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ON THE EXISTENCE OF SOLUTIONS OF ONE-DIMENSIONAL FOURTH-ORDER EQUATIONS

ПРО ІСНУВАННЯ РОЗВ'ЯЗКІВ ОДНОВИМІРНИХ РІВНЯНЬ ЧЕТВЕРТОГО ПОРЯДКУ

Using variational methods and critical point theorems, we prove the existence of nontrivial solutions for one-dimensional fourth-order equations. Multiplicity results are also pointed out.

За допомогою варіаційних методів та теореми про критичні точки доведено існування нетривіальних розв'язків одновимірних рівнянь четвертого порядку. Також наведено відповідні результати щодо кратності.

1. Introduction. In this paper, we consider the following fourth-order boundary-value problem:

$$\begin{gathered} u^{iv}h(x,u') - u'' = [\lambda f(x,u) + g(u)]h(x,u') \qquad \text{in} \quad]\,0,1\,[, \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{gathered} \tag{1.1}$$

where λ is a positive parameter, $f\colon [0,1]\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function, $g\colon\mathbb{R}\to\mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L>0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, with $g(0) = 0$, and $h\colon [0,1]\times\mathbb{R}\to[0,+\infty[$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$.

Boundary-value problems for ordinary differential equations play a fundamental role both in theory and applications. To establish the existence and multiplicity of solutions to nonlinear differential problems is very important as well as the application of such results in the physical reality. In fact, it is well-known that the mathematical modelling of important questions in different fields of research, such as mechanical engineering, control systems, economics, computer science and many others, leads naturally to the consideration of nonlinear differential equations. In particular, the deformations of an elastic beam in an equilibrium state, whose two ends are simply supported, can be described by fourth-order boundary-value problems. The work of Timoshenko [21] on elasticity, the monograph by Soedel [19], the paper by Palamides [12] on deformation of elastic membrane, and the work of Dulàcska [10] on the effects of soil settlement are rich sources of such applications. Pietramala [14] presented some results on the existence of multiple positive solutions of a fourth-order differential equation subject to nonlocal and nonlinear boundary conditions that models a particular stationary state of an elastic beam with nonlinear controllers.

For this reason, the existence and multiplicity of solutions for this kind of problems have been widely investigated (see, for instance, [1, 4, 6–9, 18, 20] and references therein).

In the above papers, the right-hand side of equation is independent of u'. The main novelty of our paper is the fact that we apply a recent critical-points result to a fourth-order equation given in the form of (1.1), in which the right-hand side is dependent on u'. Such equations are also studied by many authors, but most of them are of the second-order (see, for instance, [3, 11, 22]).

In the present paper, based on a recent critical point theorem of Bonanno (see Theorem 2.1 below), we obtain the existence of at least one solution for problem (1.1). It is worth noticing that, usually, to obtain the existence of one solution, asymptotic conditions both at zero and at infinity on the nonlinear term are requested, while, here, it is assumed only a unique algebraic condition (see (A_7) in Corollary 3.2). As a consequence, by combining with the classical Ambrosetti – Rabinowitz condition (see [2]), the existence of two solutions is obtained (see Theorem 3.5).

As an example, we state here the following special case of our results.

Theorem 1.1. *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonnegative continuous function such that*

$$4\int\limits_0^{64} f(x)\,dx < 27\int\limits_0^{1} f(x)\,dx.$$

Then, for each

$$\lambda \in \left]\frac{2^{13}}{27}\frac{(\pi^4+\pi^2+1)}{\pi^4\displaystyle\int_0^1 f(x)\,dx},\ \frac{2^{11}(\pi^4+\pi^2+1)}{\pi^4\displaystyle\int_0^{64} f(x)\,dx}\right[,$$

the problem

$$u^{iv} - u'' + u = \lambda f(u) \qquad \text{in} \quad]\,0,1\,[,$$

$$u(0) = u(1) = 0 = u''(0) = u''(1),$$

admits at least one positive classical solution $\bar{u}$ such that $\bar{u}(x) < 64$ for all $x \in [0,1]$.

2. Preliminaries. Our main tool is Ricceri's variational principle [17] (Theorem 2.5) as given in [5] (Theorem 5.1) which is below recalled (see also [5], Proposition 2.1).

For a given nonempty set X, and two functionals Φ, $\Psi : X \to \mathbb{R}$, we define the following functions:

$$\beta(r_1,r_2) = \inf_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\sup_{u\in\Phi^{-1}(]r_1,r_2[)}\Psi(u) - \Psi(v)}{r_2 - \Phi(v)}$$

and

$$\rho(r_1,r_2) = \sup_{v\in\Phi^{-1}(]r_1,r_2[)} \frac{\Psi(v) - \sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$.

Theorem 2.1 ([5], Theorem 5.1)**.** *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are r_1, $r_2 \in \mathbb{R}$, with $r_1 < r_2$, such that*

$$\beta(r_1,r_2) < \rho(r_1,r_2). \tag{2.1}$$

Then setting $I_\lambda := \Phi - \lambda\Psi$ *for each* $\lambda \in \left]\dfrac{1}{\rho(r_1,r_2)}, \dfrac{1}{\beta(r_1,r_2)}\right[$ *there is* $u_{0,\lambda} \in \Phi^{-1}\big(]r_1,r_2[\big)$ *such that* $I_\lambda(u_{0,\lambda}) \le I_\lambda(u)$ *for all* $u \in \Phi^{-1}\big(]r_1,r_2[\big)$ *and* $I'_\lambda(u_{0,\lambda}) = 0$.

Let us introduce some notation which will be used later. Define

$$H_0^1([0,1]) := \left\{u \in L^2([0,1]) : u' \in L^2([0,1]),\ u(0) = u(1) = 0\right\},$$

$$H^2([0,1]) := \left\{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\right\}.$$

Let $X := H^2([0,1]) \cap H_0^1([0,1])$ be the Sobolev space endowed with the usual norm defined as follows:

$$\|u\| := \left(\int\limits_0^1 |u''(t)|^2\,dt\right)^{1/2}.$$

We recall the following Poincaré type inequalities (see, for instance, [13], Lemma 2.3):

$$\|u'\|^2_{L^2([0,1])} \le \frac{1}{\pi^2}\|u\|^2, \tag{2.2}$$

$$\|u\|^2_{L^2([0,1])} \le \frac{1}{\pi^4}\|u\|^2 \tag{2.3}$$

for all $u \in X$. For the norm in $C^1([0,1])$,

$$\|u\|_\infty := \max\left\{\max_{x\in[0,1]}|u(x)|,\ \max_{x\in[0,1]}|u'(x)|\right\},$$

we have the following relation.

Proposition 2.1. *Let* $u \in X$. *Then*

$$\|u\|_\infty \le \frac{1}{2\pi}\|u\|. \tag{2.4}$$

Proof. Taking (2.2) into account, the conclusion follows from the known inequality $\|u\|_\infty \le \le \dfrac{1}{2}\|u'\|_{L^2([0,1])}$.

Let $g\colon \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for every $t_1,\ t_2 \in \mathbb{R}$, and $g(0) = 0$, $h\colon [0,1]\times\mathbb{R} \to [0,+\infty]$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$, and $f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function.

We recall that $f\colon [0,1]\times\mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;

(b) the mapping $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;

(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that

$$\sup_{|\xi|\le\rho} |f(x,\xi)| \le l_\rho(x)$$

for almost every $x \in [0,1]$.

Corresponding to $f,$ g and h we introduce the functions $F\colon [0,1]\times\mathbb{R}\to\mathbb{R}$, $G\colon\mathbb{R}\to\mathbb{R}$ and $H\colon [0,1]\times\mathbb{R}\to[0,+\infty]$, respectively, as follows:

$$F(x,t):=\int\limits_0^t f(x,\xi)\,d\xi,\qquad G(t):=-\int\limits_0^t g(\xi)\,d\xi,$$

$$H(x,t):=\int\limits_0^t\left(\int\limits_0^\tau\frac{1}{h(x,\delta)}d\delta\right)d\tau$$

for all $x\in[0,1]$ and $t\in\mathbb{R}$.

In the following, suppose that the Lipschitz constant $L>0$ of the function g satisfies the condition $L<\pi^4$.

We say that a function $u\in X$ is a *weak solution* of problem (1.1) if

$$\int\limits_0^1 u''(x)v''(x)\,dx+\int\limits_0^1\left(\int\limits_0^{u'(x)}\frac{1}{h(x,\tau)}d\tau\right)v'(x)\,dx-$$

$$-\lambda\int\limits_0^1 f(x,u(x))v(x)\,dx-\int\limits_0^1 g(u(x))v(x)\,dx=0$$

holds for all $v\in X$.

By standard regularity results, if f is continuous in $[0,1]\times\mathbb{R}$, then weak solutions of problem (1.1) belong to $C^2([0,1])$, thus they are classical solutions.

3. Main results. In this section we present our main results. Put

$$A:=\frac{\pi^4-L}{2\pi^4},\qquad B:=\frac{\pi^2+m(\pi^4+L)}{2m\pi^4},$$

and suppose that $B\le 4\pi^2A$. Given a nonnegative constant c_1 and two positive constants c_2 and d with $c_1^2<\dfrac{4096}{27}d^2<c_2^2$, put

$$a(c_2,d):=\frac{\int_0^1\max_{|t|\le c_2}F(x,t)\,dx-\int_{3/8}^{5/8}F(x,d)\,dx}{27Bc_2^2-4096Bd^2},$$

$$b(c_1,d):=\frac{\int_{3/8}^{5/8}F(x,d)\,dx-\int_0^1\max_{|t|\le c_1}F(x,t)\,dx}{4096Bd^2-27Ac_1^2}.$$

Theorem 3.1. *Assume that there exist a nonnegative constant c_1 and two positive constants c_2 and d, with $c_1^2<\dfrac{4096}{27}d^2<c_2^2$, such that*

(A$_1$) $F(x,t)\ge 0$ *for all* $(x,t)\in\left(\left[0,\dfrac{3}{8}\right]\bigcup\left[\dfrac{5}{8},1\right]\right)\times[0,d]$;

(A_2) $a(c_2, d) < b(c_1, d)$.

Then, for each $\lambda \in \left] \frac{1}{27\, b(c_1, d)}, \frac{1}{27\, a(c_2, d)} \right[$, *problem* (1.1) *admits at least one nontrivial weak solution* $\bar{u} \in X$ *such that*

$$\frac{A}{B} c_1^2 < \|\bar{u}\|^2 < \frac{B}{A} c_2^2.$$

Proof. Our aim is to apply Theorem 2.1 to our problem. To this end, for every $u \in X$, we introduce the functionals $\Phi,\ \Psi \colon X \to \mathbb{R}$ by setting

$$\Phi(u) := \frac{1}{2}\|u\|^2 + \int_0^1 H(x, u'(x))\, dx + \int_0^1 G(u(x))\, dx,$$

$$\Psi(u) := \int_0^1 F(x, u(x))\, dx$$

and put

$$I_\lambda(u) := \Phi(u) - \lambda \Psi(u) \quad \forall u \in X.$$

Note that the weak solutions of (1.1) are exactly the critical points of I_λ. The functionals Φ and Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by

$$\Phi'(u)(v) = \int_0^1 u''(x) v''(x)\, dx + \int_0^1 \left(\int_0^{u'(x)} \frac{1}{h(x,\tau)} d\tau \right) v'(x)\, dx - \int_0^1 g(u(x)) v(x)\, dx$$

for any $v \in X$. Furthermore, the differential $\Phi' \colon X \to X^*$ is a Lipschitzian operator. Indeed, taking (2.2) and (2.3) into account, for any $u, v \in X$, there holds

$$\|\Phi'(u) - \Phi'(v)\|_{X^*} = \sup_{\|w\| \le 1} |(\Phi'(u) - \Phi'(v), w)| \le$$

$$\le \sup_{\|w\| \le 1} \int_0^1 |u''(x) - v''(x)||w''(x)|\, dx + \sup_{\|w\| \le 1} \int_0^1 \left| \int_{u'(x)}^{v'(x)} \frac{1}{h(x,\tau)} d\tau \right| |w'(x)|\, dx +$$

$$+ \sup_{\|w\| \le 1} \int_0^1 |g(u(x)) - g(v(x))||w(x)|\, dx \le$$

$$\le \sup_{\|w\| \le 1} \|u - v\| \|w\| + \frac{1}{m} \sup_{\|w\| \le 1} \|u' - v'\|_{L^2([0,1])} \|w'\|_{L^2([0,1])} +$$

$$+ L \sup_{\|w\| \le 1} \|u - v\|_{L^2([0,1])} \|w\|_{L^2([0,1])} \le$$

$$\le \left(1 + \frac{1}{m\pi^2} + \frac{L}{\pi^4}\right) \|u - v\| = 2B\|u - v\|.$$

In particular, we derive that Φ is continuously differentiable. Also, for any $u, v \in X$, we have

$$(\Phi'(u) - \Phi'(v), u - v) = \|u - v\|^2 + \int\limits_0^1 \left(\int\limits_{u'(x)}^{v'(x)} \frac{1}{h(x,\tau)} d\tau \right) (u'(x) - v'(x))\, dx -$$

$$- \int\limits_0^1 (g(u(x)) - g(v(x)))(u(x) - v(x))\, dx \geq$$

$$\geq \|u - v\|^2 + \frac{1}{M} \|u' - v'\|^2_{L^2([0,1])} - L\|u - v\|^2_{L^2([0,1])} \geq$$

$$\geq \|u - v\|^2 - \frac{L}{\pi^4} \|u - v\|^2 = 2A\|u - v\|^2.$$

By the assumption $L < \pi^4$, it turns out that Φ' is a strongly monotone operator. So, by applying Minty–Browder theorem (Theorem 26.A of [23]), $\Phi' \colon X \to X^*$ admits a Lipschitz continuous inverse. On the other hand, the fact that X is compactly embedded into $C^0([0,1])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int\limits_0^1 f(x, u(x)) v(x)\, dx$$

for any $v \in X$.

Since g is Lipschitz continuous and satisfies $g(0) = 0$, while h is bounded away from zero, the inequalities (2.2) and (2.3) yield for any $u \in X$ the estimate

$$A\|u\|^2 \leq \Phi(u) \leq B\|u\|^2. \tag{3.1}$$

Now, put

$$r_1 := Ac_1^2, \qquad r_2 := Bc_2^2$$

and

$$w(x) = \begin{cases} -\dfrac{64d}{9}\left(x^2 - \dfrac{3}{4}x\right), & x \in \left]0, \dfrac{3}{8}\right[, \\ d, & x \in \left]\dfrac{3}{8}, \dfrac{5}{8}\right[, \\ -\dfrac{64d}{9}\left(x^2 - \dfrac{5}{4}x + \dfrac{1}{4}\right), & x \in \left]\dfrac{5}{8}, 1\right[. \end{cases}$$

It is easy to verify that $w \in X$ and, in particular,

$$\|w\|^2 = \frac{4096}{27} d^2.$$

So, taking (3.1) into account, we deduce

$$\frac{4096}{27}Ad^2 \leq \Phi(w) \leq \frac{4096}{27}Bd^2.$$

From the condition $c_1^2 < \dfrac{4096}{27}d^2 < c_2^2$, we obtain $r_1 < \Phi(w) < r_2$. Since $B \leq 4\pi^2 A$, for all $u \in X$ such that $u \in \Phi^{-1}\big(]-\infty, r_2[\big)$, from (2.4), one has $|u(x)| < c_2$ for all $x \in [0,1]$, which implies

$$\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u) = \sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \int\limits_0^1 F(x,u(x))\,dx \leq \int\limits_0^1 \max_{|t|\leq c_2} F(x,t)\,dx.$$

Arguing as before, we obtain

$$\sup_{u\in\Phi^{-1}(]-\infty,r_1])} \Psi(u) \leq \int\limits_0^1 \max_{|t|\leq c_1} F(x,t)\,dx.$$

Since $0 \leq w(x) \leq d$ for each $x \in [0,1]$, assumption (A$_1$) ensures that

$$\int\limits_0^{3/8} F(x,w(x))\,dx + \int\limits_{5/8}^{1} F(x,w(x))\,dx \geq 0,$$

and

$$\Psi(w) \geq \int\limits_{3/8}^{5/8} F(x,d)\,dx.$$

Therefore, one has

$$\beta(r_1,r_2) \leq \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u) - \Psi(w)}{r_2 - \Phi(w)} \leq$$

$$\leq \frac{27\left(\int_0^1 \max_{|t|\leq c_2} F(x,t)\,dx - \int_{3/8}^{5/8} F(x,d)\,dx\right)}{27Bc_2^2 - 4096Bd^2} =$$

$$= 27a(c_2,d).$$

On the other hand, we have

$$\rho(r_1,r_2) \geq \frac{\Psi(w) - \sup_{u\in\Phi^{-1}(]-\infty,r_1])}\Psi(u)}{\Phi(w) - r_1} \geq$$

$$\geq \frac{27\left(\int_{3/8}^{5/8} F(x,d)\,dx - \int_0^1 \max_{|t|\leq c_1} F(x,t)\,dx\right)}{4096Bd^2 - 27Ac_1^2} =$$

$$= 27b(c_1,d).$$

So, from assumption (A_2), it follows that $\beta(r_1,r_2) < \rho(r_1,r_2)$. Therefore, from Theorem 2.1, for each $\lambda \in \left]\dfrac{1}{27b(c_1,d)}, \dfrac{1}{27a(c_2,d)}\right[$, the functional I_λ admits at least one critical point $\bar{u}$ such that

$$r_1 < \Phi(\bar{u}) < r_2,$$

that is,

$$\frac{A}{B}c_1^2 < \|\bar{u}\|^2 < \frac{B}{A}c_2^2.$$

Theorem 3.1 is proved.

Now, we point out the following consequence of Theorem 3.1.

Theorem 3.2. *Assume that there exist two positive constants* c *and* d, *with* $64\,d < 3\sqrt{3}\,c$, *such that assumption* (A_1) *in Theorem* 3.1 *holds. Furthermore, suppose that*

$$(\mathrm{A}_3)\qquad \frac{\displaystyle\int_0^1 \max_{|t|\le c} F(x,t)\,dx}{c^2} < \frac{27}{4096}\frac{\displaystyle\int_{3/8}^{5/8} F(x,d)\,dx}{d^2}.$$

Then, for each

$$\lambda \in \left]\frac{4096}{27}\frac{Bd^2}{\displaystyle\int_{3/8}^{5/8} F(x,d)\,dx}, \frac{Bc^2}{\displaystyle\int_0^1 \max_{|t|\le c} F(x,t)\,dx}\right[,$$

problem (1.1) *admits at least one nontrivial weak solution* $\bar{u} \in X$ *such that* $|\bar{u}(x)| < c$ *for all* $x \in [0,1]$.

Proof. The conclusion follows from Theorem 3.1, by taking $c_1 = 0$ and $c_2 = c$. Indeed, owing to assumption (A_3), one has

$$a(c,d) = \frac{\displaystyle\int_0^1 \max_{|t|\le c} F(x,t)\,dx - \int_{3/8}^{5/8} F(x,d)\,dx}{27Bc^2 - 4096Bd^2} <$$

$$< \frac{\left(1 - \dfrac{4096d^2}{27c^2}\right)\displaystyle\int_0^1 \max_{|t|\le c} F(x,t)\,dx}{B(27c^2 - 4096d^2)} = \frac{1}{27Bc^2}\int_0^1 \max_{|t|\le c} F(x,t)\,dx.$$

On the other hand,

$$b(0,d) = \frac{\displaystyle\int_{3/8}^{5/8} F(x,d)\,dx}{4096Bd^2}.$$

Now, owing to assumption (A_3) and (2.4), it is sufficient to invoke Theorem 3.1 for concluding the proof.

The following result gives the existence of at least one nontrivial weak solution in X to problem (1.1) in the autonomous case. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function and put $F(t) := \displaystyle\int_0^t f(\xi)\,d\xi$ for all $t \in \mathbb{R}$.

***Corollary* 3.1.** *Assume that there exist a nonnegative constant* c_1 *and two positive constants* c_2 *and* d, *with* $c_1^2 < \dfrac{4096}{27} d^2 < c_2^2$, *such that*

(A$_4$) $f(t) \geq 0$ *for all* $t \in [-c_2, \max\{c_2, d\}]$;

(A$_5$) $\dfrac{F(c_2) - \frac{1}{4}F(d)}{27Bc_2^2 - 4096Bd^2} < \dfrac{F(c_1) - \frac{1}{4}F(d)}{27Ac_1^2 - 4096Bd^2}$.

Then, for each

$$\lambda \in \left] \frac{1}{27} \frac{27Ac_1^2 - 4096Bd^2}{F(c_1) - \frac{1}{4}F(d)}, \ \frac{1}{27} \frac{27Bc_2^2 - 4096Bd^2}{F(c_2) - \frac{1}{4}F(d)} \right[,$$

the problem

$$u^{iv} h(x, u') - u'' = [\lambda f(u) + g(u)] h(x, u') \qquad \text{in} \quad]\,0, 1\,[,$$
$$u(0) = u(1) = 0 = u''(0) = u''(1),$$

admits at least one nontrivial weak solution $\bar{u} \in X$ *such that*

$$\frac{A}{B} c_1^2 < \|\bar{u}\|^2 < \frac{B}{A} c_2^2.$$

Proof. From the condition $c_1^2 < \dfrac{4096}{27} d^2 < c_2^2$, we obtain $c_1 < c_2$. Therefore, assumption (A$_4$) means $f(t) \geq 0$ for each $t \in [-c_1, c_1]$ and $f(t) \geq 0$ for each $t \in [-c_2, c_2]$, which implies

$$\max_{t \in [-c_1, c_1]} F(t) = F(c_1) \qquad \text{and} \qquad \max_{t \in [-c_2, c_2]} F(t) = F(c_2).$$

So, from assumptions (A$_4$) and (A$_5$), we arrive at assumptions (A$_1$) and (A$_2$), respectively. Hence, Theorem 3.1 yields the conclusion.

Here, we point out a special situation of our main result when the nonlinear term has separable variables. To be precise, let $\alpha \in L^1([0, 1])$ be such that $\alpha(x) \geq 0$ a.e. $x \in [0, 1]$, $\alpha \not\equiv 0$, and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Consider the following Dirichlet boundary-value problem:

$$\begin{gathered} u^{iv} h(x, u') - u'' = [\lambda \alpha(x) \gamma(u) + g(u)] h(x, u') \qquad \text{in} \quad]\,0, 1\,[, \\ u(0) = u(1) = 0 = u''(0) = u''(1). \end{gathered} \tag{3.2}$$

Put

$$\Gamma(t) := \int_0^t \gamma(\xi)\, d\xi \qquad \text{for all} \quad t \in \mathbb{R},$$

and set

$$\|\alpha\|_1 := \int_0^1 \alpha(x)\, dx, \qquad \alpha_0 := \int_{3/8}^{5/8} \alpha(x)\, dx.$$

Theorem 3.3. *Assume that there exist a nonnegative constant* c_1 *and two positive constants* c_2 *and* d, *with* $c_1^2 < \dfrac{4096}{27} d^2 < c_2^2$, *such that*

(A$_6$) $\dfrac{\Gamma(c_2)\|\alpha\|_1 - \Gamma(d)\alpha_0}{27Bc_2^2 - 4096Bd^2} < \dfrac{\Gamma(d)\alpha_0 - \Gamma(c_1)\|\alpha\|_1}{4096Bd^2 - 27Ac_1^2}$.

Then, for each

$$\lambda \in \left] \frac{1}{27} \frac{4096Bd^2 - 27Ac_1^2}{\Gamma(d)\alpha_0 - \Gamma(c_1)\|\alpha\|_1}, \ \frac{1}{27} \frac{27Bc_2^2 - 4096Bd^2}{\Gamma(c_2)\|\alpha\|_1 - \Gamma(d)\alpha_0} \right[,$$

problem (3.2) *admits at least one positive weak solution* $\bar{u} \in X$, *such that*

$$\frac{A}{B} c_1^2 < \|\bar{u}\|^2 < \frac{B}{A} c_2^2.$$

Proof. Put $f(x,\xi) := \alpha(x)\gamma(\xi)$ for all $(x,\xi) \in [0,1] \times \mathbb{R}$. Clearly, $F(x,t) = \alpha(x)\Gamma(t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$. Therefore, applying Theorem 3.1, problem (3.2) admits at least one nontrivial weak solution $\bar{u} \in X$ such that

$$\frac{A}{B} c_1^2 < \|\bar{u}\|^2 < \frac{B}{A} c_2^2.$$

Now, we prove here that the attained solution is positive. Arguing by contradiction, if we assume that $\bar{u}$ is negative at a point of $[0,1]$, the set

$$\Omega := \{x \in [0,1] : \bar{u}(x) < 0\}$$

is nonempty and open. Moreover, let us consider $\bar{v} := \min\{\bar{u}, 0\}$, one has $\bar{v} \in X$. So, taking into account that $\bar{u}$ is a weak solution and by choosing $v = \bar{v}$, from our assumptions, one has

$$0 \geq \lambda \int\limits_{\Omega} \alpha(x)\gamma(\bar{u}(x))\bar{u}(x)\, dx =$$

$$= \int\limits_{\Omega} |\bar{u}''(x)|^2\, dx + \int\limits_{\Omega} \left(\int\limits_0^{\bar{u}'(x)} \frac{1}{h(x,\tau)} d\tau \right) \bar{u}'(x)\, dx - \int\limits_{\Omega} g(\bar{u}(x))\bar{u}(x)\, dx \geq$$

$$\geq \frac{\pi^4 - L}{\pi^4} \|\bar{u}\|^2_{H^2(\Omega) \cap H_0^1(\Omega)}.$$

Therefore, $\|\bar{u}\|_{H^2(\Omega) \cap H_0^1(\Omega)} = 0$ which is absurd. Hence, owing to the strong maximum principle (see, e.g., [15], Theorem 11.1) the weak solution $\bar{u}$, being nontrivial, is positive and the conclusion is achieved.

Theorem 3.3 is proved.

An immediate consequence of Theorem 3.3 is the following.

***Corollary* 3.2.** *Assume that there exist positive constants* c *and* d, *with* $64\, d < 3\sqrt{3}\, c$, *such that*

(A$_7$) $\dfrac{\Gamma(c)\|\alpha\|_1}{c^2} < \dfrac{27}{4096} \dfrac{\Gamma(d)\alpha_0}{d^2}$.

Then, for each

$$\lambda \in \left] \frac{4096}{27} \frac{Bd^2}{\Gamma(d)\alpha_0}, \ \frac{Bc^2}{\Gamma(c)\|\alpha\|_1} \right[,$$

problem (3.2) *admits at least one positive weak solution* $\bar{u} \in X$, *such that* $\bar{u}(x) < c$ *for all* $x \in [0,1]$.

Proof. This follows directly from Theorem 3.2.

***Remark* 3.1.** Theorem 1.1 in the introduction is an immediate consequence of Corollary 3.2, on choosing $g(u) = -u,\ h \equiv 1,\ c = 64$ and $d = 1$.

Here, we point out another relevant consequence of Corollary 3.2.

Theorem 3.4. *Assume that*

(A_8) $\lim_{t\to 0^+} \dfrac{\gamma(t)}{t} = +\infty,$

and put $\lambda^\star := \dfrac{B}{\|\alpha\|_1} \sup_{c>0} \dfrac{c^2}{\Gamma(c)}$. *Then, for each* $\lambda \in \,]0, \lambda^\star[$, *problem* (3.2) *admits at least one positive weak solution.*

Proof. For fixed λ as in the conclusion, there exists a positive constant c such that

$$\lambda < \frac{Bc^2}{\Gamma(c)\|\alpha\|_1}. \tag{3.3}$$

Moreover, assumption (A_8) implies that $\lim_{t\to 0^+} \dfrac{\Gamma(t)}{t^2} = +\infty$. Therefore, there is $d < \dfrac{3\sqrt{3}}{64}c$ such that

$$\frac{27}{4096}\frac{\Gamma(d)\alpha_0}{Bd^2} > \frac{1}{\lambda}.$$

Hence, Corollary 3.2 implies the conclusion.

***Remark* 3.2.** Taking (A_8) into account, fix $\rho > 0$ such that $\gamma(t) > 0$ for all $t \in]0, \rho[$. Then put

$$\lambda_\rho := \frac{B}{\|\alpha\|_1} \sup_{c\in]0,\rho[} \frac{c^2}{\Gamma(c)}.$$

Theorem 3.4 for every $\lambda \in]0, \lambda_\rho[$ holds with $\bar{u}(x) < \rho$ for all $x \in [0,1]$, where $\bar{u}$ is the ensured positive weak solution in X.

Here, we present the following example to illustrate the applicability of our results.

***Example* 3.1.** Let $\alpha(x) = 1 + x,\ \gamma(t) = e^t,\ g(t) = -t$ and $h(x,t) = (2 + x + \cos t)^{-1}$ for all $x \in [0,1]$ and $t \in \mathbb{R}$. It is clear that $\lim_{t\to 0^+} \gamma(t)/t = +\infty$. Pick $\rho = 1$. Hence, taking Remark 3.2 into account, by applying Theorem 3.4, since

$$B = \frac{\pi^4 + 4\pi^2 + 1}{2\pi^4},$$

for every

$$\lambda \in \left]0,\ \frac{\pi^4 + 4\pi^2 + 1}{3\pi^4(e-1)}\right[,$$

problem

$$u^{iv} - u''(2 + x + \cos u') + u = \lambda e^u(1 + x) \qquad \text{in} \quad]\,0,1\,[,$$

$$u(0) = u(1) = 0 = u''(0) = u''(1),$$

has at least one positive weak solution $\bar{u} \in X$ such that $\|\bar{u}\|_\infty < 1$.

Next, as consequence of Theorem 3.1, taking into account the classical theorem of Ambrosetti and Rabinowitz, we have the following multiplicity result.

Theorem 3.5. *Let the assumptions of Theorem* 3.1 *be satisfied, and* $f(\cdot,0)\neq 0$ *in* $]\,0,1\,[$. *Moreover, let*

(A_9) *there exist positive constants* ν *and* R *such that* $\nu A > 2B$, *and for all* $|t|\geq R$ *and for all* $x\in[0,1]$, *one has*

$$0<\nu F(x,t)\leq t\cdot f(x,t).$$

Then, for each $\lambda\in\left]\dfrac{1}{27b(c_1,d)},\dfrac{1}{27a(c_2,d)}\right[$, *problem* (1.1) *admits at least two nontrivial weak solutions* $\bar u_1,\bar u_2$, *such that*

$$\frac{A}{B}c_1^2<\|\bar u_1\|^2<\frac{B}{A}c_2^2. \tag{3.4}$$

Proof. Fix λ as in the conclusion. So, Theorem 3.1 ensures that problem (1.1) admits at least one nontrivial weak solution $\bar u_1$ satisfying the condition (3.4) which is a local minimum of the functional I_λ.

Now, we prove the existence of the second solution distinct from the first one. To this end, we must show that the functional I_λ satisfies the hypotheses of the mountain pass theorem.

Clearly, the functional I_λ is of class C^1 and $I_\lambda(0)=0$.

We can assume that $\bar u_1$ is a strict local minimum for I_λ in X. Therefore, there is $\rho>0$ such that $\inf_{\|u-\bar u_1\|=\rho} I_\lambda(u)>I_\lambda(\bar u_1)$, so condition [16] (($I_1$), Theorem 2.2) is verified.

From (A_9), by standard computations, there is a positive constant C such that

$$F(x,t)\geq C|t|^\nu \tag{3.5}$$

for all $x\in[0,1]$ and $|t|>R$. In fact, setting $a(x):=\min_{|\xi|=R}F(x,\xi)$ and

$$\varphi_t(s):=F(x,st)\quad \forall s>0, \tag{3.6}$$

by (A_9), for every $x\in[0,1]$ and $|t|>R$ one has

$$0<\nu\varphi_t(s)=\nu F(x,st)\leq st\cdot f(x,st)=s\varphi_t'(s)\quad \forall s>\frac{R}{|t|}.$$

Therefore,

$$\int\limits_{R/|t|}^{1}\frac{\varphi_t'(s)}{\varphi_t(s)}ds\geq\int\limits_{R/|t|}^{1}\frac{\nu}{s}ds.$$

Then

$$\varphi_t(1)\geq\varphi_t\left(\frac{R}{|t|}\right)\frac{|t|^\nu}{R^\nu}.$$

Taking into account of (3.6), we obtain

$$F(x,t)\geq F\left(x,\frac{R}{|t|}t\right)\frac{|t|^\nu}{R^\nu}\geq a(x)\frac{|t|^\nu}{R^\nu}\geq C|t|^\nu,$$

where $C>0$ is a constant. Thus, (3.5) is proved. Now, choosing any $u\in X\setminus\{0\}$, one has

$$I_\lambda(tu)=(\Phi-\lambda\Psi)(tu)\leq$$

$$\leq Bt^2\|u\|^2 - \lambda t^\nu C\int\limits_0^1 |u(x)|^\nu dx \to -\infty$$

as $t \to +\infty$ (since $\nu > 2$). So, the functional I_λ is unbounded from below and condition [16] ((I_2), Theorem 2.2) is verified. Therefore, I_λ satisfies the geometry of mountain pass.

Now, to verify the Palais – Smale condition it is sufficient to prove that any sequence of Palais – Smale is bounded. To this end, taking into account (A$_9$) one has

$$\nu I_\lambda(u_n) - \|I'_\lambda(u_n)\|_{X^*}\|u_n\| \geq \nu I_\lambda(u_n) - I'_\lambda(u_n)(u_n) =$$

$$= \nu\Phi(u_n) - \lambda\nu\Psi(u_n) - \Phi'(u_n)(u_n) + \lambda\Psi'(u_n)(u_n) \geq$$

$$\geq (\nu A - 2B)\,\|u_n\|^2 - \lambda\int\limits_0^1 \left[\nu F(x,u_n(x)) - f(x,u_n(x))u_n(x)\right] dx \geq$$

$$\geq (\nu A - 2B)\,\|u_n\|^2. \tag{3.7}$$

If $\{u_n\}$ is not bounded, from (3.7) we have a contradiction. Thus, I_λ satisfies the Palais – Smale condition.

Hence, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point $\bar{u}_2$ of I_λ such that $I_\lambda(\bar{u}_2) > I_\lambda(\bar{u}_1)$. So, $\bar{u}_1$ and $\bar{u}_2$ are two distinct weak solutions of (1.1).

Theorem 3.5 is proved.

***Corollary* 3.3.** *Assume that there exist two positive constants* c, d, *with* $64d < 3\sqrt{3}c$, *such that* (A$_7$) *holds. Assume also that*

(A$_{10}$) *there exist positive constants* ν *and* R *such that* $\nu A > 2B$, *and for all* $|t| \geq R$, *one has*

$$0 < \nu\Gamma(t) \leq t\cdot\gamma(t).$$

Then, for each

$$\lambda \in \left]\frac{4096}{27}\frac{Bd^2}{\Gamma(d)\alpha_0},\ \frac{Bc^2}{\Gamma(c)\|\alpha\|_1}\right[,$$

problem (3.2) *admits at least two nonnegative weak solutions* $\bar{u}_1$, $\bar{u}_2$, *such that* $\bar{u}_1(x) < c$ *for all* $x \in [0,1]$.

***Corollary* 3.4.** *Assume that* (A$_8$) *and* (A$_{10}$) *are satisfied. Then, for each* $\lambda \in\,]\,0, \lambda^\star[$, *problem* (3.2) *admits at least two nonnegative weak solutions.*

***Remark* 3.3.** If $\gamma(0) \neq 0$, Corollaries 3.3 and 3.4 ensure two positive weak solutions (see proof of Theorem 3.3).

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