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THE RADICAL FORMULA FOR NONCOMMUTATIVE RINGS*

РАДИКАЛЬНА ФОРМУЛА ДЛЯ КІЛЕЦЬ, ЩО НЕ КОМУТУЮТЬ

We determine some classes of left modules satisfying the radical formula in a noncommutative ring. We also show that, under a certain condition, a finitely generated module over an HNP-ring (the generalization of Dedekind domain) both satisfies the radical formula and can be decomposed into a direct sum of torsion modules and extending modules.

Визначено деякі класи лівих модулів, які задовольняють радикальну формулу в кільці, що не комутує. Також показано, що (за деякої умови) скінченнопороджений модуль над HNP-кільцем (що є узагальненням області Дедекінда) не лише задовольняє радикальну формулу, але й може бути розкладений у пряму суму модулів кручення та модулів розтягу.

1. Introduction. It is well known that the set of nilpotent elements of a commutative ring R with unity forms an ideal which is equal to the intersection of all the prime ideals. This notion has been generalized in [9] to modules. Let N be a proper submodule of an R-module M. The radical of N in M, denoted by $\mathrm{rad}_M(N)$, is defined to be the intersection of all prime submodules of M containing N. If there is no prime submodule containing N, then we put $\mathrm{rad}_M(N) = M$. The envelope submodule $RE_M(N)$ of N in M is a submodule of M generated by the set

$$E_M(N) = \{rm \colon r \in R \text{ and } m \in M \text{ such that } r^n m \in N \text{ for some } n \in \mathbb{N}\}.$$

Then N is said to satisfy the radical formula in M if $\mathrm{rad}_M(N) = RE_M(N)$. Unfortunately, not every module satisfies the radical formula. The radical formula and relations between Dedekind domain and the radical formula were studied in many papers (see, for example, [1, 2, 4, 10, 11, 14–16]). Hence, by the use of these concepts, some characterizations for Dedekind domains and modules were obtained in many results. Unfortunately, in noncommutative case, there are not enough useful results between the radical formula and an HNP-ring, which is one of the generalizations of Dedekind domian.

Let R be a noncommutative ring with unity. Then an ideal P of R is called prime if, for $a, b \in R$, either a or b is in P whenever $aRb \subseteq P$. For an ideal I of R, the radical of I, denoted by $\mathrm{Rad}\,(I)$, is defined as the intersection of all prime ideals containing I. The radical of an ideal is characterized by m-system in [6] (Theorem 10.7). A nonempty set $S \subseteq R$ is called an m-system if, for any $a, b \in S$, there exists $r \in R$ such that $arb \in S$. Then, for any ideal I of any ring R, it follows that

$$\mathrm{Rad}\,(I) = \{s \in R \colon \text{every } m\text{-system containing } s \text{ meets } I\}.$$

The concepts of prime ideals and radicals of a noncommutative ring have been generalized to modules in [13]. Let M be a module over a noncommutative ring. A proper submodule P of M is prime if, for any $r \in R$ and $m \in M$ such that $rRm \subseteq P$, either $rM \subseteq P$ or $m \in P$. Then similarly, $\mathrm{rad}_M(N)$ is defined as the intersection of all prime submodules containing N for a submodule N

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of M. As in commutative ring theory, it is natural to ask whether the radical of a submodule of a module over a noncommutative ring has a simple description and there is any relation among the radical formula, the HNP-ring.

In this paper, we are interested in the radical of a submodule of a module M over a noncommutative ring and describe a submodule $W_M(N)$ generated by a strongly nilpotent element on a submodule N. In fact, this submodule is the generalization of the envelope of a submodule over a commutative ring. Then we prove that the radical of a projective module has a simple description and so show that a projective module satisfies the radical formula. Moreover, we give a simple description for radical of a submodule N of a module M that $\mathrm{rad}_M(N) = W_M(N) = W_R(0)M + N$ for a ring R such that $\bar{R} = R/\mathrm{rad}_R(0)$ is semisimple.

In the last section, we deal with an HNP-ring to show that under a condition, a finitely generated module over an HNP- ring, the generalization of Dedekind domain, both satisfies the radical formula and can be decomposed into a direct sum of a torsion module and an extending module. (The notion of extending modules is one of the other important concepts in module theory and many authors focus on this topic [5, 12].) Moreover, by using the prime submodule, we also prove that the left socle of a left extending ring is in the Jacobson radical under the condition.

2. The radical of a submodule. Throughout the paper R will denote a ring with identity and M be an unital left module over R. We start to prove some properties of a prime submodule of $_RR$.

Lemma 2.1. *Let P be a left ideal of R. Then*:

(i) *If a nonzero homomorphism f of $\mathrm{End}_R(R/P)$ is injective, then P is a prime submodule of $_RR$.*

(ii) *If P is a prime submodule such that $xRy \subseteq P$ whenever $xy \in P$, then every homomorphism f of $\mathrm{End}_R(R/P)$ is injective.*

Proof. Let $xRy \subseteq P$ for $y \in R$ and $x \in R$. If y is not in P, define a homomorphism $f \in \mathrm{End}_R(R/P)$ such that $f(l + P) = ly + P$ so $f(xr + P) = 0$ for all $r \in R$. This means that $xR \subseteq P$.

Let $f \in \mathrm{End}_R(R/P)$. Then there is $y \in R$ such that $f(1 + P) = y + P$. If $x + P \in \mathrm{Ker}\, f$, then $xy \in P$ and so $xRy \subseteq P$. Hence, $y \in P$ or $xR \subseteq P$ and so $\mathrm{Ker}\, f = 0$.

***Corollary* 2.1.** *Let P be a left ideal of R such that $xRy \subseteq P$ whenever $xy \in P$. If P is a prime submodule of $_RR$, then R/P is an indecomposable R-module.*

Proof. Let $R/P = A/P \oplus B/P$ for some $A, B \subseteq R$. Consider the projection homomorphism from $f(\overline{a}, \overline{b}) = \overline{a}$ where $\overline{a} = a + P$. Then $\mathrm{Ker}\, f = 0$ by Lemma 2.1.

Now we define the concept of strongly nilpotent element on a submodule. Let N be a submodule of a module M. An element $x = am$ of an R-module M on N is called strongly nilpotent if, for each sequence

$$\{a_i \in R \colon a_{i+1} \in a_i R a_i \ \text{ and } \ a_0 = a, \ \ i \in \mathbb{Z}\},$$

there is a positive integer k such that $a_k Rm \subseteq N$. We use the notation $S_M(N)$ to denote the set of strongly nilpotent elements of M on N. In generally, $S_M(N)$ does not need to be a submodule. Then the submodule generated by $S_M(N)$ is denoted by $W_M(N)$. In the commutative ring, it is easy prove that $W_M(N) = RE_M(N)$. Therefore, it is a generalization of the envelope of a submodule and so the radical of a submodule N of a module may not equal to $W_M(N)$ (see, for example, [11]). Similarly, N is said to satisfy the radical formula when $W_M(N) = \mathrm{rad}_M(N)$. If every submodule of a module M satisfies the radical formula, then M is said to satisfy the radical formula. If every R-module M satisfies the radical formula, then R is said to satisfy the radical formula.

Let I be a submodule of ${}_RR$. It is clear that a is a strongly nilpotent element of ${}_RR$ on I if, for each sequence

$$\{a_i \in R\colon a_{i+1} \in a_iRa_i \text{ and } a_0 = a, \ i \in \mathbb{Z}\},$$

there is a positive integer k such that $a_kR \subseteq I$. It is also easy to see that every element of a left T-nilpotent ideal L of R is strongly nilpotent and so L is in $W_R(0)$.

Lemma 2.2. *Let M be any finitely generated R-module and N, L be submodules of M. Then $W_M(N) + W_M(L) = M$ if and only if $N + L = M$.*

Proof. We know that $S_M(N) \cup S_M(L)$ generates $W_M(N) + W_M(L)$. Let $N + L \neq M$. Since M is a finitely generated R-module, there exists a maximal submodule T of M such that $N + L \subseteq T$. Since T is also a prime submodule of M, we have $W_M(N) \subseteq T$ and $W_M(L) \subseteq T$ and so $W_M(N) + W_M(L) \subseteq T$. This is a contradiction. Then $N + L = M$.

Since $N \subseteq W_M(N)$, $L \subseteq W_M(L)$ and $N + L = M$, it follows that $W_M(N) + W_M(L) = M$.

Lemma 2.3. *Let M, N be R-modules and B be a submodule of M. Then for an R-module homomorphism $f\colon M \to N$, $f(W_M(B)) \subseteq W_N(f(B))$.*

In particular, if f is an epimorphism and $\ker f \subseteq B$, the inverse inclusion holds.

Proof. Let $x = am \in S_M(B)$ where $m \in M$ and $a \in R$. Then, for each sequence $a_0, a_1, a_2, \ldots$ such that $a_0 = a$ and $a_{i+1} \in a_iRa_i$, there is a positive integer k such that $a_kRm \subseteq B$. Hence, $f(a_kRm) = a_kRf(m) \subseteq f(B)$ and so $f(x) \in S_N(f(B))$. Therefore, $f(W_M(B)) \subseteq W_N(f(B))$.

Let $x = tn \in S_N(f(B))$ where $n \in N$ and $t \in R$. Let $n = f(b)$ for some $b \in B$. Then, for each sequence $a_0, a_1, a_2, \ldots$ such that $a_0 = t$ and $a_{i+1} \in a_iRa_i$, there is a positive integer k such that $a_kRn \subseteq f(B)$. Thus we get $a_krn = f(b_r)$ where $b_r \in B$ and $r \in R$. Then $f(a_krb - b_r) = 0$ and so, for all $r \in R$, it follows that $a_kRb \subseteq B$. We have $x \in f(S_M(B))$. This means that $S_N(f(B)) \subseteq f(S_M(B))$. Therefore, $W_N(f(B)) \subseteq f(W_M(B))$.

Lemma 2.4. *Let M be an R-module and let N be a submodule of M. Then $W_R(N : M)M \subseteq W_M(N)$.*

Proof. Let $x = am$ where $a \in S_R(N : M)$ and $m \in M$. Since $a \in S_R(N : M)$, for each sequence a_0, $a_1, a_2, \ldots$, where $a_0 = a$ and $a_{i+1} \in a_iRa_i$, there is a positive integer k such that $a_kR \subseteq (N : M)$. Therefore, we get that $a_kRM \subseteq N$ and so $a_kRm \subseteq N$. Then $am \in S_M(N)$ and so it follows that $W_R(N : M)M \subseteq W_M(N)$.

Lemma 2.5. *Let N be a submodule of a module M. Then we have $W_M(N) \subseteq \mathrm{rad}_M(N)$.*

Proof. Let $x = a_0m \in S_M(N)$ and assume that $x \notin \mathrm{rad}_M(N)$. Then there exists a prime submodule P of M containing N such that $a_0m \notin P$. Since P is a prime submodule of M, we get that $a_0Ra_0m \subseteq P$ and so there exists an element $a_1 \in a_0Ra_0$ such that $a_1m \notin P$. Similarly, by the hypothesis on P, we also get that $a_1Ra_1m \subseteq P$. So there exists an element $a_2 \in a_1Ra_1$ such that $a_2m \notin P$. Therefore, we obtain the sequence $a_0, a_1, a_2, \ldots$ such that $a = a_0$ and $a_{i+1} \in a_iRa_i$, $i = 0, 1, 2, 3, \ldots$, but there does not exist any positive integer k such that a_km is in P and so a_kRm is not in N. This means that a_0m is not a strongly nilpotent element of M on N, a contradiction. Then $S_M(N) \subseteq \mathrm{rad}_M(N)$ and the proof is completed.

Theorem 2.1. *Let I be a left ideal of a ring R. Then*

$$W_R(I) \subseteq \mathrm{rad}_R(I) \subseteq \mathrm{rad}_R(IR) = \mathrm{Rad}(IR) = W_R(IR). \tag{2.1}$$

Proof. Since every prime ideal of R is a prime submodule of ${}_RR$, we get that $\mathrm{rad}_R(I) \subseteq \subseteq \mathrm{Rad}(IR)$. Therefore, we have $W_R(I) \subseteq \mathrm{rad}_R(I) \subseteq \mathrm{Rad}(IR)$ and $W_R(IR) \subseteq \mathrm{rad}_R(IR) \subseteq \subseteq \mathrm{Rad}(IR)$.

Let $x \notin S_R(IR)$. Then there is a sequence

$$\{x_i \in R\colon x_{i+1} \in x_iRx_i \quad \text{and} \quad x_0 = x, \quad i \in \mathbb{Z}\}$$

such that, for all positive integer k, $x_kR \subseteq IR$ and so $x_k \notin IR$. It is clear that $S = \{x_0, x_1, x_2, \ldots\}$ is m-system but S does not meet IR. Therefore, we get that $x \notin \mathrm{Rad}(IR)$. Thus $\mathrm{Rad}(IR) \subseteq \subseteq W_R(IR)$ and this completes the proof.

***Corollary* 2.2.** *Let I be a submodule of ${}_RR$. If I satisfies one of the following conditions*:

(i) *every element of IR is strongly nilpotent on I*,

(ii) *I is an ideal of a ring R*,

then $\mathrm{rad}_R(I) = W_R(I)$.

Proof. (i) Let x be in $W_R(IR)$. Then, for each sequence $\{x_i \in R\colon x_{i+1} \in x_iRx_i$ and $x_0 = x$, $i \in \mathbb{Z}\}$, there is a positive integer k such that $x_kR \subseteq IR$ and so $x_k \in IR$. Hence, by the hypothesis, for each sequence $\{a_i \in R\colon a_{i+1} \in a_iRa_i$ and $a_0 = x_k$, $i \in \mathbb{Z}\}$, there is a positive integer t such that $a_tR \subseteq I$. This means that x is strongly nilpotent element on I and so $x \in W_R(I)$.

(ii) It is clear from (i).

Theorem 2.2. *Let $M = Rm$ be an R-module. Then we have*:

(i) $W_M(0) = W(\mathrm{ann}(m))m$,

(ii) *if every element of* $\mathrm{ann}(m)R$ *is strongly nilpotent on* $\mathrm{ann}(m)$, *then* $\mathrm{rad}_M(0) = W_M(0)$.

Proof. (i) Let $M = Rm$ and $I = \mathrm{ann}(m)$. Then I is an left ideal of R. Let $x = rdm \in S_M(0)$. Then, for each sequence $a_0,\ a_1,\ a_2, \ldots$, where $a_0 = rd$ and $a_{i+1} \in a_iRa_i$, there is a positive integer k such that $a_kRm = 0$ and so $a_kR \subseteq \mathrm{ann}(m)$. This means that rd is in $S_R(\mathrm{ann}(m))$ and so $W_M(0) \subseteq W(\mathrm{ann}(m))m$. The converse is clear and we get the first equality.

(ii) Since $M = Rm$ is isomorphic to R/I, it follows that $W_M(0)$ is isomorphic to $W_{R/I}(0) = = W_R(I)/I$. Then similarly, we get that $\mathrm{rad}_M(0)$ is isomorphic to $\mathrm{rad}_{R/I}(0) = \mathrm{rad}_R(I)/I$. On the other hand, by Corollary 2.2, we have that $\mathrm{rad}_R(I) = W_R(I)$ and so $\mathrm{rad}_M(0)$ is isomorphic to $W_M(0)$. Therefore, by Lemma 2.3, we get $\mathrm{rad}_M(0) = W_M(0)$.

Lemma 2.6. *Let M be an R-module and N be a submodule of M. Then we have*:

(i) $W_{M/N}(0) = (W_M(N))/N$,

(ii) $\mathrm{rad}_{M/N}(0) = \mathrm{rad}_M(N)/N$.

Proof. (i) It is sufficient to show that $S_{M/N}(0) = \{r(x + N) : rx \in S_M(N)\}$.

Let $\bar{m} = r(m + N) \in S_{M/N}(0)$. Then, for each sequence $a_0, a_1, a_2, \ldots$, where $a_0 = r$ and $a_{i+1} \in a_iRa_i$, there is a positive integer k such that $a_kR\bar{m} = \bar{0}$. Thus, $a_kR\bar{m} = (a_kRm+N)/N = 0$ and so $a_kRm \subseteq N$. This means that $rm \in S_M(N)$. Therefore, $\bar{m} = rm + N \in \{rx + N\colon rx \in S_M(N)\}$.

Let $\bar{m} = m + N \in \{rx + N : rx \in S_M(N)\}$. We may assume that $m = rx \in S_M(N)$. Then, for each sequence $a_0, a_1, a_2, \ldots$, where $a_0 = r$ and $a_{i+1} \in a_iRa_i$, there is a positive integer k such that $a_kRx \subseteq N$ and so we have $a_kR(x + N) = 0 + N$. Hence it follows that $r(x + N) \in S_{M/N}(0)$.

(ii) It is clear.

Lemma 2.7. *Let M and M^* be R-modules. Then we have*:

(i) $W_M(0) \oplus W_{M^*}(0) = W_{M\oplus M^*}(0)$,

(ii) $\mathrm{rad}_M(0) \oplus \mathrm{rad}_{M^*}(0) = \mathrm{rad}_{M\oplus M^*}(0)$.

Proof. It is enough to prove (i).

It is clear that the set $A = \{(ra, 0) : ra \in S_M(0)\}$ (the set $B = \{(0, kb) : kb \in S_{M^*}(0)\}$) generates the submodule $W_M(0) \oplus 0$ ($0 \oplus W_{M^*}(0)$, respectively).

Let $x \in A \cup B$. We may assume that $x = (rm, 0) \in A$ and so $rm \in S_M(0)$. Then, for each sequence $a_0, a_1, a_2, \ldots$, such that $a_0 = r$ and $a_{i+1} \in a_i R a_i$, there is a positive integer k such that $a_k Rm = 0$ and so $a_k R(m, 0) = 0$. Then $r(m, 0) \in S_{M\oplus M^*}(0)$. This means that $W_M(0) \oplus \oplus W_{M^*}(0) \subseteq W_{M\oplus M^*}(0)$.

Let $x = r(m, n) \in S_{M\oplus M^*}(0)$. Then, for every sequence $a_0, a_1, a_2, \ldots$ such that $a_0 = r$ and $a_{i+1} \in a_i R a_i$, there is a positive integer k such that $a_k R(m, n) = 0$, and it follows that $a_k Rm = 0$ and $a_k Rn = 0$. Therefore, $r(m, n) = r(m, 0) + r(0, n) \in W_M(0) \oplus W_{M^*}(0)$. This means that $W_{M\oplus M^*}(0) \subseteq W_M(0) \oplus W_{M^*}(0)$.

By using the fact that every module is a homorphic image of a free module and Lemma 2.7, we get the following result.

Theorem 2.3. *Let R be a ring. If every free R-module satisfies the radical formula, then so does every R-module.*

Proposition 2.1. *Let M be a projective R-module. Then we have*

$$W_R(0)M = W_M(0) = \mathrm{rad}_M(0) = \mathrm{rad}_R(0)M.$$

Proof. Let M be a projective R-module. Then there exists a free R-module F and an R-module A such that $F = M \oplus A$.

Firstly, we prove that our claim is true for F. Let $\{x_i : i \in I\}$ be a basis for F. Then $F = \oplus Rx_i$ and so each $x \in F$ has a unique expansion $x = \sum_{i\in I} r_i x_i$, where $r_i \in R$ and almost all $r_i = 0$. Define a homomorphism φ_i from F to R by $\varphi_i(x) = r_i$. Then φ_i is an epimorphism for all $i \in I$ and we obtain $x = \sum_{i\in I} \varphi_i(x)x_i$.

Let $u = \sum_{i\in I} r_i x_i \in W_F(0)$ where $r_i \in R$ and almost all $r_i = 0$. Thus, $u = \sum_{i\in I} \varphi_i(u)x_i$ and, by Lemma 2.3, we have $u = \sum_{i\in I} \varphi_i(u)x_i \in W_R(0)F$. Now, we get $W_F(0) \subseteq W_R(0)F$ and so $W_F(0) = W_R(0)F$.

Take $m \in W_M(0)$. By Lemma 2.7, it follows that $W_F(0) = W_M(0) \oplus W_A(0)$ and we have $m \in W_F(0) = W_R(0)F = W_R(0)M \oplus W_R(0)A$. This implies that $m = \sum r_i m_i + \sum k_j a_j$ where r_i, $k_j \in W_R(0)$, $m_i \in M$ and $a_j \in A$. Therefore, we get $m = \sum r_i m_i \in W_R(0)M$ and $W_R(0)M = W_M(0)$.

By the using the similar argument, we get $\mathrm{rad}_M(0) = \mathrm{rad}_R(0)M$. Therefore, we complete the proof since $\mathrm{rad}_R(0) = W_R(0)$.

Theorem 2.4. *Let M/N be a projective R-module and N be a submodule of M. Then we get*

$$\mathrm{rad}_M(N) = W_M(N) = W_R(0)M + N.$$

Proof. We know that $\mathrm{rad}_{M/N}(0) = (\mathrm{rad}_M(N))/N$ and $W_M(N)/N = W_{M/N}(0)$. Since M/N is a projective R-module, we have $\mathrm{rad}_{M/N}(0) = W_{M/N}(0) = W_R(0)(M/N)$ and so $W_M(N)/N = = (W_R(0)M + N)/N$. Therefore, we obtain $W_M(N) = W_R(0)M + N = \mathrm{rad}_M(N)$.

***Corollary* 2.3.** *Let N be a submodule of an R-module M such that M/N is a projective R-module and $W_R(0)M \subseteq N$. Then we have* $\mathrm{rad}_M(N) = W_M(N) = N$.

Proof. It is clear by Theorem 2.4.

Theorem 2.5. *Let R be a ring such that $\bar{R} = R/\mathrm{rad}_R(0)$ is semisimple and let N be a submodule of an R-module M. Then we have* $\mathrm{rad}_M(N) = W_M(N) = W_R(0)M + N$.

Proof. Assume that $N = 0$. Then it will be enough to show that $\mathrm{rad}_M(0) = W_R(0)M$. Since $\bar{R}$ is semisimple, we know that $\bar{M} = M/W_R(0)M$ is a semisimple and so $W_R(0)\bar{M} = \mathrm{rad}_{\bar{M}}(0) = \bar{0}$. On the other hand, since $W_R(0)M \subseteq \mathrm{rad}_R(0)M = \mathrm{rad}_M(0)$, we get

$$\mathrm{rad}_{\bar{M}}(0) = \mathrm{rad}_M(0)/W_R(0)M = \bar{0}.$$

This means that $\mathrm{rad}_M(0) = W_R(0)M$.

Now let $N \neq 0$. Then we obtain

$$\mathrm{rad}_{M/N}(0) = W_R(0)(M/N).$$

Therefore, $\mathrm{rad}_M(N) = W_R(0)M + N = W_M(N)$.

***Corollary* 2.4.** *Let R be a ring such that $\bar{R} = R/W_R(0)$ is semisimple. Then R satisfies the radical formula.*

Proof. Let M be any R-module. Then, by Theorem 2.5, we get $\mathrm{rad}_M(0) = W_R(0)M$. If $W_R(0)M \subseteq W_M(0) \subseteq \mathrm{rad}_M(0)$, we have $W_R(0)M = W_M(0) = \mathrm{rad}_M(0)$. This means that M satisfies radical formula.

3. Modules over $\boldsymbol{HNP}$-rings. Let M be a finitely generated module over an HNP-ring R and $M = \oplus_{i=1}^n Rm_i \oplus K$ where K is a projective module for some element $m_i \in M$. Assume that $\mathrm{ann}(m_i) = \mathrm{ann}(rm_i)$ for all non zero elements $r \in R$ and $i \in \{1, \ldots, n\}$.

Theorem 3.1. *With the above notations, we have*

$$W_R\left(\bigcap_{i=1}^n \mathrm{ann}(m_i)\right)\left(\bigoplus_{i=1}^n Rm_i\right) \oplus W_R(0)K = W_M(0).$$

Proof. Let M be a module. Then $M = \oplus_{i=1}^n Rm_i \oplus K$ for some elements $m_1, \ldots, m_t$ of M and a projective module K.

We show that $W_R\big(\bigcap_{i=1}^n \mathrm{ann}(m_i)\big)\big(\bigoplus_{i=1}^n Rm_i\big) = W_{(\oplus_{i=1}^n Rm_i)}(0)$.

We may assume that $n = 2$.

Let $rm \in S_{(R_{m_1} \oplus Rm_2)}(0)$. Then, for each sequence $a_0, a_1, a_2, \ldots$ such that $a_0 = r$ and $a_{i+1} \in a_i R a_i$, there is a positive integer k such that $a_k R(d_1, e_1) = 0$ where $m = (d_1, e_1)$. It follows $a_k R d_1 = 0$ and $a_k R e_1 = 0$ and, by the hypothesis, we get that $a_k R \subseteq \mathrm{ann}(m_1) \cap \mathrm{ann}(m_2)$. Therefore, it follows that $r \in S_R(\mathrm{ann}(m_1) \cap \mathrm{ann}(m_2))$ and so $W_{(R_{m_1} \oplus Rm_2)}(0) \subseteq W_R(\mathrm{ann}(m_1) \cap$ $\cap\, \mathrm{ann}(m_2))(Rm_1 \oplus Rm_2)$.

For the converse, take an element $n = a(d_1, e_1) \in S_R(\mathrm{ann}(m_1) \cap \mathrm{ann}(m_2))(Rm_1 \oplus Rm_2)$ where $a \in S_R(\mathrm{ann}(m_1) \cap \mathrm{ann}(m_2))$. Then, for each sequence $a_0, a_1, \ldots$ such that $a_0 = a$ and $a_{i+1} \in a_i R a_i$, there is a positive integer k such that $a_k R \subseteq \mathrm{ann}(m_1) \cap \mathrm{ann}(m_2)$ and so $a_k R(m_1, m_2) = 0$. Therefore, we get the equality.

By Proposition 2.1, we get

$$W_M(0) = W_{(\oplus_{i=1}^n Rm_i)}(0) \oplus W_K(0) = W_{(\oplus_{i=1}^n Rm_i)}(0) \oplus W_R(0)K.$$

An element m of a module M over a ring R is called a torsion element of the module if there exists a regular element r of the ring (an element that is neither a left nor a right zero divisor) that annihilates m (i.e., $r\,m = 0$) and if $T(M) = M$, then M is called torsion module and if $T(M) = 0$, then M is called torsion free module.

As well known, the concept of Dedekind domain is important for the commutative ring theory and it is well known that a Dedekind domain satisfies radical formula. In noncommutative ring theory, one of its generalization is called an HNP-ring (a right and left Noetherian prime ring in which every right and every left ideal are a projective module). From [7] (Lemma 7.4), we note that if M is a finitely generated R-module over an HNP-ring, then $M = T(M) \oplus K$ where K is isomorphic to $M/T(M)$.

To recall the definition of an extending module, we need the definition of an essential submodule. A submodule N of a module M is called essential in M if, for all $m \in M$, there is an element $r \in R$ such that $0 \neq rm \in N$. A submodule N is said to be closed in M if N has no proper essential extension in M. Let N and K be submodules of M such that N is essential in K. If K is closed in M, then K is called the closure of N. A module M is called extending if every closed submodule of M is a direct summand of M.

Lemma 3.1. *Let R be a ring such that every element of R is regular and let N be a submodule of a module M such that $T(M)$ is a submodule of N. Then the closure L of N is of the form $T(M/N) = L/N$.*

In particular, if M is finitely generated over an HNP-ring, then L is a direct summand of M.

Proof. Let N and K be submodules of M such that N is essential in K. Then it is clear that K/N is a submodule of $T(M/N)$.

Let N and L be submodules such that $T(M)$ is a submodule of N and $L/N = T(M/N)$. Take an element $0 \neq x + N \in T(M/N)$ and so there is an element $r \in R$ such that $0 \neq rx \in N$. Otherwise $x + N = 0$. Hence, L is an essential extension of N and also L is a closed submodule of M. Moreover, we derive that M/L is torsion free since M/L is isomorphic to $(M/N)/T(M/N)$. If M is a finitely generated module over an HNP-ring, then M/L is projective and so L is a direct summand of a module M.

For any prime ideal P of R, we say that P has the properties S if $xRy \subseteq P$ whenever $xy \in P$ where $x, y \in R$. We call that a ring R is $HNPS$-ring if R is HNP-ring and the zero ideal satisfies S-property. It is clear that $HNPS$-ring is a generalization of Dedekind domain.

***Corollary* 3.1.** *Let R be a left extending ring. Then*:

(i) *Let P be a prime submodule of ${}_RR$ with the S-property. Then P is essential in R.*

(ii) *If every prime ideal has the S-property, then* $\mathrm{Soc}\,({}_RR) \subseteq \mathrm{rad}_R(0) \subseteq J(R)$ *where $J(R)$ is the Jacobson radical of R.*

Proof. (i) Let P be a prime submodule of ${}_RR$ and L be a closure of P. Then there is a decomposition $R = L \oplus K$ for a submodule K of R since R is extending. On the other hand, by Corollary 2.1, it follows that R/P is indecomposable and so $K = 0$. Hence, P is essential in R.

(ii) It is clear since $\mathrm{Soc}({}_RR)$ is the intersection of essential left ideals of R.

Theorem 3.2. *Let R be an $HNPS$-ring. Then a finitely generated module is the direct sum of a torsion module and an extending module.*

Proof. By [7] (Lemma 7.4), it is known that $M = T(M) \oplus K$ where K is isomorphic to $M/T(M)$ if M is a finitely generated R-module over an HNP-ring. Let A be a submodule of K. Then L is a closure of A where $T(K/A) = L/A$ and so L is a summand of K. This means that K is an extending module.

Theorem 3.3. *Let M be a finitely generated module over an HNP-ring R. If for any element m of M, every element of* $\mathrm{ann}(m)R$ *is strongly nilpotent on* $\mathrm{ann}(m)$, *then M satisfies the radical formula.*

Proof. Let M be a finitely generated module over an HNP-ring R. By [3] (Lemma 7), we get that M is direct sum of cyclic modules and projective module. Then, by the use of Theorem 2.2 and Proposition 2.1, the proof is completed.

In the commutative theory, every module over a Dedekind domain satisfies the radical formula. Therefore, we wonder whether the condition in Theorem 3.3 can be removed or not.

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