, \ldots, X_{2n-2}$ are defined by $X_j = \operatorname{Re} L_j$, $X_{n+j-1} = \operatorname{Im} L_j$, $j = 1, \ldots, n-1$. For each finite sequence $i_1, \ldots, i_k$ of integers with $1 \le i_j \le$ $\le 2n-2$, setting $I = (i_1, \ldots, i_k)$ and the length $|I| = k$. We can write the commutator

$$[X_{i_k}, [X_{i_{k-1}}, \ldots, [X_{i_2}, X_{i_1}], \ldots]] = \lambda_{i_1\ldots i_k} T, \qquad \mod (X_1, \ldots, X_{2n-2}),$$

where $\lambda_{i_1\ldots i_k} \in C^\infty(U)$.

Definition 3.1. *For $x \in U$ and $r > 0$, the size functions are defined by*

$$\Lambda_l(x) = \left(\sum_{2\le|I|\le l} |\lambda_{i_1\ldots i_k}(x)|^2 \right)^{\frac{1}{2}}, \quad l \ge 2,$$

and

$$\Lambda(x, r) = \sum_{l=2}^{m} \Lambda_l(x) r^l.$$

Definition 3.2. *For each $x,\ y \in U$, the natural nonisotropic distance $\rho_M(x,y)$ corresponding to the vector fields $X_1,\ldots,X_{2n-2}$ is defined by*

$$\rho_M(x,y) = \inf\Big\{\delta > 0 : \text{there exists a continuous, piecewise smooth map}$$

$$\phi\colon [0,1] \to U \ \text{ such that } \phi(0) = x, \ \ \phi(1) = y,$$

$$\text{and } \ \phi'(t) = \sum\nolimits_{j=1}^{2n-2} \alpha_j(t) X_j \ \text{ almost everywhere},$$

$$\text{with } \ |\alpha_j(t)| < \delta, \ \text{for } \ j = 1,\ldots,2n-2\Big\}.$$

The nonisotropic ball centered at $x \in U$, with radius $r > 0$ is given by

$$B_M(x,r) = \{y \in U : \rho_M(x,y) < r\}.$$

For any $x,\ y \in U$, we also define $V(x,y) = |B_M(x,\rho_M(x,y))|$.

Let $\mathbb{B}_0$ denote the unit ball (defined by the Euclidean metric) in $\mathbb{R}^{2n-1}$. For $x \in U$ and $r > 0$, we set

$$\Phi_{x,r}(u) = \exp(ru_1X_1 + \ldots + ru_{2n-2}X_{2n-2} + \Lambda(x,r)u_{2n-1}T)(x),$$

where $u = (u_1,\ldots,u_{2n-1}) \in \mathbb{B}_0$. There is $R_0 > 0$ depending on the manifold M so that for all $0 < r < R_0$, the map $\Phi_{x,r}$ is a diffeomorphism of the unit ball $\mathbb{B}_0$ to its image. Hereafter, $0 < r < R_0$ when we have calculations on the exponential map $\Phi_{x,r}$. Now, let

$$\widetilde{B}_M(x,r) = \Phi_{x,r}(\mathbb{B}_0),$$

that is

$$\widetilde{B}_M(x,r) = \Big\{y \in U : y = \exp(a_1X_1 + \ldots + a_{2n-2}X_{2n-2} + aT)(x),$$

$$\text{where } |a_j| < r \text{ for } j = 1,\ldots,2n-2, \text{ and } |a| < \Lambda(x,r)\Big\}.$$

We have the following facts about the size function Λ and the above families of nonisotropic balls, which were proved in [9].

Theorem 3.1. *Assume that $U \subset M$ of finite commutator type, there exists $R_0 > 0$ such that*:

(1) *There are positive constants C_1, C_2 so that for all $x \in U$ and $0 < r < R_0$,*

$$B_M(x,C_1r) \subset \widetilde{B}_M(x,r) \subset B_M(x,C_2r).$$

(2) *There are two constants $C_3, C_4 > 0$ such that for all $x,y \in U$*

$$C_3 \le \frac{V_M(x,y)}{(\rho_M(x,y))^{2n-2}\Lambda(x,\rho_M(x,y))} \le C_4.$$

(3) *Let $J_{x,r}(u)$ denote the Jacobian matrix of $\Phi_{x,r}(u)$. Then $|\det(J_{x,r}(u))| \approx r^{2n-2}\Lambda(x,r)$ uniformly in x and $0 < r < R_0$.*

(4) $\left|\dfrac{\partial^\alpha}{\partial u^\alpha}\det(J_{x,r}(u))\right| \lesssim r^{2n-2}\Lambda(x,r)$ *uniformly in x and $0 < r < R_0$, for each multiindex α.*

For any function $f \in C^1(\mathbb{B}_0)$, the scaled pullbacks to $\mathbb{B}_0$ of the vector fields X_j are defined by

$$(\widehat{X}_j f)(u) = (\widehat{X}_j f)_{x,r}(u) = r(X_j \check{f})(\Phi_{x,r}(u)), \quad j = 1, \ldots, 2n-2,$$

where $\check{f}(y) = f \circ \Phi_{x,r}^{-1}(y)$ for $y \in \widetilde{B}_M(x,r)$. Therefore, $\widehat{X}_1, \ldots, \widehat{X}_{2n-2}$ may be written (in the u-coordinates) as linear combinations of the vector fields $\dfrac{\partial}{\partial u_1}, \ldots, \dfrac{\partial}{\partial u_{2n-1}}$. Also, we define the scaled pullback to $\mathbb{B}_0$ of the function ϕ on $\widetilde{B}_M(x,r)$ by

$$\widehat{\phi}(u) = \phi(\Phi_{x,r}(u))$$

for $u \in \mathbb{B}_0$.

The following facts are also from [2, 9].

Theorem 3.2. (1) *The coefficients of the $\widehat{X}_j$ (expressed in u-coordinates), together with their derivatives, are bounded above uniformly in x and r.*

(2) *The vector fields $\widehat{X}_1, \ldots, \widehat{X}_{2n-2}$ are of finite commutator type on $\mathbb{B}_0$, and*

$$\left| \det(\widehat{X}_1, \ldots, \widehat{X}_{2n-2}, Z) \right| > C$$

for a commutator Z (of the $\widehat{X}_j$) of length $\leq m$ such that $\widehat{X}_1, \ldots, \widehat{X}_{2n-2}, Z$ span the tangent space ($C > 0$ is independent of x and r).

In particular, we can write

$$\frac{\partial}{\partial u_j} = \sum_{l=1}^{2n-2} b_{jl} \widehat{X}_l + b_{j,2n-1} Z,$$

so that $b_{j,l}$ and its derivatives are bounded above uniformly in x and r, for j, $l = 1, \ldots, 2n-1$.

Now, let us define the CR structure on $\mathbb{B}_0$ determined by the following vector fields:

$$\widehat{L}_j = \widehat{X}_j + i\widehat{X}_{n+j-1}, \qquad \widehat{\bar{L}}_j = \widehat{X}_j - i\widehat{X}_{n+j-1},$$

and the basis of $(0,1)$-forms dual to $\widehat{\bar{L}}_1, \ldots, \widehat{\bar{L}}_{n-1}$ by $\widehat{\bar{\omega}}_1, \ldots, \widehat{\bar{\omega}}_{n-1}$.

We consider the equation on $\widetilde{B}(x,r)$

$$\bar{L}_j \phi = f.$$

From the definition,

$$\widehat{\bar{L}}_j \widehat{\phi} = r(\bar{L}_j \phi)(\Phi_{x,r}(u)) = r f(\Phi_{x,r}(u)) = r\widehat{f} = r\widehat{\bar{L}_j \phi}.$$

So, $\widehat{\bar{L}_j \phi} = r^{-1} \widehat{\bar{L}}_j \widehat{\phi}$ is the scaled pullback of the equation $\bar{L}_j \phi = f$. Now, $\bar{L}_j^* = -L_j + a_j$, for some $a_j \in C^\infty(U)$, we also define

$$\widehat{\bar{L}_j^*} = -\widehat{L}_j + r\widehat{a}_j.$$

Similarly, we obtain $\widehat{\bar{L}_j^* \phi} = r^{-1} \widehat{\bar{L}}_j^* \widehat{\phi}$. We also define the pullbacks $\widehat{\bar{\partial}}_b$ and $\widehat{\bar{\partial}_b^*}$ of the operators $\bar{\partial}_b$ and $\bar{\partial}_b^*$ by

$$\widehat{(\bar{\partial}_b \phi)} = r^{-1} \widehat{\bar{\partial}}_b \widehat{\phi},$$

and

$$\widehat{(\bar{\partial}_b^* \phi)} = r^{-1} \widehat{\bar{\partial}_b^*} \widehat{\phi},$$

respectively.

Lemma 3.1 [6]. *Let $(\bar{\partial}_b)^B$ and $(\bar{\partial}_b^*)^B$ be the operators defined in the terms of the CR structure on the unit ball $\mathbb{B}_0$ which is determined by the vector fields $\{\widehat{L}_j, \widehat{\bar{L}}_j\}$. Then $\widehat{\bar{\partial}}_b = (\bar{\partial}_b)^B$ and $r\widehat{\bar{\partial}_b^*} - (\bar{\partial}_b^*)^B$ is a differential operator of order zero on $\mathbb{B}_0$ uniformly in x and r.*

Finally, we also extend the map $\Phi_{x,r}$ on $\mathbb{B}_0$ to the map $\Phi_{(s,x),r}$ on $\mathbb{R} \times \mathbb{B}_0$ by

$$\Phi_{(s,x),r}(s,u) = \big(r^{-2}s, \Phi_{x,r}(u)\big)$$

with $0 < r < R_0$.

The scaled pullback of the heat equation on $\mathbb{R} \times M$ to $\mathbb{R} \times \mathbb{B}_0$ is

$$\left(\left(\frac{\partial}{\partial s} + \Box_b\right)\phi(s,x)\right)^{\widehat{}} = r^{-2}\frac{\partial}{\partial s}\widehat{\phi}(s,u) + r^{-2}\widehat{\Box_b}\widehat{\phi}(s,u),$$

where $\widehat{\Box_b} = \widehat{\bar{\partial}}_b\widehat{\bar{\partial}_b^*} + \widehat{\bar{\partial}_b^*}\widehat{\bar{\partial}}_b$ is defined on $\mathbb{B}_0$.

Next, under the condition of finite commutator type on M, we also define the parabolic non-isotropic metric on $\mathbb{R} \times M$. Recall that the family of dilation δ_λ on $\mathbb{R} \times M$ is

$$\delta_\lambda(s,x) = (\lambda^2 s, \lambda x),$$

for $s \in \mathbb{R}, x \in M$ and the parameter $\lambda > 0$.

Definition 3.3. *Denote by Y the vector field $\dfrac{\partial}{\partial s}$ on $\mathbb{R}$, then the family of the vector fields $\{Y, X_1, \ldots, X_{2n-2}\}$ also satisfies the condition of finite commutator type. For every $p = (s,x)$, $q = (t,y) \in \mathbb{R} \times U$, the following function is finite*:

$$\rho_{\mathbb{R}\times M}(p,q) = \rho_M(x,y) + \sqrt{|s-t|}.$$

This distance associates to the corresponding balls $B_{\mathbb{R}\times M}((s,x),r)$ on $\mathbb{R} \times M$. Note that

$$\big|B_{\mathbb{R}\times M}((s,x),|t-s|)\big| \approx (t-s)^2|B_M(x,|t-s|)|.$$

Let $\widetilde{\mathbb{B}}_0$ denote the unit ball in $\mathbb{R}^{2n}$. For each $(u,u_0) \in \widetilde{\mathbb{B}}_0$ the exponential mapping on $\mathbb{R} \times M$ is

$$\widetilde{\Phi}_{(s,x),r}(u,u_0) = \exp(r^2u_0Y + ru_1X_1 + \ldots + ru_{2n-2}X_{2n-2} + \Lambda(x,r)u_{2n-1}T)(s,x)$$

and

$$\widetilde{B}_{\mathbb{R}\times M}((s,x),r) = \widetilde{\Phi}_{(s,x),r}(\widetilde{\mathbb{B}}_0).$$

Now, we briefly recall the definition of the class of NIS operators on the CR manifold M. For more discussions, see [6–8].

Let $\mathcal{D}'(M)$ be the space of distributions defined on M and $\mathbb{I}_k$ be the set of multiindexes $(\alpha_1, \ldots, \alpha_{2n-2})$ such that $\sum_{j=1}^{2n-2}\alpha_j = k$ for $k = 0, 1, \ldots$.

Definition 3.4. *An operator $\mathcal{T} : C_0^\infty(M) \to \mathcal{D}'(M)$ is called a NIS operator of order $\kappa \geq 0$ if the following conditions hold*:

(1) *There is a function $T_0(x,y) \in C^\infty(M \times M \setminus \Delta_M)$ (smooth of the diagonal) so that if $\phi, \psi \in C_0^\infty(M)$ have disjoint supports,*

$$\big\langle \mathcal{T}[\phi], \psi \big\rangle = \int\limits_M \int\limits_M \phi(y)\psi(x)T_0(x,y)dV(x)dV(y).$$

(2) *For any* $s \geq 0$, *there exist parameters* $\alpha(s) < \infty, \beta < \infty$ *such that if* $\zeta, \zeta' \in C^\infty(M)$, $\zeta \prec \zeta'$, *then there is a constant* C_s *so that*

$$\|\zeta \mathcal{T}[f]\|_s \leq C_s(\|\zeta' f\|_{\alpha(s)} + \|f\|_\beta) \tag{3.1}$$

for all $f \in C^\infty(M)$.

(3) *For any* $\alpha \in \mathbb{I}_k, \beta \in \mathbb{I}_l$, *there exists a constant* $C_{k,l}$ *so that*

$$|X_x^\alpha X_y^\beta T_0(x,y)| \leq C_{k,l}\rho_M(x,y)^{\kappa-k-l}V_M(x,y)^{-1}.$$

(4) *For any ball* $B_M(x_0,r) \subset U$, *for each integer* $k \geq 0$, *there is a positive integer* N_k *and a constant* C_k *so that if* $\phi \in C_0^\infty(B_M(x_0,r))$ *and* $\alpha \in \mathbb{I}_k$, *we have*

$$\sup_{x \in B_M(x_0,r)} |X_x^\alpha \mathcal{T}[\phi](x)| \leq C_k r^{\kappa-k} \sup_{y \in M} \sum_{|J| \leq N_k} r^{|J|}|X^J[\phi](y)|.$$

(5) *The above conditions also hold for the adjoint operator* T^* *with kernel* $\overline{T_0(y,x)}$.

We also generalize this definition to a class of operators defined on $(0,q)$-forms. Let $\mathcal{T}$ be an operator from $C^\infty_{0,q_1}(M)$ into $C^\infty_{0,q_2}(M)$ and $\phi = \sum'_{|I|=q_1} \phi_I \bar{\omega}_I \in C^\infty_{0,q_1}(U)$, then

$$\mathcal{T}[\phi](x) = \sum_{|J|=q_2}{}' (\mathcal{T}[\phi])_J(x)\bar{\omega}_J,$$

where

$$(\mathcal{T}[\phi])_J(x) = \sum_{|I|=q_1}{}' \big\langle \mathcal{T}[\phi_I(x)\bar{\omega}_I], \bar{\omega}_J\big\rangle_{L^2}.$$

Then we define $\mathcal{T}^{IJ}[g](x) = \langle \mathcal{T}[g(x)\bar{\omega}_I], \bar{\omega}_J\rangle_{L^2}$ for $g \in C_0^\infty(U)$. We say that $\mathcal{T}$ is a NIS operator of order κ on $(0,q_1)$-form if and only if $\mathcal{T}$ and $\mathcal{T}^*$ satisfy the estimate in condition (2) of the Definition 3.4 and each $\mathcal{T}^{IJ}$ is a NIS operator of order κ on functions.

***Example* 3.1** (Szegö projections)**.** Let $\mathcal{S}_q$ and $\mathcal{S}'_q$ denote the orthogonal projections in $L^2_{(0,q)}(M)$ onto ker($\bar{\partial}_b^{0,q}$) and ker($\bar{\partial}_b^{*0,q}$), respectively, where $\bar{\partial}_b^{0,q}$ and $\bar{\partial}_b^{*0,q}$ mean $\bar{\partial}_b$, $\bar{\partial}_b^*$ acting on $(0,q)$-forms. We can rewrite these operators by

$$\mathcal{S}_q[\phi](x) = \sum_{|J|=q_2}{}' \sum_{|I|=q_1}{}' \langle \mathcal{S}_q[\phi_I(x)\bar{\omega}_I], \bar{\omega}_J\rangle_{L^2}\bar{\omega}_J = \sum_{|J|=q_2}{}' \left(\sum_{|I|=q_1}{}' \mathcal{S}_q^{IJ}[\phi_I](x)\right)\bar{\omega}_J,$$

$$\mathcal{S}'_q[\phi](x) = \sum_{|J|=q_2}{}' \sum_{|I|=q_1}{}' \langle \mathcal{S}'_q[\phi_I(x)\bar{\omega}_I], \bar{\omega}_J\rangle_{L^2}\bar{\omega}_J = \sum_{|J|=q_2}{}' \left(\sum_{|I|=q_1}{}' (\mathcal{S}_q^{IJ})'[\phi_I](x)\right)\bar{\omega}_J$$

for $\phi = \sum_{|I|=q_1} \phi_I \bar{\omega}_I$. Now, by the Riesz representation theorem,

$$\mathcal{S}_q[\phi](x) = \sum_{|J|=q_2}{}' \left(\sum_{|I|=q_1}{}' \int_M S_q^{IJ}(x,y)\phi_I(y)dV(y)\right)\bar{\omega}_J,$$

$$\mathcal{S}_q'[\phi](x) = \sum_{|J|=q_2}{}' \left(\sum_{|I|=q_1}{}' \int\limits_M (S_q^{IJ})'(x,y)\phi_I(y)dV(y) \right) \bar{\omega}_J,$$

where $S_q^{IJ}(x,y)$ and $(S^{IJ})'(x,y)$ are the respective Schwartz kernels of $\mathcal{S}_q^{IJ}[\cdot]$ and $(\mathcal{S}_q^{IJ})'[\cdot]$. In [6], the author showed that the operators $\mathcal{S}_q$ and $\mathcal{S}_q'$ are the NIS operators of order zero with $q_0 \le q \le n-1-q_0$.

As a consequence, we have the following proposition.

Proposition 3.1. *Let α be a multiindex with $|\alpha| = k \ge 1$. For $0 \le j \le [k/2]$, there are NIS operators $A_{j,1}, \ldots, A_{j,2n-2}, A_{j,2n-1}$ smoothing of order zero such that*

$$X^\alpha(I - \mathcal{H}_q) = \sum_{j=0}^{[k/2]} \left(\sum_{l=1}^{2n-2} (A_{j,l}X_l) + A_{j,2n-1} \right) \Box_b^j.$$

Here the operator $\mathcal{H}_q$ is the orthogonal projection in $L^2_{0,q}$ onto its harmonic subspace. In particular, if $k = 2j$, $A_{j,1} = A_{j,2} = \ldots = A_{j,2n-2} = 0$.

Proof. The proof uses the fact (which is established in [6]) that the relative inverse K to $\Box_b$ is a NIS operator of order 2, in the cases of comparable Levi form. Here the modification to Proposition 3.4.7 in [8] is that now $\Box_b K = I - \mathcal{H}_q$ instead of $I - \mathcal{S}_q$ in dimension $n = 2$.

The scaling method also provides following Sobolev type theorem.

Theorem 3.3. *Assume that the $D^\epsilon(q)$ and finite type conditions hold on M. Then there are a constant C and an even integer L_m so that if $f \in C^\infty(U)$, then, for all $x \in U$ and all $r \le r_0$,*

$$\sup_{B_M(x,r)} |f| \le C|B_M(x,r)|^{-\frac{1}{2}} \sum_{0 \le |I| \le L_m, |I| \text{ even}} r^{|I|} \|X^I f\|_{L^2(B_M(x,2r))}.$$

Let $f \in \Lambda^{0,q'}(C^\infty(M)) \cap L^2_{0,q'}(M)$ with $q \le q' \le n-1-q$. Moreover, if $f \in (\ker(\Box_b))^\perp$, then

$$\sup_{B_M(x,r)} |f| \le C|B_M(x,r)|^{-\frac{1}{2}} \sum_{j=0}^{L_m/2} r^{2j} \|\Box_b^j f\|_{L^2_{(0,q)}}.$$

Proof. We apply the scaling method introduced above. From the property (1) in Theorem 3.1, we have

$$\sup_{y \in B_M(x,r)} |f(y)| \le \sup_{y \in \widetilde{B}_M(x,C_1 r)} |f(y)| \le \sup_{u \in \mathbb{B}_0} |f(\Phi_{x,C_2 r}(u))|.$$

Set $F(u) = f(\Phi_{x,C_2 r}(u))$ for $u \in \mathbb{B}_0$. Let $G(u) = F(u)\theta(u)$, where $\theta \in C_0^\infty(\mathbb{R}^{2n-1})$, $\theta = 1$ on $\mathbb{B}_0$, and $\theta = 0$ outside the ball $\mathbb{B}(0,2) \subset \mathbb{R}^{2n-1}$. So,

$$\sup_{u \in \mathbb{B}_0} |F(u)| \le \sup_{u \in \mathbb{R}^{2n-1}} |G(u)| \le \int\limits_{\mathbb{R}^{2n-1}} |\widehat{G}(\xi)| dV(\xi) \le$$

$$\le \|(1 + |\xi|^4)^{1/N} \widehat{G}(\xi)\|_{L^2(\mathbb{R}^{2n-1})} \|(1 + |\xi|^4)^{-1/N}\|_{L^2(\mathbb{R}^{2n-1})} \le$$

(N can be chosen large enough to guarantee integrability)

$$\le C \sum_{0 \le |I| \le 2, \, |I| \text{ even}} \left\| \left(\frac{\partial}{\partial u} \right)^I F \right\|_{L^2(\mathbb{B}(0,2))}.$$

Now, from the statement (2) in Theorem 3.2, there is a positive integer number l depending on m (the constant l should be the higher power to make the Hölder's inequality work) such that

$$\sup_{\mathbb{B}_0} |F| \leq C \sum_{0\leq |I| \leq 2l,\, |I| \text{ even}} \|(\widehat{X})^I F\|_{L^2(\mathbb{B}(0,2))}.$$

Then, after rescaling pullback, by Theorem 3.1, we have the first statement. The analogue of the first statement for forms is immediately obvious. In order to estimate $X^I f$ in the terms of $\Box_b^j f$, with $f = f - \mathcal{H}_q[f]$, we will apply the basic decomposition in Proposition 3.1. Since $|I|$ is even, there exists a NIS operator of smoothing of order zero A_I such that

$$X^I(I - \mathcal{H}_q) = A_I \Box_b^{\frac{|I|}{2}}.$$

Therefore, if f is orthogonal to the null space of $\Box_b$, we obtain the second assertion.

Theorem 3.3 is proved.

4. Proof of the main theorem. ***4.1. The heat kernel.*** For convenience, we recall some results for the heat semigroup of unbounded operators $e^{-s\Box_b}$ via Hilbert space theory.

Theorem 4.1. *Let $\phi \in L^2_{(0,q)}(M)$ for $q \in [q_0, n-1-q_0]$, then*:

(1) $\lim_{s\to 0} \|e^{-s\Box_b}[\phi] - \phi\|_{L^2_{(0,q)}(M)} = 0$;

(2) *for $s > 0$,* $\|e^{-s\Box_b}[\phi]\|_{L^2_{(0,q)}(M)} \leq \|\phi\|_{L^2_{(0,q)}(M)}$;

(3) *if $\phi \in \mathrm{Dom}(\Box_b)$, then* $\|e^{-s\Box_b}[\phi] - \phi\|_{L^2_{(0,q)}(M)} \leq s\|\Box_b[\phi]\|_{L^2_{(0,q)}(M)}$;

(4) *for $s > 0$ and j nonnegative integer,* $\|(\Box_b)^j e^{-s\Box_b}[\phi]\|_{L^2_{(0,q)}(M)} \leq \left(\dfrac{j}{e}\right)^j s^{-j} \|\phi\|_{L^2_{(0,q)}(M)}$;

(5) $e^{-s\Box_b}\mathcal{H}_q[\phi] = \mathcal{H}_q e^{-s\Box_b}[\phi] = \mathcal{H}_q[\phi]$;

(6) $e^{-s\Box_b}[\phi] = (I - \mathcal{H}_q)e^{-s\Box_b}[\phi] + \mathcal{H}_q[\phi] = e^{-s\Box_b}(I - \mathcal{H}_q)[\phi] + \mathcal{H}_q[\phi]$;

(7) *for any $\phi \in L^2_{(0,q)}(M)$ and any $s > 0$, the Hilbert space valued form $e^{-s\Box_b}[\phi]$ satisfies*

$$[\partial_s + \Box_b][e^{-s\Box_b}[\phi]] = 0 \quad \textit{for} \quad s > 0,$$

$$\lim_{s\to 0} e^{-s\Box_b}[\phi] = \phi \quad \textit{in} \quad L^2_{(0,q)}((0,\infty)\times M);$$

(8) *for any $s \geq 0$, $e^{-s\Box_b}$ is a self-adjoint operator on $L^2_{(0,q)}(M)$.*

From Proposition 4.1 in [6], the operator $\mathcal{H}_q$ does exactly equal to $\mathcal{S}_q + \mathcal{S}'_q - I$. Therefore, $\mathcal{H}_q$ is a NIS operator of order zero, with $q_0 \leq q \leq n-1-q_0$. This fact and Theorem 4.1 imply the pseudolocal property (3.1) for $\mathcal{T} = e^{-s\Box_b}$ uniformly in $s > 0$.

Lemma 4.1. *Let M be a pseudoconvex finite type compact CR manifold for which the range of $\bar{\partial}_b$ is closed in L^2 and which satisfies the $D^\epsilon(q_0)$ condition. Let $|\alpha| = a \geq 0$ and $K \subset M$ be a compact set. Choose an integer N so that $N\epsilon_0 > 2n - 1 + a$. Then there is a constant C such that for each $s > 0$, if $x \in K$, and for all $\phi \in L^2_{(0,q)}(M)$ with $q_0 \leq q \leq n-1-q_0$,*

$$|X^\alpha e^{-s\Box_b}[\phi](x)| \leq C(1 + s^{-N})\|\phi\|_{L^2_{(0,q)}(M)}.$$

As a consequence of the condition of finite commutator type, for any derivative D on M,

$$|D^\alpha e^{-s\Box_b}[\phi](x)| \leq C(1 + s^{-N})\|\phi\|_{L^2_{(0,q)}(M)}.$$

Proof. Choose $\zeta \in C_0^\infty(M)$ with $\zeta(x) = 1$ for all $x \in K$. Then, pick cutoff functions $\zeta \prec \zeta_1 \prec \ldots \prec \zeta_N = \zeta'$. By Sobolev imbedding heorem, we have

$$|X^\alpha e^{-s\Box_b}[\phi](x)| = |X^\alpha \zeta(x) e^{-s\Box_b}[\phi](x)| \leq C\|\zeta e^{-s\Box_b}[\phi]\|_{H^{2n-1+a}}.$$

Applying the basic subelliptic estimate, we get

$$\|\zeta e^{-s\Box_b}[\phi]\|_{H^{2n-1+a}} \leq C\left[\|\zeta_1 \Box_b e^{-s\Box_b}[\phi]\|_{H^{2n-1+a-\epsilon_0}} + \|\zeta_1 e^{-s\Box_b}[\phi]\|_{L^2}\right].$$

If we repeat this argument N times, by (4) in Theorem 4.1, we will obtain

$$\|\zeta e^{-s\Box_b}[\phi]\|_{H^{2n-1+a}} \leq C\sum_{j=0}^{N} \|\zeta' \Box_b^j e^{-s\Box_b}[\phi]\|_{L^2_{(0,q)}(M)} \leq C(1 + s^{-N})\|\phi\|_{L^2_{(0,q)}(M)}.$$

Lemma 4.1 is proved.

We also have integral kernels for $X^\alpha e^{-s\Box_b}[\phi]$.

Lemma 4.2. *For each $x \in M$ and $s > 0$, and for each multiindex $|\alpha| = a$, there exist unique functions $H^{IJ}_{s,x,\alpha} \in L^2(M)$, where $|I| = |J| = q$, $q \in [q_0, n-1-q_0]$, so that*

$$X^\alpha e^{-s\Box_b}[\phi](x) = \sum_{|J|=q}{}' \left(\sum_{|I|=q}{}' \int_M H^{IJ}_{s,x,\alpha}(y)\phi_I(y)dV(y)\right)\bar{\omega}_J$$

or in short

$$X^\alpha e^{-s\Box_b}[\phi](x) = \int_M H_{s,x,\alpha}(y)\phi(y)dV(y),$$

where $\phi = \sum'_{|I|=q}\phi_I\bar{\omega}_I$. Moreover, if $K \subset M$ is compact and if C is the corresponding constant in Lemma 4.1, *then, if $x \in K$,*

$$\sum_{I,J} \int_M |H^{IJ}_{s,x,\alpha}(y)|^2 dy \leq C^2(1 + s^{-N})^2.$$

Proof. For each $s > 0$, $x \in M$, we define the mapping $\phi \mapsto X^\alpha e^{-s\Box_b}[\phi](x)$. By Lemma 4.1, this functional is bounded. Moreover, since

$$X^\alpha e^{-s\Box_b}[\phi](x) = \sum_{|J|=q}{}' \sum_{|I|=q}{}' \langle X^\alpha e^{-s\Box_b}[\phi_I\bar{\omega}_I], \bar{\omega}_J\rangle_{L^2}(x)\bar{\omega}_J,$$

and so by the Riesz representation theorem, there exist functions $H^{IJ}_{s,x,\alpha} \in L^2(M)$ so that

$$X^\alpha((e^{-s\Box_b})_{IJ}[\phi_I](x)) = \langle X^\alpha e^{-s\Box_b}[\phi_I\bar{\omega}_I], \bar{\omega}_J\rangle(x) = \int_M H^{IJ}_{s,x,\alpha}(y)\phi_I(y)dV(y).$$

Hence, by duality, we obtain

$$\sum_{I,J}{}' \int_M |H^{IJ}_{s,x,\alpha}(y)|^2 dy \leq C^2(1 + s^{-N})^2.$$

Lemma 4.2 is proved.

Definition 4.1. *We define the following* $(0,q)$*-forms, for* $q \in [q_0, n-1-q_0]$,

$$H_y^I(s,x) = \sum_{|J|=q}{}' H^{IJ}(s,x,y)\bar{\omega}_J(x),$$

$$H_x^J(s,y) = \sum_{|I|=q}{}' H^{IJ}(s,x,y)\bar{\omega}_I(y),$$

and, for $s > 0$,

$$H_s(x,y) = \sum_{\substack{|I|=q\\|J|=q}}{}' H^{IJ}(s,x,y)\overline{\omega}_J(x) \otimes \overline{\omega}_I(y).$$

It turns out that $e^{-s\Box_b}[\phi](x) = \sum_{|J|=q}' \langle H_x^J(s,\cdot),\phi\rangle_{L^2_{(0,q)}(M)}\bar{\omega}_J$.

For fixed $s > 0$, since $H^{IJ}(s,x,y) = H^{IJ}_{s,x,0}$, for all I, J, Lemma 4.2 says that the maps $y \mapsto H^{IJ}(s,x,y)$ belong to $L^2(M)$ for all I, J, and

$$e^{-s\Box_b}[\phi](x) = \sum_{|J|=q}{}' \left(\sum_{|I|=q}{}' \int_M H^{IJ}(s,x,y)\phi_I(y)dV(y) \right)\bar{\omega}_J.$$

We denote these sums as $\int_M H(s,x,y)\phi(y)dV(y)$.

Theorem 4.2. *For each fixed* $s > 0$ *and* $x \in M$, *the function* $y \mapsto H^{IJ}(s,x,y)$ *belongs to* $L^2(M)$, *so each integral above converges absolutely. Moreover, each component* $H^{IJ}(s,x,y)$ *of* $H(s,x,y)$ *satisfies*

(1) *for* $s > 0$ *and* $x, y \in M$, $H^{IJ}(s,x,y) = \overline{H^{JI}(s,y,x)}$;

(2) $[\partial_s + (\Box_b)_x][H_y^I](s,x) = [\partial_s + (\Box_b)_y][H_x^J](s,y) = 0$ *and, hence,*

$$[\partial_s + (\Box_b)_x][H_s(x,y)] = [\partial_s + (\Box_b)_y][H_s(x,y)] = 0;$$

(3) *for any integer* $j, k \geq 0$,

$$(\Box_b)_x^j(\Box_b)_y^k H_s^{IJ}(x,y) = (\Box_b)_x^{j+k} H_y^I(s,x) = (\Box_b)_y^{j+k} H_x^J(s,y);$$

(4) *for each* $s > 0$ *and* $y \in M$, *for any nonnegative integer* j, *each function*

$$x \mapsto (\Box_b)_x^j H_y^I(s,x)$$

is orthogonal to $\ker(\Box_b)$.

Proof. The proof of Theorem 4.2 is almost identical to the one in [8] (Theorem 5.1.2), hence it is omitted.

These heat kernels also provide the fundamental solutions for the initial-value problem on the whole space $\mathbb{R} \times M$.

Definition 4.2. *For $\psi \in \Lambda^{0,q}(C_0^\infty(\mathbb{R}\times M))$, we set*

$$\langle \mathbb{H}_x, \psi\rangle = \lim_{\epsilon\to 0^+} \int\limits_\epsilon^\infty \int\limits_M \widetilde{H}(s,x,y)\psi(s,y)dV(y)\,ds = \lim_{\epsilon\to 0^+} \int\limits_\epsilon^\infty \int\limits_M H(s,x,y)\psi(s,y)dV(y)\,ds,$$

where $\widetilde{H}$'s components are

$$\widetilde{H}^{IJ}(s,x,y) = \begin{cases} H^{IJ}(s,x,y), & \textit{if } s>0,\\ 0, & \textit{if } s\le 0.\end{cases}$$

Proposition 4.1. *The limit defining $\mathbb{H}_x$ exists. Moreover,*

$$[\partial_s + (\Box_b)_y][\mathbb{H}_x] = \delta_0\otimes\delta_x$$

in the sense of distributions (in component-wise), i.e.,

$$\big\langle \mathbb{H}_x, [-\partial_s + \Box_b]\psi\big\rangle = \psi(0,x).$$

Proof. Setting $\psi_s(y) = \psi(s,y)$, so $\psi_s \in \Lambda^{0,q}(C^\infty(M))$. Choose a positive integer N so that $N\epsilon > \dfrac{2n-1}{2}$. Choose $\zeta \prec \zeta_1 \prec \ldots \prec \zeta_N = \zeta'$ with $\zeta(x) = 1$. Then, once again, by Sobolev imbedding theorem and the basic subelliptic estimate applied N times, we obtain

$$\left|\int\limits_M H(s,x,y)\psi(y)dV(y)\right| = \left|\zeta e^{-s\Box_b}[\psi_s](x)\right| \le$$

$$\le C\|\zeta e^{-s\Box_b}[\psi_s]\|_{N\epsilon} \le$$

$$\le C[\|\zeta_1\Box_b[e^{-s\Box_b}[\psi_s]]\|_{(N-1)\epsilon} + \|\zeta_1 e^{-s\Box_b}[\psi_s]\|_0] \le$$

$$\le \ldots \text{repeating } N\text{-times as above} \ldots \le$$

$$\le C\sum_{j=0}^N \|\zeta'\Box_b^j[e^{-s\Box_b}[\psi_s]]\|_0.$$

Moreover, since the operators $\Box_b$ and $e^{-s\Box_b}$ commute,

$$\left|\int\limits_M H(s,x,y)\psi(y)dV(y)\right| \le C\sum_{j=0}^N \|\zeta' e^{-s\Box_b}[\Box_b^j\psi_s]\|_0 \le C\sum_{j=0}^N \|\Box_b^j\psi_s\|_0.$$

The right-hand side is uniformly bounded in s, and then, taking integral on $[\eta_1,\eta_2]$, we have

$$\left|\int\limits_{\eta_1}^{\eta_2}\int\limits_M H(s,x,y)\psi(y)dV(y)\,ds\right| \le C|\eta_2-\eta_1|\sup_s\sum_{j=0}^N \|\Box_b\psi_s\|_0.$$

We see that the left-hand side goes to zero as $\eta_2\to 0$, so the limit defining $\mathbb{H}_x$ exists. Again, let $\psi\in\Lambda^{0,q}(C_0^\infty(\mathbb{R}\times M)$, then

$$\big\langle \mathbb{H}_x, [-\partial_s + \square_b]\psi_s \big\rangle = \lim_{\epsilon\to 0} \int_\epsilon^\infty e^{-s\square_b}[[-\partial_s + \square_b]\psi_s]\, ds =$$

$$= -\lim_{\epsilon\to 0} \int_\epsilon^\infty \int_M H(s,x,y)\partial_s\psi(s,x)dV(y)\, ds +$$

$$+\lim_{\epsilon\to 0} \int_\epsilon^\infty \int_M H(s,x,y)\square_b\psi(s,y)dV(y)\, ds. \tag{4.1}$$

Now, for the first term, we get

$$-\int_\epsilon^\infty \int_M H(s,x,y)\partial_s\psi(s,x)dV(y)\, ds =$$

$$= {\sum_{|J|=q}}' \left({\sum_{|I|=q}}' \int_M H^{IJ}(\epsilon,x,y)\psi_I(\epsilon,y)dV(y) \right) \bar{\omega}_J +$$

$$+ \int_\epsilon^\infty {\sum_{|J|=q}}' \langle \partial_s H_x^J(s,\cdot), \psi(s,\cdot)\rangle \bar{\omega}_J\, ds. \tag{4.2}$$

For the second term,

$$\int_\epsilon^\infty \int_M H(s,x,y)\square_b\psi(s,y)dV(y)\, ds = \int_\epsilon^\infty {\sum_{|J|=q}}' \langle H_x^J(s,\cdot), \square_b\psi(s,\cdot)\rangle \bar{\omega}_J\, ds =$$

$$= \int_\epsilon^\infty {\sum_{|J|=q}}' \langle (\square_b)_y H_x^J(s,\cdot), \psi(s,\cdot)\rangle \bar{\omega}_J\, ds. \tag{4.3}$$

Hence, since $[\partial_s + (\square_b)_y]H_x^J(s,y) = 0$, (4.1), (4.2), and (4.3) imply

$$\langle \mathbb{H}_x, [-\partial_s + \square_b]\psi_s\rangle = \psi(0,x) = \langle \delta_0 \otimes \delta_x, \psi\rangle.$$

Proposition 4.1 is proved.

A remark that, by translation, we also have

$$\langle \mathbb{H}_x, [-\partial_{s+t} + \square_b]\psi\rangle = \psi(t,x).$$

4.2. Pointwise estimates for the heat kernel. We begin by recalling the scaled pullback of the heat equation on $\mathbb{R}\times M$ to $\mathbb{R}\times\mathbb{B}_0$ by

$$((\partial_s + \widehat{\square_b})\widehat{\phi}(s,u)) = r^2[(\partial_s + \square_b)\phi(s,x)]^\wedge.$$

Now, using the above changing of the variable $\Phi_{(s,x_0),r}$, with $s>0,\ x_0\in M,\ u,v\in\mathbb{B}_0$, we define the pullback of the heat kernel $H(s,x,y)$:

$$W^{IJ}(s,u,v) = W^{IJ}_{x_0,r}(s,u,v) = H^{IJ}(r^2 s, \Phi_{x_0,r}(u), \Phi_{x_0,r}(v))$$

for each $|I| = |J| = q,\ q_0 \leq q \leq n-1-q_0$, and $0 < r < R_0$. Hence, from previous sections, we have

$$[\partial_s + (\widehat{\Box_b})_u][W^I_v](s,u) = 0,$$

$$[\partial_s + (\widehat{\Box_b})_v][W^J_u](s,v) = 0,$$

where $W^I_v(s,u)$ and $W^J_u(s,v)$ are defined by the same formulas as $H^I_y(s,x)$ and $H^J_x(s,y)$.

In the same way, for $s > 0$, and $\phi \in \Lambda^{0,q}(C_0^\infty(\mathbb{B}_0))$, we define the scaled pullback of $e^{-s\Box_b}$ as

$$\mathbb{W}_s[\phi](u) = \int\limits_{\mathbb{B}_0} W(s,u,v)\phi(v)dv = \int\limits_{\mathbb{B}_0} H(r^2 s, \Phi_{x_0,r}(u), \Phi_{x_0,r}(v))\phi(v)dv.$$

The key point is that the norm of the operator $\mathbb{W}_s$ is bounded on $L^2_{(0,q)}(\mathbb{B}_0)$.

Lemma 4.3. *There is a constant C which is independent of x_0, r and $s > 0$ so that*

$$\|\mathbb{W}_s[\phi]\|_{L^2_{(0,q)}(\mathbb{B}_0)} \leq C|B(x_0,r)|^{-1}\|\phi\|_{L^2_{(0,q)}(\mathbb{B}_0)}.$$

Proof. Let $x \in B(x_0,r)$, by changing of variables, we have

$$\mathbb{W}_s[\phi](\Phi^{-1}_{x_0,r}(x)) = \int\limits_{\mathbb{B}_0} H(r^2 s, \Phi_{x_0,r}(\Phi^{-1}_{x_0,r}(x)), \Phi_{x_0,r}(v))\phi(v)dv =$$

$$= \int\limits_M H(r^2 s, x, y)(\phi \circ \Phi^{-1}_{x_0,r})(y) J_{x_0,r}(\Phi^{-1}_{x_0,r})\Phi^{-1}_{x_0,r}(y)(y)dV(y) =$$

$$= e^{-r^2 s\Box_b}[(\phi \circ \Phi^{-1}_{x_0,r})J_{x_0,r}\Phi^{-1}_{x_0,r}](x).$$

Since $\|e^{-r^2 s\Box_b}[(\phi \circ \Phi^{-1}_{x_0,r})J_{x_0,r}\Phi^{-1}_{x_0,r}]\|_{L^2} \leq \|(\phi \circ \Phi^{-1}_{x_0,r})J_{x_0,r}\Phi^{-1}_{x_0,r}\|_{L^2}$, it follows that

$$\int\limits_M |\mathbb{W}_s[\phi](\Phi^{-1}_{x_0,r}(x))|^2 dV(x) \leq \int\limits_M |\phi(\Phi^{-1}_{x_0,r}(x))|^2 (J_{x_0,r}(\Phi^{-1}_{x_0,r}(x)))^2 dV(x) =$$

$$= \int\limits_{\mathbb{B}_0} |\phi(u)|^2 (J_{x_0,r}\Phi^{-1}_{x_0,r}(\Phi_{x_0,r}(u)))^2 J_{x_0,r}\Phi_{x_0,r}(u)du \leq$$

$$\leq C|B(x_0,r)|^{-1} \int\limits_{\mathbb{B}_0} |\phi(u)|^2 du.$$

The last inequality is derived from the facts that $J_{x_0,r}\Phi^{-1}_{x_0,r}(\Phi_{x_0,r}(u)) = J_{x_0,r}\Phi_{x_0,r}(u)^{-1}$, and $J_{x_0,R}\Phi_{x_0,R}(u) \geq C^{-1}|B(x_0,r)|$ for $0 < r < R_0$ according to Theorem 3.1.

On the other hand,

$$\int\limits_M |\mathbb{W}_s[\phi](\Phi^{-1}_{x_0,r}(x))|^2 dV(x) \geq C^{-1}|B(x_0,r)| \int\limits_{\mathbb{B}_0} |\mathbb{W}_s[\phi](u)|^2 du.$$

Hence, we obtain

$$\|\mathbb{W}_s[\phi]\|_{L^2_{(0,q)}(\mathbb{B}_0)} \le C|B(x_0,r)|^{-1}\|\phi\|_{L^2_{(0,q)}(\mathbb{B}_0)}.$$

Lemma 4.3 is proved.

Next, we will obtain local estimates for the functions H^{IJ}'s, $|I|=|J|=q$, $q\in[q_0,n-1-q_0]$.

Theorem 4.3. *Let j, k, l be nonnegative integers. For every positive integer N, there is a constant $C_N=C_{N,j,k,l}$ so that if $|\alpha|=k$, $|\beta|=l$,*

$$|\partial_s^j X_x^\alpha X_y^\beta H^{IJ}(s,x,y)| \le \begin{cases} C_N\rho(x,y)^{-2j-k-l}|B(x,\rho(x,y))|^{-1}\left(\dfrac{s}{\rho(x,y)^2}\right)^N, & \text{if } s\le\rho(x,y)^2,\\ C_N s^{-j-k/2-l/2}|B(x,\sqrt{s})|^{-1}, & \text{if } s\ge\rho(x,y)^2,\end{cases}$$

for all (s,x,y) with $\rho_{\mathbb{R}\times M}((s,x),(0,y))=|s|^{1/2}+\rho(x,y)\le 1$.

The proof is based the scaling method which was introduced M. Christ [2], and then developed in higher dimensions by K. Koenig [6]. We need the following subelliptic estimate for the scaled pullback of $\Box_b$ operator on $\mathbb{B}_0$ which is a consequence of the subelliptic estimate of $\Box_b$ and Theorem 3.1.

Proposition 4.2. *Fix $\zeta,\zeta'\in C_0^\infty(\mathbb{B}_0)$ with $\zeta\prec\zeta'$. For smooth $(0,q)$-forms, $q\in[q_0,n-1-q_0]$, $\phi=\sum'_{|K|=q}\phi_K\widehat{\bar\omega}_K$ on $\mathbb{B}_0$ and $\delta\ge 0$, we have*

$$\|\zeta\phi\|^2_{\delta+\epsilon}\le C_\delta\left(\|\zeta'\widehat{\bar\partial}_b\phi\|^2_\delta+\|\zeta'\widehat{\bar\partial}^*_b\phi\|^2_\delta+\|\zeta'\phi\|^2_0\right),$$

where C_δ is a positive constant independent of x and $0<r<R_0$.

As a consequence, for $q\in[q_0,n-1-q_0]$, the heat operator $\partial_s+\widehat{\Box_b}$ also satisfies the subelliptic estimate

$$\|\zeta\phi\|^2_{\delta+\epsilon}\le C_\delta\left(\|\zeta'[\partial_s+\widehat{\Box_b}]\phi\|^2_\delta+\|\zeta'\phi\|^2_0\right),$$

for all smooth $(0,q)$-forms ϕ on $\mathbb{R}\times\mathbb{B}_0$.

***Proof of Theorem* 4.3.** We will prove the theorem with $N=0$ first. By compactness, if $R_0\le|s|^{1/2}+\rho(x,y)\le 1$, the estimates are trivial. Hence, it suffices to show that the estimates hold when $|s|^{1/2}+\rho(x,y)\le R_0$. Now, let fix $(s_0,x_0)\in\mathbb{R}\times M$, and let $(s,x)\in\mathbb{R}\times M$ be another point so that $\rho_{\mathbb{R}\times M}((s_0,x_0),(s,x))=r\le R_0$. There exists a unique point $(t_0,v_0)\in(-1,1)\times\mathbb{B}_0$ such that $(s,x)=(s_0+r^2t_0,\Phi_{x_0,r}(v_0))$. Let $\tau>0$ such that $|t_0|^{1/2}+|v_0|\ge\tau$.

For $(t_1,u),(t_2,v)\in(-1,1)\times\mathbb{B}_0$, we put

$$W^\#((t_1,u),(t_2,v))=H(r^2(t_2-t_1),\Phi_{x_0,r}(u),\Phi_{x_0,r}(v)),$$

in the sense that $\left(W^\#\right)^{IJ}((t_1,u),(t_2,v))=H^{IJ}(r^2(t_2-t_1),\Phi_{x_0,r}(u),\Phi_{x_0,r}(v))$ for all $|I|=$ $=|J|=q$. Then

$$[-\partial_{t_1}+(\widehat{\Box_b})_u][(W^\#)^I_v]=0,$$

$$[\partial_{t_2}+(\widehat{\Box_b})_v][(W^\#)^J_u]=0,$$

and

$$[\partial_s^jX_x^\alpha X_y^\beta H](r^2(t_2-t_1),\Phi_{x_0,r}(u),\Phi_{x_0,r}(v))=r^{-2j-k-l}[\partial_{t_2}^j\widehat{X}_u^\alpha\widehat{X}_v^\beta W^\#]((t_1,u),(t_2,v)).$$

Now, for $\phi \in C_0^\infty((-1,1)\times\mathbb{B}_0)$, set

$$\mathcal{T}^{\#}[\phi](t_1,u) = \iint\limits_{\mathbb{R}\times\mathbb{B}_0} W^{\#}((t_1,u),(t_2,v))\phi(t_2,v)dvdt_2$$

in the sense as above, i.e.,

$$\left(\mathcal{T}^{\#}[\phi](t_1,u)\right)_J = \left(\sum_{|I|=q}{}' \iint\limits_{\mathbb{R}\times\mathbb{B}_0} (W^{\#})^{IJ}((t_1,u),(t_2,v))\phi_I(t_2,v)dvdt_2\right)_J.$$

Then

$$\mathcal{T}^{\#}[\phi](t_1,u) = \sum_{|J|=q}{}' \left(\mathcal{T}^{\#}[\phi](t_1,u)\right)_J \widehat{\omega}_J.$$

The nonisotropic balls

$$B_1 = \left\{(t_1,u) : |t_1|^{1/2} + |u| < \frac{1}{3}\tau\right\},$$

$$B_2 = \left\{(t_2,v) : |t_2 - t_0|^{1/2} + |v - v_0| < \frac{1}{3}\tau\right\}$$

are disjoint.

Choose cutoff functions $\zeta \prec \zeta' \prec \zeta'' \in C_0^\infty(B_2)$ with $\zeta(t_0,v_0)=1$, and $\eta \prec \eta' \in C_0^\infty(B_1)$ with $\eta(0,0)=1$. Then, by the Sobolev inequality and the basic subelliptic estimate for the operator $\partial_{t_2} + \widehat{\Box_b}$, we have

$$\left|[\partial_{t_2}^j \widehat{X}_u^\alpha \widehat{X}_v^\beta (W^{\#})^J]((0,0),(t_0,v_0))\right| = \left|\zeta(t_0,s_0)[\partial_{t_2}^j \widehat{X}_u^\alpha \widehat{X}_v^\beta (W^{\#})^J]((0,0),(t_0,v_0))\right| \le$$

$$\le C \left\|\zeta'(W^{\#})^J((0,0),(\cdot,\cdot))\right\|_{2n+j+k+l} \le$$

$$\le C \left[\left\|\zeta''[\partial_{t_2} + (\widehat{\Box_b})_v](W^{\#})^J((0,0),(\cdot,\cdot))\right\|_{2n+j+k+l-\epsilon} + \right.$$

$$\left. + \left\|\zeta''(W^{\#})^J((0,0),(\cdot,\cdot))\right\|_0\right] \le$$

$$\le C \left\|\zeta''(W^{\#})^J((0,0),(\cdot,\cdot))\right\|_0 \le$$

$$\le C \sup_{\substack{\phi\in C^\infty(B_2)\\ \|\phi\|=1}} \left|\mathcal{T}^{\#}[\zeta'\phi](0,0)\right|,$$

where the last estimate follows from the fact that $[\partial_{t_2} + \widehat{\Box_b}]W^{\#}((0,0),(t,s)) = 0$ on B_1 containing the support of ζ'.

Now, to estimate the term with the supremum sign, again, we use the basic subelliptic for $-\partial_{t_1} + (\widehat{\Box_b})_u$,

$$\sup_{\substack{\phi\in C^\infty(B_2)\\ \|\phi\|=1}} \left|\mathcal{T}^\#[\zeta'\phi](0,0)\right| = \sup_{\substack{\phi\in C^\infty(B_2)\\ \|\phi\|=1}} \left|\eta(0,0)\mathcal{T}^\#[\zeta'\phi](0,0)\right| \le$$

$$\le C \sup_{\substack{\phi\in C^\infty(B_2)\\ \|\phi\|=1}} \left\|\eta\mathcal{T}^\#[\zeta'\phi]\right\|_{2n} \le$$

$$\le C \sup_{\substack{\phi\in C^\infty(B_2)\\ \|\phi\|=1}} \left[\left\|\eta' \underbrace{[-\partial_{t_1} + (\widehat{\Box_b})_u]\mathcal{T}^\#[\zeta'\phi]}_{=0 \text{ on } B_1 \text{ containing supp}(\eta')}\right\|_{2n-\epsilon} + \left\|\eta'\mathcal{T}^\#[\zeta'\phi]\right\|_0\right] \le$$

$$\le C \sup_{\substack{\phi\in C^\infty(B_2)\\ \|\phi\|=1}} \left\|\mathcal{T}^\#[\zeta'\phi]\right\|_0 \le$$

$$\le C\|\mathcal{T}^\#\|.$$

Therefore, we have shown that

$$\left|\partial_s^j X_x^\alpha X_y^\beta H(r^2 t_0, x_0, x)\right| \le C r^{-2j-k-l}\|\mathcal{T}^\#\|.$$

The last step is to estimate the norm $\|\mathcal{T}^\#\|$. Let ϕ, ψ be $(0,q)$-forms whose coefficients are $C_0^\infty((-1,1)\times\mathbb{B}_0)$, and let $\phi_s(v) = \phi(s,v)$, $\psi_t(u) = \psi(t,u)$. Then, in the sense as above, we get

$$\left|\iint\limits_{\mathbb{R}\times\mathbb{B}_0} \mathcal{T}^\#[\phi](t,u)\psi(t,u)\,du\,dt\right| =$$

$$= \left|\sum_{|J|=q}{}' \iint\limits_{\mathbb{R}\times\mathbb{B}_0} \left(\sum_{|I|=q}{}' \iint\limits_{\mathbb{R}\times\mathbb{B}_0} (W^\#)^{IJ}((t,u),(s,v))\phi_I(s,v)dv\,ds\right)\psi_J(t,u)\,du\,dt\right| =$$

$$= \left|\sum_{|J|=q}{}' \sum_{|I|=q}{}' \iint\limits_{\mathbb{R}\times\mathbb{B}_0}\iint\limits_{\mathbb{R}\times\mathbb{B}_0} (W^\#)^{IJ}((t,u),(s,v))\phi_I(s,v)\psi_J(t,u)\,ds\,dt\,du\,dv\right| =$$

$$= \left|\sum_{|J|=q}{}' \sum_{|I|=q}{}' \iiiint H^{IJ}(r^2(s-t), \Phi_{x_0,x}(u), \Phi_{x_0,r}(v))\phi_I(s,v)\psi_J(r,u)\,ds\,dt\,du\,dv\right| =$$

$$= \left|\sum_{|J|=q}{}' \sum_{|I|=q}{}' \iiiint H^{IJ}(r^2 s, \Phi_{x_0,x}(u), \Phi_{x_0,r}(v))\phi_I(s+t,v)\psi_J(t,u)\,ds\,dt\,du\,dv\right| \le$$

$$\le C\iint\limits_{\mathbb{R}^2} \|\mathbb{W}_s[\phi_{s+t}]\|_{L^2_{(0,q)}(\mathbb{B}_0)}\|\psi_t\|_{L^2_{(0,q)}(\mathbb{B}_0)}\,dsdt.$$

Now, by Lemma 4.3, $\|\mathbb{W}_s[\phi_{s+t}]\|_{L^2_{(0,q)}} \le C|B(x_0,r)|^{-1}\|\phi_{s+t}\|_{L^2_{(0,q)}(\mathbb{B}_0)}$. Then

$$\left| \iint\limits_{\mathbb{R}\times\mathbb{B}_0} \mathcal{T}^{\#}[\phi](t,u)\psi(t,u)\,du\,dt \right| \leq$$

$$\leq C|B(x_0,r)|^{-1}\|\phi\|_{\mathbb{R}\times\mathbb{B}_0}\|\psi\|_{\mathbb{R}\times\mathbb{B}_0}.$$

Hence, we obtain

$$\left|\partial_s^j X_x^\alpha X_y^\beta H(s,x,y)\right| \leq C(\rho_{\mathbb{R}\times M}(s,x),(0,y))^{-2j-k-l}|B(x,\rho_{\mathbb{R}\times M}(s,x),(0,y))|^{-1}. \qquad (4.4)$$

This implies the statement of Theorem 4.3 when $N=0$. To deal with the case $N>0$, we must use the facts that $s\mapsto H^{IJ}(s,x,y)$ is a smooth function when $x\neq y$, and vanishes to infinite order as $s\to 0$ by Proposition 4.1. Hence, applying Taylor's formula and integrating by parts, we get

$$|H^{IJ}(s,x,y)| \leq \frac{1}{(N-1)!}\int\limits_0^s |\partial_t^N H(t,x,y)|(s-t)^{N-1}dt \leq$$

$$\leq C_0\frac{1}{(N-1)!}\rho(x,y)^{-2N}|B(x,\rho(x,y)|^{-1}\int\limits_0^s (s-t)^{N-1}dt \leq$$

$$\leq C_0\frac{1}{N!}\left(\frac{s}{\rho(x,y)^2}\right)^N |B(x,\rho(x,y)|^{-1},$$

when $s\leq\rho(x,y)$, and replace $\rho(x,y)$ by $s^{1/2}$ to obtain the expected estimate. This argument also provides the same results when $s\geq\rho(x,y)$. Finally, applying (4.4), estimates for other derivatives of $H^{IJ}(s,x,y)$ are handled in the same way.

Theorem 4.3 is proved.

Next, the action of the heat operator on bump functions is provided.

Theorem 4.4. *Fix $s>0$, for each multiindex α, there is an integer N_α and a constant C_α so that if $\phi\in\Lambda^{0,q}(C_0^\infty(B(x,r)))$, then*

$$|X_x^\alpha e^{-s\Box_b}[\phi](x)| \leq C_\alpha r^{-|\alpha|}\sup_{y\in M}\sum_{|\beta|\leq N_\alpha} r^{|\beta|}|X^\beta\phi(y)|. \qquad (4.5)$$

Proof. By Sobolev type Theorem 3.3 and the argument before, for $\phi\in(\ker\Box_b)^\perp$ and $0<r<$ $<R_0$, we have

$$r^{|\alpha|}|X^\alpha e^{-s\Box_b}[\phi](x)| \leq$$

$$\leq C|B_M(x,r)|^{-1/2}\sum_{0\leq|\beta|\leq L_m,|\beta|\,\text{even}} r^{|\beta|+|\alpha|}\|X^{\alpha+\beta}e^{-s\Box_b}[\phi]\|_{L^2_{(0,q)}} \leq$$

$$\leq C|B_M(x,r)|^{-1/2}\sum_{l=0,\,l\,\text{even}}^{L_m} r^{l+|\alpha|}\sum_{j=0}^{\frac{|\alpha|+l}{2}}\|e^{-s\Box_b}[(\Box_b)^j\phi]\|_{L^2_{(0,q)}} \leq$$

$$\leq C|B_M(x,r)|^{-1/2}\sum_{l=0,\,l\,\text{even}}^{L_m} r^{l+|\alpha|}\sum_{|\beta|=0}^{l+|\alpha|}\|X^\beta\phi\|_{L^2_{(0,q)}},$$

which yields the desired estimate in this case.

If $r \geq R_0$, applying Theorem 3.1 for $r = R_0$, we obtain

$$|X^\alpha e^{-s\Box_b}[\phi](x)| \leq Cr^{-|\alpha|}|B_M(x, R_0)|^{-1/2} \sum_{l=0,\, l \text{ even}}^{L_m} R_0^{l+|\alpha|} \sum_{|\beta|=0}^{l+|\alpha|} \|X^\beta \phi\|_{L^2_{(0,q)}(B_M(x,2r))} \leq$$

$$\leq C'r^{-|\alpha|} \sum_{|\beta|=0}^{L_m+|\alpha|} r^{|\beta|} \sup_{B(x,2r)} |X^\beta \varphi|.$$

For the last inequality, note that $r \leq cR_0$ since M is compact. This yields (4.5) for $\phi \in (\ker \Box_b)^\perp$. The general case follows from Theorem 4.1 and the fact that $\mathcal{H}_q$ is NIS of order zero.

Theorem 4.4 is proved.

References

1. *Beals R., Greiner P. C., Stanton N. K.* The heat equation on a CR manifold // J. Different. Geom. – 1984. – **20**. – P. 343 – 387.
2. *Christ M.* Regularity properties of the $\bar{\partial}_b$ equation on weakly pseudoconvex CR manifolds of dimension 3 // J. Amer. Math. Soc. – 1988. – **1**. – P. 587 – 646.
3. *Chen S. C., Shaw M. C.* Partial differential equations in several complex variables // Stud. Adv. Math. – 2001. – **21**.
4. *Fefferman C.* On Kohn's microlocalization of $\bar{\partial}$ problem // Modern Methods in Complex Analysis: Princeton Conf. in Honor of Gunning and Kohn / Eds T. Bloom, D. W. Catlin, J. P. D'Angelo, Y. T. Siu. – Princeton Univ. Press, 1995. – P. 119 – 134.
5. *Hörmander L.* Hypoelliptic second order differential equations // Acta. Math. – 1967. – **119**. – P. 147 – 171.
6. *Koenig K. D.* On maximal Sobolev and Hölder estimates for tangential Cauchy – Riemann operator and boundary Laplacian // Amer. J. Math. – 2002. – **124**. – P. 129 – 197.
7. *Nagel A.* Analysis and geometry on Carnot – Caratheodory spaces // http://www.math.wisCedu/ nagel/2005Book.pdf.
8. *Nagel A., Stein E. M.* The $\Box_b$-heat equation on pseudoconvex manifolds of finite type in $\mathbb{C}^2$ // Math. Z. – 2001. – **238**. – S. 37 – 88.
9. *Nagel A., Stein E. M., Wainger S.* Balls and metrics defined by vector fields I: Basic properties // Acta Math. – 1985. – **155**. – P. 103 – 147.
10. *Zampieri G.* Complex analysis and CR geometry // Univ. Lect. Ser. – Providence: Amer. Math. Soc., 2008. – **43**. – 200 p.

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