DOI: 10.37863/umzh.v72i10.6032

UDC 517.9

H. Mahmoudi, A. Esfahani (School Math. and Comput. Sci., Damghan Univ., Iran)

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO AN EVOLUTION EQUATION FOR BIDIRECTIONAL SURFACE WAVES IN A CONVECTING FLUID

АСИМПТОТИЧНА ПОВЕДІНКА РОЗВ'ЯЗКІВ ЕВОЛЮЦІЙНОГО РІВНЯННЯ ДЛЯ ДВОНАПРАВЛЕНИХ ПОВЕРХНЕВИХ ХВИЛЬ У РІДИНІ З КОНВЕКЦІЄЮ

We consider the Cauchy problem for an evolution equation modeling bidirectional surface waves in a convecting fluid. We study the existence, uniqueness, and asymptotic properties of global solutions to the initial value problem associated with this equation in $\mathbb{R}^n$. We obtain some polynomial decay estimates of the energy.

Розглядається задача Коші для еволюційного рівняння, що моделює двонаправлені поверхневі хвилі у рідині з конвекцією. Вивчаються існування, єдиність та асимптотичні властивості глобальних розв'язків початкової задачі, що пов'язана з цим рівнянням у $\mathbb{R}^n$. Отримано деякі поліноміальні оцінки спадання енергії.

1. Introduction and preliminaries. In this paper, we study

$$u_{tt} - \varepsilon\Delta u_{tt} - \Delta + \Delta^2 u + \alpha\Delta u_t + \Delta^2 u_t + u_t = \Delta f(u) + \beta\Delta g(u_t) \qquad (1.1)$$

which arises as the phase equation in the study of the stability of one-dimensional periodic patterns in systems with Galilean invariance. Also, it was derived to describe the oscillatory instability of convective rolls and elastic beams [3, 6, 7]. Equation (1.1) is a higher order wave model in which the terms $\alpha\Delta u_t + \Delta^2 u_t + u_t$ represent the frictional dissipation. Equation (1.1) can be viewed as a generalized the Cahn – Hilliard equation with an inertial term which models nonequilibrium decompositions caused by deep supercooling in certain glasses [8 – 10, 17].

Here $u = u(t, x)$ is the unknown function of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and $t > 0$, $\varepsilon > 0, \alpha < 0$ as well as β are real constants. In addition, the nonlinear terms $g(u)$ and $f(u)$ are like $O(|u|^p)$. We assume that f and g are continuously differentiable in $\mathbb{R}$, and satisfy the following hypothesis:

$$|(f^{(j)})(a)| \le k_j |a|^{p-j} \quad \text{and} \quad |(g^{(j)})(a)| \le k_j' |a|^{p-j}$$

for all $a \in \mathbb{R}$ and $j = 0, 1, 2$, where k_j and k_j' are real positive constants. It is well-known that equation (1.1) is closely related to several wave-type equations. For example, the Boussinesq equation

$$u_{tt} - \Delta + \Delta^2 u = \Delta f(u), \qquad (1.2)$$

which was derived by Boussinesq in 1872 to describe shallow water waves. The counterpart of equation (1.2), i.e., the improved Boussinesq equation, can be presented as follows:

$$u_{tt} - \Delta - \Delta u_{tt} = \Delta f(u).$$

In [19], the authors considered the Cauchy problem associated with the Cahn – Hilliard equation with the inertial term

$$u_{tt} + \Delta u - \Delta^2 u + u_t = \Delta f(u).$$

Combining the high/low-frequency techniques and energy methods, they obtained the global exis-

© H. MAHMOUDI, A. ESFAHANI, 2020

tence and asymptotic behavior of the solutions. In [18], the authors investigated a fourth wave equation that is of the regularity-loss type. Based on the decay property of the solution operators, the global existence and asymptotic behavior of solutions are derived. See also [12, 13] and references therein for the global existence and asymptotic behavior of solutions to higher order wave-type and dissipative hyperbolic-type equations.

In this work, we show decay estimates in time for the total energy of the Cauchy problem associated with (1.1) and the L^2-norm of the solution. To this end, we study the associated linear problem in detail as well as the behavior of the solutions. Subsequently, by using the formula of the variation of parameters, we implement that result to the semilinear problem.

2. Main result. Before stating the main result, we give some notations which are used in this paper.

For $1 \leq p \leq \infty$, $L^p = L^p(\mathbb{R}^n)$ denotes the usual Lebesgue space with the norm $\|\cdot\|_p$. We also use $\|\cdot\|$ as the norm of $L^2(\mathbb{R}^n)$. The inner product in $L^2(\mathbb{R}^n)$ will be indicated by $\langle\cdot,\cdot\rangle$. Let s be a nonnegative integer. Then $H^s = H^s(\mathbb{R}^n)$ denotes the Sobolev space of L^2 functions, equipped with the norm $\|\cdot\|_{H^s}$. Also, $C^k(I; H^s)$ denotes the space of k-times continuously differentiable functions on the interval I with values in the Sobolev space $H^s = H^s(\mathbb{R}^n)$.

To state our main result, we consider the weak solution u of (1.1) with the initial data $u(0,x) = u_0(x)$ and $u_t(0,t) = u_1(x)$. More precisely,

$$\begin{gathered}\langle u_{tt}(t),\psi\rangle + \varepsilon\,\langle\nabla u_{tt}(t),\nabla\psi\rangle + \langle\nabla u(t),\nabla\psi\rangle + \langle\Delta u(t),\Delta\psi\rangle - \\ -\alpha\,\langle\nabla u_t(t),\nabla\psi\rangle + \langle\Delta u_t(t),\Delta\psi\rangle = \langle\Delta f(u(t)) + \beta\Delta g(u_t(t)),\psi\rangle\,, \\ u(0,x) = u_0(x), \\ u_t(0,t) = u_1(x).\end{gathered} \tag{2.1}$$

The energy associated with the linear problem of (1.1) (see equation (3.1)) is defined by

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{\varepsilon}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\Delta u\|^2.$$

Theorem 2.1. *Let $p > 2$ and $n \leq 3$. Suppose that $(u_0,u_1) \in H^3 \times H^2$ with $I_0 < \delta$. Then there exists $\delta > 0$ such that problem* (2.1) *has a unique global solution $u \in C([0,\infty); H^3) \cap C^1([0,\infty); H^2) \cap C^2([0,\infty); H^1)$ such that $\|u(t)\|^2 \leq CI_0$ and $E(t) \leq CI_0(1+t)^{-1}$, where*

$$I_0 = \|u_0\|_{H^3}^2 + \|u_1\|_{H^2}^2.$$

3. Linear equation. This section is devoted to studying the existence and uniqueness of the weak solution of the linear equation in $\mathbb{R}^n$:

$$\begin{gathered}v_{tt}(t,x) - \varepsilon\Delta v_{tt}(t,x) - \Delta v(t,x) + \Delta^2 v(t,x) + \alpha\Delta v_t(t,x) + \Delta^2 v_t(t,x) + v_t(t,x) = 0, \\ v(0,x) = v_0(x), \\ v_t(0,x) = v_1(x).\end{gathered} \tag{3.1}$$

Throughout this section, we assume that $n \geq 1$.

Theorem 3.1. *For* $(v_0, v_1) \in H^{3+k} \times H^{2+k}, k = 0, 1, 2,$ *the linear problem* (3.1) *admits unique solution*

$$v \in C([0,\infty); H^{3+k}) \cap C^1([0,\infty); H^{2+k}) \cap C^2([0,\infty); H^{1+k})$$

such that there holds for all $\psi \in H^2$ *that*

$$\langle v_{tt}, \psi\rangle + \varepsilon \langle \nabla v_{tt}, \nabla\psi\rangle + \langle \nabla v, \nabla\psi\rangle + \langle \Delta v, \Delta\psi\rangle + \alpha \langle \nabla v, \nabla\psi\rangle + \langle \Delta v_t, \Delta\psi\rangle + \langle v_t(t), \psi\rangle = 0. \tag{3.2}$$

The proof of the existence is obtained by employing the semigroup theory.

***Proof of Theorem* 3.1.** Equation (3.2) can be rewritten as

$$v_{tt} + (I - \varepsilon\Delta)^{-1}(\Delta^2 - \Delta + I)v = (I - \varepsilon\Delta)^{-1}((-\alpha\Delta - \Delta^2 - I)v_t + v).$$

Define $D(A)$ to be the subspace of all $v \in H^2$ such that there exists $y = y_v \in H^1$ satisfying

$$\langle \Delta v, \Delta\psi\rangle + \langle \nabla v, \nabla\psi\rangle + \langle v, \psi\rangle = \langle y, \psi\rangle + \varepsilon \langle \nabla y, \nabla\psi\rangle$$

for all $\psi \in H^2$. Therefore, it is natural to define an operator A from the definition of $D(A)$ as follows:

$$A\colon D(A) \longrightarrow H^1,$$

$$Av = y_v.$$

Indeed, A is formally the operator $(I - \varepsilon\Delta)^{-1}(\Delta^2 - \Delta + I)$. In addition, it is straightforward to see from the definition of A that $D(A) = H^3$, and there exists $C > 0$ such that $\|v\|_{H^3} \le C\|Av\|_{H^1}$ for all $v \in D(A)$.

Now, we complete the proof of the existence for the linear problem.

Let $z \in H^1$. It follows from the Lax – Milgram lemma that there exists a unique $\tilde{z} \in H^1$ such that

$$-\langle \tilde{z}, \psi\rangle - \varepsilon \langle \nabla\tilde{z}, \nabla\psi\rangle = \langle z, \psi\rangle$$

for all $\psi \in H^1$. So we define the function $h\colon H^1 \longrightarrow H^1$ such that for each $z \in H^1$, $h(z)$ is given by the equation

$$-\langle h(z), \psi\rangle - \varepsilon \langle \nabla h(z), \nabla\psi\rangle = \langle z, \psi\rangle \tag{3.3}$$

for all $\psi \in H^1$. Furthermore, $\varepsilon\Delta h(z) - h(z) = z$ and $h(z) = (I - \varepsilon\Delta)^{-1}(-z)$. For $(v(t), w(t)) \in$ $\in H^2 \times H^1$ with $t \ge 0$ such that $v(0) = v_0, w(0) = v_1$, we define

$$Z(t) = [v(t), w(t)]\ ;\ H(Z(t)) = [0, h(\alpha\Delta w + \Delta^2 w + w) - h(v)],$$

in which $Z(0) = (v_0, v_1) \in H^3 \times H^2$. Consider the operator $B\colon D(B) \longrightarrow H^2 \times H^1$ defined by

$$B(v, w) = (w, -Av), \tag{3.4}$$

where $D(B) = H^3 \times H^2$. The operator B generates a C_0 semigroup of contractions in $H^2 \times H^1$. See Lemma 5.2 in the Appendix for the proof of this fact. Consequently, equation (3.1) is equivalent to

$$\frac{d}{dt}Z(t) = B(Z(t)) + H(Z(t)), \tag{3.5}$$
$$Z(0) = Z_0 \in D(B).$$

It should be mentioned that H is linear and continuous in $X = H^2 \times H^1$. Hence by the semigroups theory, the operator $\mathcal{L} := B + H$ is the generator of an infinitesimal C_0 semigroup. Therefore, (3.5) has a unique solution

$$Z \in C([0, \infty); D(\mathcal{L})) \cap C^1([0, \infty), X).$$

That is, there exists a unique function $v \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2) \cap C^2([0, \infty), H^1)$ satisfying (3.2), where $v = v(t)$ is the first component of $Z = Z(t)$. The cases $k = 1$ and $k = 2$ of Theorem 3.1 are similar.

Theorem 3.1 is proved.

To give our estimates of the linear problem we define

$$\begin{gathered} I_1 = \|v_0\|^2_{H^4} + \|v_1\|^2_{H^3}, \\ I_2 = \|v_0\|^2_{H^4} + \|v_1\|^2_{H^3} + \||x|^2(v_0 + v_1)\|^2, \\ I_3 = \|v_0\|^2_{H^4} + \|v_1\|^2_{H^3} + \|v_0 + v_1\|^2_{L^{\frac{2n}{n-4}}}. \end{gathered} \tag{3.6}$$

Theorem 3.2. (i) *If $(v_0, v_1) \in H^3 \times H^2$, then the solution $v \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2) \cap$ $\cap\, C^2([0, \infty); H^1)$ of the linear problem* (3.2) *satisfies*

$$E(t) \le C I_0 (1 + t)^{-1}$$

and

$$\|v(t)\|^2 \le C I_0,$$

where I_0 and E are defined in Section 2.

(ii) *If $(v_0, v_1) \in H^4 \times H^3$ such that $|x|^2(v_0 + v_1) \in L^2$, then the solution $v \in C([0, \infty); H^4) \cap$ $\cap\, C^1([0, \infty); H^3) \cap C^2([0, \infty); H^2)$ of* (3.2) *satisfies*

$$E(t) \le C I_2 (1 + t)^{-2}$$

and

$$\|v(t)\|^2 + \|\Delta v_t(t)\|^2 + \|\Delta(\nabla v(t))\|^2 \le C I_2 (1 + t)^{-1}.$$

(iii) *If $(v_0, v_1) \in H^4 \times H^3$ such that $(v_0 + v_1) \in L^{\frac{2n}{n-4}}$, then the solution $v \in C([0, \infty); H^4) \cap$ $\cap\, C^1([0, \infty); H^3) \cap C^2([0, \infty); H^2)$ of* (3.2) *satisfies*

$$E(t) \le C I_3 (1 + t)^{-2}$$

and

$$\|v(t)\|^2 + \|\Delta v_t(t)\|^2 + \|\Delta(\nabla v(t))\|^2 \le C I_3 (1 + t)^{-1}.$$

To prove Theorem 3.2, we need the following lemmas.

Lemma 3.1. *Suppose that the hypotheses of Theorem* 3.2 *hold. Then, the solution v of problem* (3.2) *satisfies*

$$\|v_t\|^2 + (1+t)E(t) \le CI_0 + C\int\limits_0^t \|\nabla v_t(s)\|^2 ds,$$

where $C > 0$ is a constant which does not depend on the initial data.

Proof. We know from Theorem 3.1 for $(v_0, v_1) \in H^3 \times H^2$ that there exists a unique function v satisfying (3.2). Thus, we have

$$\frac{d}{dt}E(t) + \|v_t(t)\|^2 + \|\Delta v_t(t)\|^2 - \alpha\|\nabla v_t(t)\|^2 = 0,$$

and consequently

$$E(t) + \int\limits_0^t (\|v_t(s)\|^2 + \|\Delta v_t(s)\|^2 - \alpha\|\nabla v_t(s)\|^2) ds = E(0). \tag{3.7}$$

On the other hand, multiplying (3.7) by $(1+t)$ and integrating on $[0,t]$, we get

$$(1+t)E(t) + \int\limits_0^t (1+s)(\|v_t(s)\|^2 + \|\Delta v_t(s)\|^2 - \alpha\|\nabla v_t(s)\|^2) ds = E(0) + \int\limits_0^t E(s) ds. \tag{3.8}$$

By substituting $\psi = v$ in (3.2) and integrating on $[0,t]$, we deduce

$$\|v(t)\|^2 + \int\limits_0^t \|\Delta v(s)\|^2 ds + \int\limits_0^t \|\nabla v(s)\| ds + \int\limits_0^t (\|v_t(s)\|^2 + \Delta v_t(s)\|^2 - \alpha\nabla v_t(s)\|^2) ds \le$$

$$\le CI_0 + C\int\limits_0^t \|\nabla v_t(s)\|^2 ds. \tag{3.9}$$

The proof follows from (3.7) – (3.9).

Lemma 3.2. *Suppose that the hypotheses of Theorem* 3.2 *hold. Then the solution v of* (3.2) *satisfies*

$$\|\Delta v_t(t)\|^2 + \|\Delta v(t)\|^2 + \|\Delta(\nabla v(t))\|^2 + \int\limits_0^t \|\nabla v_t(s)\|^2 ds \le CI_0,$$

where the constant $C > 0$ is independent of the initial data.

Proof. Let $(\varphi_0, \varphi_1) \in H^4 \times H^3$ and $\varphi \in C([0,\infty); H^4) \cap C^1([0,\infty); H^3) \cap C^2([0,\infty); H^2)$ be the associated solution of the linear problem (3.2). The regularity of the solution φ implies for any $\beta \in \mathbb{N}^n$ with $|\beta| \le 1$ that

$$\left\langle D^\beta \varphi_{tt}(t), D^\beta \varphi_t(t) \right\rangle + \varepsilon \left\langle \nabla \varphi_{tt}(t), \nabla D^\beta \varphi_t(t) \right\rangle +$$

$$+ \left\langle \nabla D^\beta \varphi(t), \nabla D^\beta \varphi_t(t) \right\rangle + \left\langle \Delta D^\beta \varphi(t), \Delta D^\beta \varphi_t(t) \right\rangle -$$

$$-\alpha \left\langle \nabla D^\beta \varphi_t(t), \nabla D^\beta \varphi_t(t) \right\rangle + \left\langle \Delta D^\beta \varphi_t(t), \Delta D^\beta \varphi_t(t) \right\rangle + \left\langle D^\beta \varphi_t(t), D^\beta \varphi_t(t) \right\rangle = 0.$$

Let

$$E_{D^\beta\varphi}(t)=\frac{1}{2}\|D^\beta\varphi_t(t)\|^2+\frac{\varepsilon}{2}\|\nabla D^\beta\varphi_t(t)\|^2+\frac{1}{2}\|\nabla D^\beta\varphi(t)\|^2+\frac{1}{2}\|\Delta D^\beta\varphi(t)\|^2.$$

Hence,

$$E_{D^\beta\varphi}(t)+\int\limits_0^t(\|D^\beta\varphi_t(s)\|^2+\|\Delta D^\beta\varphi_t(s)\|^2-\alpha\|\nabla D^\beta\varphi_t(s)\|^2)ds=E_{D^\beta\varphi}(0).$$

By taking $\beta=(0,\ldots,0,1,0,\ldots,0)$, with 1 in the jth position, we have, for $j\in\{0,1,2,\ldots\}$,

$$\left\|\frac{\partial^2}{\partial x_j^2}\varphi_t(t)\right\|^2+\left\|\frac{\partial^2}{\partial x_j^2}\varphi(t)\right\|^2+\left\|\Delta(\frac{\partial}{\partial x_j}\varphi(t))\right\|^2+\int\limits_0^t\left\|\frac{\partial^2}{\partial x_j^2}\varphi_t(s)\right\|^2ds\leq$$

$$\leq C\left(\|\varphi_0\|_{H^3}^2+\|\varphi_1\|_{H^2}^2\right).$$

Therefore, it follows that

$$\|\Delta\varphi_t(t)\|^2+\|\Delta\varphi(t)\|^2+\|\Delta(\nabla\varphi(t))\|^2+\int\limits_0^t\|\nabla\varphi_t(s)\|^2ds\leq C(\|\varphi_0\|_{H^3}^2+\|\varphi_1\|_{H^2}^2).$$

Lemma 3.3. *Let* $(v_0,v_1)\in H^4\times H^3$. *Then the solution of* (3.2) *satisfies*

$$(1+t)\|v(t)\|^2+\int\limits_0^t(1+s)(\|\nabla v(s)\|^2+\|\Delta v(s)\|^2+\|v_t(s)\|^2+\|\Delta v_t(s)\|^2-\alpha\|\nabla v_t(s)\|^2)ds\leq$$

$$\leq CI_1+C\int\limits_0^t\|\nabla v_t(s)\|^2ds+C\int\limits_0^t\|v(s)\|^2ds.$$

Proof. By Theorem 3.1, there exists the unique function v satisfying (3.2). Similar to Lemma 3.1, we have

$$(1+t)E(t)+\int\limits_0^t(1+s)(\|v_t(s)\|^2+\|\Delta v_t(s)\|^2-\alpha\|\nabla v_t(s)\|^2)ds\leq CI_1+C\int\limits_0^t\|\nabla v_t(s)\|^2ds$$

and

$$\frac{d}{dt}\left[\langle v_t,v\rangle+\varepsilon\langle\nabla v_t,\nabla v\rangle-\frac{\alpha}{2}\langle\nabla v,\nabla v\rangle+\frac{1}{2}\langle\Delta v,\Delta v\rangle+\frac{1}{2}\langle v,v\rangle\right]-\|v_t(s)\|^2-$$

$$-\varepsilon\|\nabla v_t(s)\|^2+\|\nabla v(s)\|^2+\|\Delta v(s)\|^2=0. \qquad (3.10)$$

By multiplying (3.10) by $(1+t)$, and integrating on $[0,t]$ as well as utilizing (3.6), we obtain the desired estimate.

Lemma 3.4. *Under the hypotheses of Theorem* 3.2(ii), *the solution* v *of* (3.2) *satisfies*

$$\int_0^t \|v(s)\|^2 ds \le CI_2.$$

Proof. Consider the function w defined by $w(t) = \int_0^t v(s)ds$, where v is the solution of (3.2) with initial data $[v_0, v_1] \in H^4 \times H^3$. Then $w \in C^1([0,\infty); H^4) \cap C^2([0,\infty); H^3) \cap C^3([0,\infty); H^2)$. It can be easily found out that w is the solution of

$$\langle w_{tt}(t), \psi\rangle + \varepsilon\langle \nabla w_{tt}(t), \nabla\psi\rangle + \langle \nabla w(t), \nabla\psi\rangle + \langle \Delta w(t), \Delta\psi\rangle - \alpha\langle \nabla w_t(t), \nabla\psi\rangle + \langle \Delta w_t(t), \Delta\psi\rangle +$$
$$+ \langle w_t(t), \psi\rangle = \langle v_0 + v_1, \psi\rangle + \varepsilon\left\langle \nabla v_1, \nabla w_t\right\rangle - \alpha\left\langle \nabla v_0, \nabla w_t\right\rangle + \left\langle \Delta v_0, \Delta w_t\right\rangle, \tag{3.11}$$

in which $w(0, x) = 0$ and $w_t(0, x) = v_0$. Therefore, we have by substituting $\psi = w_t$ in (3.11) and integrating on $[0, t]$ that

$$\|\nabla w(t)\|^2 + \|\Delta w(t)\|^2 - \alpha\int_0^t \|\nabla w_t(s)\|^2 ds + \int_0^t \|\Delta w_t(s)\|^2 ds + \int_0^t \|w_t(s)\|^2 ds \le$$
$$\le CI_2 + C\left\langle v_0 + v_1, w(t)\right\rangle.$$

As a result, we get

$$\langle v_0 + v_1, w(t)\rangle = \int_{\mathbb{R}^n} |x|^2 (v_0(x) + v_1(x)) \frac{w(t,x)}{|x|^2} dx \le$$
$$\le \frac{1}{\varepsilon}\||\cdot|^2(v_0 + v_1)\|^2 + \varepsilon\int_{\mathbb{R}^n} \frac{|w(t,x)|^2}{|x|^4} dx \le \frac{1}{\varepsilon}\||\cdot|^2(v_0 + v_1)\|^2 + \varepsilon K\|\Delta w(t)\|^2,$$

where in the previous inequality, we used the Hardy-type inequality (see [5])

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^4} dx \le K\int_{\mathbb{R}^n} |\Delta u(x)|^2 dx, \quad u \in H^2.$$

By using two above estimates with ε sufficiently small, we observe that

$$\|\nabla w(t)\|^2 + \|\Delta w(t)\|^2 - \alpha\int_0^t \|\nabla w_t(s)\|^2 ds + \int_0^t \|\Delta w_t(s)\|^2 ds + \int_0^t \|w_t(s)\|^2 ds \le CI_2.$$

Since

$$(1+t)\|v(t)\|^2 + \int_0^t (1+s)[\|\nabla v(s)\|^2 + \|\Delta v(s)\|^2 + \|v_t(s)\|^2 + \|\Delta v_t(s)\|^2 - \alpha\|\nabla v_t(s)\|^2] ds \le$$
$$\le CI_1 + C\int_0^t \|\nabla v_t(s)\|^2 + C\int_0^t \|v(s)\|^2 ds \le CI_2,$$

then

$$\|v(t)\|^2 \le CI_2(1+t)^{-1}.$$

Lemma 3.5. *Under the assumptions of Lemma* 3.3, *the solution of* (3.2) *satisfies*

$$\|\Delta(\nabla v_t(t))\|^2 + (1+t)\|\Delta v_t(t)\|^2 + (1+t)\|\Delta(\nabla v_t(t))\|^2 + \int\limits_0^t (1+s)\|\nabla v_t(s)\|^2 ds \le CI_1.$$

Proof. Let $(\varphi_0, \varphi_1) \in H^5 \times H^4$ and $\varphi \in C([0,\infty); H^5) \cap C^1([0,\infty); H^4) \cap C^2([0,\infty); H^3)$ be the solution of (3.2) with initial data (φ_0, φ_1). Then, for each $\beta \in N^n$ with $|\beta| \le 2$, we have, for all $\psi \in H^2$,

$$\left\langle D^\beta \varphi_{tt}(t), \psi \right\rangle + \varepsilon \left\langle \nabla D^\beta \varphi_{tt}(t), \nabla\psi \right\rangle + \left\langle \nabla D^\beta \varphi(t), \nabla\psi \right\rangle + \left\langle \Delta D^\beta \varphi(t), \Delta\psi \right\rangle -$$

$$-\alpha \left\langle \nabla D^\beta \varphi_t(t), \nabla\psi \right\rangle + \left\langle \Delta D^\beta \varphi_t(t), \Delta\psi \right\rangle + \left\langle D^\beta \varphi_t(t), \psi \right\rangle = 0. \tag{3.12}$$

Here, $D^\beta\varphi$ is denoted by w^β. Therefore, the energy of $w^\beta(t,x)$ is given by

$$E_{w^\beta}(t) = \frac{1}{2}\|w_t^\beta\|^2 + \frac{\varepsilon}{2}\|\nabla w_t^\beta\|^2 + \frac{1}{2}\|\nabla w^\beta\|^2 + \frac{1}{2}\|\Delta w^\beta\|^2.$$

We obtain by substituting $\psi = D^\beta\varphi_t(t)$ in (3.12) that

$$\frac{d}{dt}E_{w^\beta}(t) + \|w_t^\beta\|^2 + \|\Delta w_t^\beta\|^2 - \alpha\|\nabla w_t^\beta\|^2 = 0. \tag{3.13}$$

By integrating the above identity on $[0,t]$, we get

$$E_{w^\beta}(t) + \int\limits_0^t (\|w_t^\beta(s)\|^2 + \|\Delta w_t^\beta(s)\|^2 - \alpha\|\nabla w_t^\beta(s)\|^2) ds = E_{w^\beta}(0), \tag{3.14}$$

where $|\beta| \le 2$. Since $w^\beta = D^\beta\varphi$, we see from the definition of the energy that

$$\|D^{2e_i}(D^{e_j}\varphi_t(t))\|^2 + \int\limits_0^t \|D^{e_i}(D^{e_j}\varphi_t(s))\|^2 ds \le CE_{w^\beta}(0)$$

for all $i, j = 1, 2, \ldots, n$, where e_i is the ith basis vector of $\mathbb{R}^n$. Thus,

$$\|\Delta(D^{e_j}\varphi_t(t))\|^2 + \int\limits_0^t \|\nabla(D^{e_j}\varphi_t(s))\|^2 ds \le CE_{w^\beta}(0).$$

Multiplying (3.13) by $(1+t)$ and integrating on $[0,t]$, we obtain

$$(1+t)E_{w^\beta}(t) + \int\limits_0^t (1+s)(\|w_t^\beta(s)\|^2 + \|\Delta w_t^\beta(s)\|^2 - \alpha\|\nabla w_t^\beta(s)\|^2) ds =$$

$$= E_{w^\beta}(0) + \int\limits_0^t E_{w^\beta}(s) ds. \tag{3.15}$$

On the other hand, by substituting $\psi = D^\beta\varphi(t)$ in (3.12) as well as integrating on $[0,t]$, and using (3.14), we derive that

$$\|w^\alpha(t)\|^2 + (1+t)E_{w^\alpha}(t) \le C(\|\varphi_0\|^2_{H^4} + \|\varphi_1\|^2_{H^3}) + C\int_0^t \|\nabla w_t^\beta(s)\|^2 ds.$$

By combining this estimate with (3.14) and (3.15), we have

$$(1+t)E_{w^\beta}(t) + \int_0^t (1+s)(\|w_t^\alpha(s)\|^2 + \|\Delta w_t^\alpha(s)\|^2 - \alpha\|\nabla w_t^\alpha(s)\|^2)ds \le$$

$$\le C(\|\varphi_0\|^2_{H^4} + \|\varphi_1\|^2_{H^3}) + C\int_0^t \|\nabla w_t^\alpha(s)\|^2 ds.$$

Moreover, by employing the definition of the energy once again, we can conclude the following inequality as

$$(1+t)\|D^{2e_j}\varphi_t(t)\|^2 + (1+t)\|\Delta D^{e_j}\varphi(t)\|^2 + \int_0^t (1+s)\|D^{e_j}\varphi_t(s)\|^2 ds \le$$

$$\le C(\|\varphi_0\|^2_{H^4} + \|\varphi_1\|^2_{H^3}) + C\int_0^t \|\nabla w_t^\alpha(s)\|^2 ds.$$

The above-mentioned estimates lead to

$$\|\Delta D^{e_j}\varphi_t(t)\|^2 + (1+t)\|D^{2e_j}\varphi_t(t)\|^2 + (1+t)\|\Delta D^{e_j}\varphi(t)\|^2 + \int_0^t (1+s)\|D^{e_j}\varphi_t(s)\|^2 ds \le$$

$$\le C(\|\varphi_0\|^2_{H^4} + \|\varphi_1\|^2_{H^3}).$$

By summing on i, it follows that

$$\|\Delta(\nabla\varphi_t(t))\|^2 + (1+t)\|\Delta\varphi_t(t)\|^2 + (1+t)\|\Delta(\nabla\varphi(t))\|^2 + \int_0^t (1+s)\|\nabla\varphi_t(s)\|^2 ds \le$$

$$\le C(\|\varphi_0\|^2_{H^4} + \|\varphi_1\|^2_{H^3}),$$

where φ is the solution of (3.2) with initial data $(\varphi_0, \varphi_1) \in H^5 \times H^4$. The density argument completes the proof.

***Proof of Theorem* 3.2.** (i) The proof is an immediate consequence of Lemmas 3.1 and 3.2.

(ii) We use the fact that the energy $E(t)$ is a nonincreasing function. Thus,

$$\frac{d}{dt}[(1+t)^2E(t)] = 2(1+t)E(t) + (1+t)^2E'(t) \le 2(1+t)E(t), \qquad t \ge 0.$$

By integrating on $[0,t]$, we can conclude that

$$E(t) \le CI_2(1+t)^{-2}.$$

Lemmas 3.3, 3.4 and 3.5 give the estimates of (ii).

(iii) The proof is similar to the one of (ii) under the assumptions of (iii) and the following fact (see Lemma 3.4)

$$\int_0^t \|v(s)\|^2 ds \leq CI_3.$$

4. Global existence and asymptotic estimate. In this section, we study the existence of the local solution for the semilinear problem (1.1). One can easily find out that $\Delta f(u), \beta\Delta g(u) \in L^2$ for all $u \in H^2$. We now consider two functions $k_1 : H^2 \to H^2$ and $k_2 : H^2 \to H^2$ defined by

$$\begin{aligned} \langle k_1(u), \psi\rangle + \varepsilon \langle \nabla k_1(u), \nabla\psi\rangle &= \langle \Delta f(u), \psi\rangle, \\ \langle k_2(u), \psi\rangle + \varepsilon \langle \nabla k_2(u), \nabla\psi\rangle &= \langle \beta\Delta g(u), \psi\rangle. \end{aligned} \tag{4.1}$$

The functions $k_1, k_2 \in H^2$ are well-defined from the Lax – Milgram lemma. Also, there is $C > 0$ such that

$$\|k_1(u)\|_{H^2} \leq C\|\Delta f(u)\|_{L^2},$$

$$\|k_2(u)\|_{H^2} \leq C\|\beta\Delta g(u)\|_{L^2}$$

for all $u \in H^2$. In addition, as a result of the elliptic regularity and the uniqueness of (4.1), we can obtain the following inequalities as

$$\|k_1(u_1) - k_1(u_2)\|_{H^2} \leq C\|\Delta f(u_1) - \Delta f(u_2)\|_{L^2},$$

$$\|k_2(u_1) - k_2(u_2)\|_{H^2} \leq C\|\beta\Delta g(u_1) - \beta\Delta g(u_2)\|_{L^2}.$$

Denote $U(t) = (u(t), v(t))$ and $F(U(t)) = (0, k_1(u) + k_2(v))$, where k_1 and k_2 are defined in (4.1). Consider now the problem

$$\frac{d}{dt}U(t) = B(U(t)) + F(U(t)),$$

$$U(0) = U_0,$$

where $B : D(B) \subset X \longrightarrow X$ is defined in Section 2, $U_0 = (u_0, u_1)$, $X = H^2 \times H^1$ and $F(\cdot)$ as above. The following result follows from the well-known classical semigroup theorem.

Theorem 4.1 [2]. *Let $(u_0, u_1) \in H^3 \times H^2$. Then there exists $T_m > 0$, and a unique solution $u \in C([0, T_m); H^3) \cap C^1([0, T_m); H^2) \cap C^2([0, T_m); H^2)$ of* (2.1) *with $u(0, x) = u_0(x)$ and $u_t(0, x) = u_1(x)$. Moreover, $T_m = +\infty$ or $T_m < +\infty$ and*

$$\lim_{t\to T_m} \|(u(t), u_t(t))\|_{H^3\times H^2} = +\infty.$$

Moreover, we have, for $(u_0, u_1) \in H^4 \times H^3$, that

$$u \in C([0, T_m); H^4) \cap C^1([0, T_m); H^3) \cap C^2([0, T_m); H^2).$$

Proof. Define the operator $B : H^3 \times H^2 \longrightarrow H^2 \times H^1$ with $B(u, v) = (v, -Au)$ and $A = (I - \varepsilon\Delta)^{-1}(\Delta^2 - \Delta + I)$. By Lemma 5.1 in the Appendix, we need to show that the function $F : D(B) = H^3 \times H^2 \to D(B)$ given by $F(u, v) = (0, k_1(u) + k_2(v))$ is Lipschitz, with the graph norm on bounded sets, where k_1, k_2 are functions defined in (4.1). The definition of (4.1) implies that $k_1(u) + k_2(v) \in H^2$. Thus, $F\colon D(B) \longrightarrow D(B)$ is well-defined. Since f and g are Lipschitz, consequently, it is obvious that F is Lipschitz on bounded sets.

We are ready now to prove Theorem 2.1.

***Proof of Theorem* 2.1.** Define the norms

$$\|(u,v)\|_E = \|v\| + \|\nabla v\| + \|\Delta u\|,$$

$$\|(u,v)\|_F = \|u\| + \|\Delta v\| + \|\Delta(\nabla u)\|.$$

We use the local solution given by Theorem 4.1, and combine it with the decay estimates of the linear problem (3.2). The solution of (2.1) can be written by the Duhamel principle as

$$U(t) = S(t)U_0 + \int\limits_0^t S(t-s)F(u(s))ds, \tag{4.2}$$

where $U(t) = (u(t), u_t(t))$, $U_0 = (u_0, u_1)$, $F(U(s)) = (0, k_1(u(s)) + k_2(u_t(s)))$, and k_1, k_2 are defined in (4.1). Also, $S(t)$ indicates the semigroup associated with the linear problem. Define $I_0 = \|k_1(u(s)) + k_2(u_t(s))\|^2_{H^2}$ for each $s \in [0,t]$ and $t \in [0, T_m)$, with T_m given by Theorem 4.1. From the Gagliardo – Nirenberg inequality and the Sobolev embedding, we have

$$I_0 \le \|k_1(u)\|^2_{H^2} + \|k_2(u_t)\|^2_{H^2} \le C\|\Delta f(u)\|^2 + C\|\Delta g(u_t)\|^2 \le$$
$$\le C\|\Delta(u)^p\|^2 + c\|\Delta(u_t)^p\|^2 \le$$
$$\le C\|u^p\|^2_{H^2} + C\|(u_t)^p\|^2_{H^2} \le C\|u\|^{2p}_{H^2} + C\|u_t\|^{2p}_{H^2} \le$$
$$\le C\sum_{|\beta|=2}(\|D^\beta u\|^{2p} + \|D^\beta u_t\|^{2p}). \tag{4.3}$$

Furthermore, we have, from the estimates

$$\|S(t)U_0\|_E \le CI_0^{1/2}(1+t)^{-1/2},$$

$$\|S(t)U_0\|_F \le CI_0^{1/2},$$

that

$$\|S(t-s)F(U(s))\|_E \le CI_0^{1/2}(s)(1+t-s)^{-1/2},$$
$$\|S(t-s)F(U(s))\|_F \le CI_0^{1/2}(s) \tag{4.4}$$

for $s \in [0,t]$ and $t \in [0, T_m)$. Choose K large enough such that $K > C$ to be fixed later, where C is the same as in (4.4). Suppose, by contradiction, that the estimates

$$(1+t)^{\frac{1}{2}}\|U(t)\|_E \le KI_0^{\frac{1}{2}},$$

$$\|U(t)\|_F \le KI_0^{\frac{1}{2}}$$

fail for all $t \in [0, T_m)$. By choosing K sufficiently large, there exists $T_0 \in (0, T_m)$ such that, for all $t \in [0, T_0)$, we get

$$(1+t)^{\frac{1}{2}}\|U(t)\|_E < KI_0^{\frac{1}{2}},$$
$$\|U(t)\|_F \le KI_0^{\frac{1}{2}}. \tag{4.5}$$

Moreover, the estimates

$$(1+T_0)^{\frac{1}{2}}\|U(T_0)\|_E = KI_0^{\frac{1}{2}}$$

and

$$\|U(T_0)\|_F = KI_0^{\frac{1}{2}}$$

hold. By estimate (4.4) and (4.2), we have

$$\|U(t)\|_E \le CI_0^{\frac{1}{2}}(1+t)^{\frac{-1}{2}} + C\int\limits_0^t (1+t-s)^{\frac{-1}{2}} I_0^{\frac{1}{2}}(s)ds.$$

We obtain by (4.3) that

$$\|U(t)\|_E \le CI_0^{\frac{1}{2}}(1+t)^{\frac{-1}{2}} + C\sum_{|\alpha|=2}\int\limits_0^t (1+t-s)^{\frac{-1}{2}}(\|D^\alpha u(s)\|^{2p} + \|D^\alpha u_t(s)\|^{2p})ds.$$

By applying (4.5) and for all $t \in [0, T_0]$, we can conclude that

$$\|U(t)\|_E \le CI_0^{\frac{1}{2}}(1+t)^{\frac{-1}{2}} + C\int\limits_0^t (1+t-s)^{\frac{-1}{2}} K^p I_0^{\frac{p}{2}}(1+s)^{\frac{-p}{2}}ds.$$

Hence, we obtain, for all $t \in [0, T_0]$,

$$\|U(t)\|_E \le CI_0^{\frac{1}{2}}(1+t)^{\frac{-1}{2}} + C_p C K^p I_0^{\frac{p}{2}}(1+t)^{\frac{-1}{2}}.$$

In the previous inequality, we used the following elementary inequality (see [11, 16])

$$\int\limits_0^t (1+t-s)^{\frac{-1}{2}}(1+s)^{-\beta}ds \le C_\beta(1+t)^{\frac{-1}{2}}, \quad \beta > 1.$$

If we take K is sufficiently large such that $K > C$, and $\delta > 0$ such that $\delta \le \left(\dfrac{K-C}{C_p C K^p}\right)^{\frac{2}{p-1}}$, then we see that, for all $t \in [0, T_0]$,

$$\|U(t)\|_E \le KI_0^{\frac{1}{2}}(1+t)^{\frac{-1}{2}}, \tag{4.6}$$

provided $I_0 < \delta$. On the other hand, we have from (4.4),

$$\|U(t)\|_F \le CI_0^{\frac{1}{2}} + C\int\limits_0^t I_0^{\frac{1}{2}}(s)ds. \tag{4.7}$$

Also, we note, from (4.3) and (4.7), that

$$\|U(t)\|_F \leq CI_0^{\frac{1}{2}} + C\sum_{|\alpha|=2}\int\limits_0^t (\|D^\alpha u(s)\|^{2p} + \|D^\alpha u_t(s)\|^{2p})ds \leq$$

$$\leq CI_0^{\frac{1}{2}} + C\int\limits_0^t K^p I_0^{\frac{p}{2}}(1+s)^{\frac{-p}{2}}ds \leq CI_0^{\frac{p}{2}} + C_1CK^pI_0^{\frac{p}{2}}$$

for all $t \in [0, T_0]$, where C_1 a positive constant. If $K > C$ and $I_0 < \delta$ such that δ satisfies $0 < \delta \leq \left(\dfrac{K-C}{C_1CK^p}\right)^{\frac{2}{p-1}}$, then, we obtain, for all $t \in [0, T_0]$,

$$\|U(t)\|_F \leq KI_0^{\frac{1}{2}}. \tag{4.8}$$

Hence, estimates (4.8) and (4.6) contradict (4.5). This proves the validity of (4.6). Therefore, there exists a constant $\delta > 0$ such that $I_0 < \delta$ and the solution $U(t)$ satisfies

$$\|U(t)\|_{H^3\times H^2} \leq C$$

for all $t \in [0, T_m]$. Hence, $T_m = +\infty$ and the solution is global. In addition, the decay estimates of (4.6) hold and the proof is now complete.

5. Appendix. We show here that the operator B, defined in (3.4), generates a C_0 semigroup of contraction in $H^2 \times H^1$. This will be deduced from the following result [14].

Lemma 5.1. *Let B be a linear operator with the dense domain $D(B)$ in a Hilbert space H. If B is dissipative and $0 \in \rho(B)$, the resolvent set of B, then B is the infinitesimal generator of a C_0 semigroup of contraction in H.*

Lemma 5.2. *The operator $B \colon D(B) \longrightarrow H^2 \times H^1$ generates a C_0 semigroup of contraction in $H^2 \times H^1$.*

Proof. We show that B satisfies the assumptions of Lemma 5.1. Let $(v, w) \in D(B)$. Therefore, $v \in H^3$, $w \in H^2$, and

$$(B(v,w),(v,w))_{H^2\times H^1} = ((w,-Av),(v,w))_{H^2\times H^1} = (w,v)_{H^2} + (-Av,w)_{H^1} \leq$$

$$\leq C(\langle w,v\rangle + \langle\nabla w,\nabla v\rangle + \langle\Delta v,\Delta w\rangle + \langle -Av,w\rangle + \varepsilon\langle\nabla(-Av),\nabla w\rangle) =$$

$$= C(\langle w,v\rangle + \langle\nabla w,\nabla v\rangle + \langle\Delta v,\Delta w\rangle - \langle(I-\varepsilon\Delta)v,w\rangle) =$$

$$= C(\langle w,v\rangle + \langle\nabla w,\nabla v\rangle + \langle\Delta v,\Delta w\rangle - \big\langle(\Delta^2-\Delta+I)v,w\big\rangle) =$$

$$= C(\langle w,v\rangle + \langle\nabla w,\nabla v\rangle + \langle\Delta v,\Delta w\rangle - \langle\Delta v,\Delta w\rangle + \langle\Delta w,v\rangle - \langle w,v\rangle) =$$

$$= C(\langle w,v\rangle - \langle\Delta w,v\rangle + \langle\Delta v,\Delta w\rangle - \langle\Delta v,\Delta w\rangle + \langle\Delta w,v\rangle - \langle w,v\rangle) = 0.$$

Hence, B is dissipative. Next, we show that $0 \in \rho(B)$. We first show for $(f, g) \in H^2 \times H^1$ that there exists $(v, w) \in D(B)$ such that $B(v, w) = (f, g)$, and consequently $w = f \in H^2$ and $-Av = g \in H^1$. Let $y = -g \in H^1$. Then, by the Lax – Milgram lemma, there exists a unique function $v \in H^2$ satisfying

$$\langle \Delta v, \Delta \psi \rangle + \langle \nabla v, \nabla \psi \rangle + \langle v, \psi \rangle = \langle y, \psi \rangle + \varepsilon \langle \nabla y, \nabla \psi \rangle$$

for all $\psi \in H^2$. Hence, we see that $v \in D(A)$ as well as $Av = y$, and the operator B is onto. The fact $\|v\|_{H^3} \le C\|Av\|_{H^1}$ implies that the operator B is one-to-one. On the other hand, we have, for $v \in D(A)$ such that $-Av = g$,

$$\|B^{-1}(f,g)\|_X^2 = \|B^{-1}(B(v,w))\|_X^2 = \|(v,w)\|_X^2 = \|v\|_{H^2}^2 + \|w\|_{H^1}^2 \le$$
$$\le \|v\|_{H^3}^2 + \|f\|_{H^1}^2 \le C\|Av\|_{H^1}^2 + \|f\|_{H^2}^2 = C\| - g\|_{H^1}^2 + \|f\|_{H^2}^2 \le C\|(f,g)\|_X^2,$$

where $X := H^2 \times H^1$. Hence, $0 \in \rho(B)$ and B^{-1} is continuous. Also, $D(B) = H^3 \times H^2$ is dense in $H^2 \times H^1$. Using again the facts $D(A) = H^3$, and $\|v\|_{H^3} \le C\|Av\|_{H^1}$ for all $v \in D(A)$ and some $C > 0$, the proof of Lemma 5.2 follows.

References

1. R. A. Adams, J. J. F. Fournier, *Sobolev spaces*, Mathematics, **140**, Acad. Press, New York (1975).
2. H. Brezis, T. Cazenave, *Nonlinear evolution equations*, Univ. Pierre et Marie Curie (1994).
3. P. Coullet, S. Fauve, *Propagative phase dynamics for systems with Galilean invariance*, Phys. Rev. Lett., **55**, 2857 – 2859 (1985).
4. C. R. da Luz, R. C. Charão, *Asymptotic properties for a semilinear plate equation in unbounded domains*, J. Hyperbolic Different. Equat., **6**, 269 – 294 (2009).
5. E. B. Davis, A. M. Hinz, *Explicit constants for Rellich inequalities in* $L^p(\Omega)$, Math. Z., **227**, 511 – 523 (1998).
6. M. C. Depassier, *Evolution equation for bidirectional surface waves in a convecting fluid*, Phys. Fluids, **18**, Article 107102 (2006).
7. S. Fauve, E.W. Bolton, M. E. Brachet, *Nonlinear oscillatory convection: A quantitative phase approach*, Phys. D, **29**, 202 – 214 (1987).
8. P. Galenko, *Phase-field model with relaxation of the diffusion flux in nonequilibrium solidification of a binary system*, Phys. Lett. A, **287**, 190 – 197 (2001).
9. P. Galenko, D. Jou, *Diffuse-interface model for rapid phase transformations in nonequilibrium systems*, Phys. Rev. E, **71**, Article 046125 (2005).
10. P. Galenko, V. Lebedev, *Analysis of the dispersion relation in spinodal decomposition of a binary system*, Philos. Mag. Lett., **87**, 821 – 827 (2007).
11. R. Ikehata, *Energy decay of solution for the semilinear dissipative wave equations in an exterior domain*, Funkc. Ekvac., **44**, 487 – 499 (2001).
12. Y. Liu, S. Kawashima, *Global existence and asymptotic behavior of solutions for quasi-linear dissipative plate equation*, Discrete Contin. Dyn. Syst. Ser. A, **29**, 1113 – 1139 (2011).
13. Y. Liu, S. Kawashima, *Global existence and decay of solutions for a quasi-linear dissipative plate equation*, J. Hyperbolic Different. Equat., **8**, 591 – 614 (2011).
14. Z. Liu, S. Zheng, *Semigroups associated with dissipative systems*, Chapman and Hall/CRC (1999).
15. Y. Sugitani, S. Kawashima, Decay estimates of solutions to a semi-linear dissipative plate equation, J. Hyperbolic Different. Equat., **7**, 471 – 501 (2010).
16. I. E. Segal, Dispersion for non-linear relativistic equations, II, Ann. Sci. École Norm. Super, **1**, 459 – 497 (1968).
17. H. Takeda, Y. Maekawa, S. Kawashima, Asymptotic profile of solutions to a hyperbolic Cahn – Hilliard equation, Bull. Inst. Math. Acad. Sin. (N.S.), **10**, 479 – 539 (2015).
18. Y. Wang, F. Liu, Y. Zhang, Global existence and asymptotic of solutions for a semi-linear wave equation, J. Math. Anal. and Appl., **385**, 836 – 853 (2012).
19. Y. Wang, Z. Wei, *Global existence and asymptotic behavior of solutions to Cahn – Hilliard equation with inertial term*, Int. J. Math., **23**, Article 1250087 (2012).

Received 14.11.17,
after revision — 02.01.18