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A SHORT NOTE ON THE NONCOPRIME REGULAR MODULE PROBLEM*

КОРОТКИЙ КОМЕНТАР ЩОДО ЗАДАЧІ ПРО РЕГУЛЯРНІ МОДУЛІ, ЩО НЕ Є ВЗАЄМНО ПРОСТИМИ

Considering a special configuration in which a finite group A acts by automorphisms on a finite group G and the semidirect product GA acts on the vector space V by linear transformations, we discuss the existence of a regular A-module in V_A.

Розглянуто спеціальну конфігурацію, в якій скінченна група A діє за допомогою автоморфізмів на скінченну групу G, а напівпрямий добуток GA — на векторний простір V за допомогою лінійних перетворень; обговорюється існування регулярного A-модуля у V_A.

1. Introduction. Let A be a finite group which acts faithfully on the vector space V by linear transformations. We say "A has a regular orbit on V" if there is a vector v in V such that $C_A(v) = 1$. In this case, the A-orbit containing v is called a regular A-orbit. Furthermore, V contains the regular A-module if a regular A-orbit happens to be linearly independent. More generally if A acts by linear transformations on the vector space V (not necessarily faithfully), then we say that A has a regular orbit on V or V contains the regular A-module if $A/C_A(V)$ does the same.

While studying the structure of a finite solvable group G admitting a certain group of automorphisms A, we are often forced to study A-invariant chief factors V of G together with the action of the semidirect product $(G/C_G(V))A$ on V. It turns out to be rather efficient to know that V contains the regular A-module or at least a regular A-orbit. Not all groups act with regular orbits although many interesting and rich classes do, especially under the additional assumptions of coprimeness that $(|G|, |A|) = 1 = (|V|, |GA|)$. There has been extensive research about the existence of regular orbits such as [1, 6–8, 11, 12] in the case of coprimeness and [2, 4, 5, 13, 14] in the noncoprime case. All the results concerning a nilpotent A are culminating in Theorem 1.1 in [14] which can be reformulated as follows:

Let G be a finite solvable group admitting a nilpotent group A as a group of automorphisms. Suppose that $C_{O_p(A)}(G) = 1$. Let V be a finite faithful kGA-module over a field k of characteristic p not dividing the order of G. Then A has at least one regular orbit on V if A involves no wreath product $\mathbb{Z}_2 \wr \mathbb{Z}_2$ and involves no wreath product $\mathbb{Z}_r \wr \mathbb{Z}_r$ for r a Mersenne prime when $p = 2$.

In the present paper, we prove a theorem which concludes the existence of a regular module without the coprimeness condition the prototype of which is Theorem 1.5 in [11]. This theorem was improved as Theorem B in [5] in case where the group GA is of odd order. For the convenience

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of the reader, we formulate the main conclusion of Theorem 1.5 in a way suitable to emphasize the similarities and differences between this theorem and Theorem B in [5] and our result.

Let PRA be a finite group where P is a p-group and R is an r-group for distinct primes p and r not dividing the order of A such that $P \lhd PRA$ and $R \lhd RA$. Assume that the following are satisfied:

(a) *P is an extraspecial p-group for some prime p where $Z(P) \leq Z(PRA)$ and $C_A(P) = 1$;*

(b) *$\bar{R} = R/R_0$ is of class at most two and of exponent r where $R_0 = C_R(P)$; suppose that $\left|C_A(\bar{R}/\Phi(\bar{R})\right|$ is either a prime or 1;*

(c) *$A/C_A(\bar{R}/\Phi(\bar{R})$ has a regular orbit in its action on $\bar{R}/\Phi(\bar{R})$;*
if $C_A(\bar{R}/\Phi(\bar{R}) \neq 1$, $[C_A(\bar{R}/\Phi(\bar{R}), P] \neq P$ and $p = 2$, assume that $\left|C_A(\bar{R}/\Phi(\bar{R})\right|$ is not a Fermat prime.

Let χ be a complex PRA-character such that χ_P is faithful. Then χ_A contains the regular A-character.

Namely we obtain the following theorem.

Theorem. *Let PRA be a finite group where P is a p-group and R is an r-group for distinct primes p and r such that $P \lhd PRA$ and $R \lhd RA$. Assume that the following are satisfied:*

(a) *P is an extraspecial p-group for some prime p where $Z(P) \leq Z(PRA)$ and $C_A(P) = 1$;*

(b) *R/R_0 is of class at most two and of exponent dividing r where $R_0 = C_R(P)$ and $A_0 = C_A(R/R_0) = 1$;*

(c) *$A = A_p \times A_r \times A_{\{p,r\}'}$ where its Sylow r-subgroup A_r and Sylow p-subgroup A_p are both cyclic and $A_{\{p,r\}'}$ acts with regular orbits on $R/\Phi(R)$;*

(d) *if $p = 2$ then r is not a Fermat prime.*

Let χ be a complex PRA-character such that χ_P is faithful. Then χ_A contains the regular A-character.

Notice that both p and r are allowed to divide the order of A.

All groups considered in this paper are finite and the notation and terminology are standard.

2. Existence of regular orbits. In this section, we present a result due to Dade [3] on the existence of regular orbits which will be applied in the proof of our theorem.

Proposition. *Let V be a faithful kA-module over a finite field k of characteristic p. Assume that $A = B \times C$ where B is a cyclic p-group and C is a p'-group which has a regular orbit on every C-invariant irreducible section of V. Then A has a regular orbit on V.*

Proof. Let $V_C = W_1 \oplus \ldots \oplus W_\ell$ be the decomposition of V into its C-homogeneous components. As B and C commute, each W_i is A-invariant. Therefore it suffices to prove that A has a regular orbit on W_i for each $i = 1, \ldots, \ell$. To see this let $w_i \in W_i$ be such that $C_A(w_i) = C_A(W_i)$ for $i = 1, \ldots, \ell$. If $v = w_1 + \ldots + w_\ell$, then

$$C_A(v) = \bigcap_{i=1}^{k} C_A(w_i) = \bigcap_{i=1}^{k} C_A(W_i) = C_A(V) = 1.$$

Thus we may assume that $\ell = 1$, that is, V_C is homogeneous. Let X be the irreducible kC-module which appears in V_C and let $B = \langle \alpha \rangle$. Then we have $kB = k[\alpha - 1]$. Set $R_j = kB/\left\langle (\alpha - 1)^j \right\rangle$ for $j = 1, \ldots, p^n$, where $p^n = |\alpha|$. Note that R_j is an indecomposable kB-module of dimension j for each j and these are the only indecomposable kB-modules by Theorem VII.5.3 in [9]. Then the decomposition of the kA-module V into indecomposable kA-modules can be given as

$$V \cong (X \otimes R_{j_1}) \oplus \ldots \oplus (X \otimes R_{j_m}) \cong X \otimes \left(\bigoplus_{i=1}^{m} R_{j_i} \right)$$

for some $j_1, \ldots, j_m$ in $\{1, \ldots, p^n\}$. To simplify the notation we set $U = \bigoplus_{i=1}^{m} R_{j_i}$. The group C has a regular orbit on X by hypothesis, that is, there is $x \in X$ such that $C_C(x) = C_C(X) = 1$. We shall observe that B has a regular orbit on U: As a consequence of the faithful action of A on V, B acts faithfully on U. Hence there is at least one indecomposable component, say R_{j_i}, on which B acts faithfully, since B is cyclic. Let

$$R_{j_i} = U_1 \supset U_2 \supset \ldots \supset U_s = 0$$

be a B-composition series of $R_{j_i} = U_1$. Each factor U_i/U_{i+1}, $i = 1, \ldots, s-1$, is isomorphic to the trivial module of dimension 1. Hence $s - 1 = \dim U_1 = j_1$ and $\left[U_1, \underbrace{\alpha, \ldots, \alpha}_{j_1-\text{times}} \right] = 0$. It follows that $\dim U_1 = j_1 \geq p^{n-1} + 1$, because otherwise $(\alpha - 1)^{p^{n-1}} = 0$ on U_1 and, hence, $\alpha^{p^{n-1}}$ is trivial on U_1, a contradiction. Pick an element u from $U_1 - U_2$. If $C_B(u) \neq 1$, then $\alpha^{p^{n-1}}$ acts trivially on u, whence the degree j_1 of the minimum polynomial of α on U_1 is at most p^{n-1}. But then $p^{n-1} + {} + 1 \leq j_1 \leq p^{n-1}$, which is impossible. This yields that $C_B(u) = 1 = C_B(U)$. As a consequence, B has a regular orbit on U. We are now ready to complete the proof of the theorem. Let $a \in C_A(x \otimes u)$. Then $a = b + c$ for some $b \in B$ and $c \in C$. As $c \in \langle a \rangle$, we have $(x \otimes u)c = xc \otimes u = x \otimes u$ and hence $xc = x$. That is, $c \in C_C(x) = C_C(X)$. Similarly, we observe that $b \in C_B(u) = C_A(U)$. Consequently, we have $a \in C_A(X \otimes U)$ and, hence, the equality $C_A(x \otimes u) = C_A(X \otimes U)$ holds. It follows that A has regular orbit on V, as claimed.

The proposition is proved.

Remark. The above proposition cannot be extended to Abelian $O_p(A)$ as the following example shows: Let V be an elementary Abelian group of order p^3 with a basis $\{v_1, v_2, v_3\}$ and A an elementary Abelian group of order p^2 of automorphisms of V generated by $\{a_1, a_2\}$ with the action $v_1^{a_1} = v_1^{a_2} = v_1$, $v_2^{a_1} = v_1 v_2$, $v_2^{a_2} = v_2$, $v_3^{a_1} = v_3$, $v_3^{a_2} = v_3 v_1$. Then every A-orbit on V has length dividing p.

3. Proof of theorem. Let (P, R, χ) be a counterexample with $|PR| + \chi(1)$ minimum. We shall proceed in a series of steps. To simplify the notation we set $G = PR$.

(1) *χ is irreducible.*

There exists an irreducible constituent χ_1 of χ which does not contain $Z(P)$ in its kernel, that is $(\chi_1)_P$ is faithful. Then we have $\chi_1 = \chi$ because otherwise χ_1 contains the regular A-character by induction.

(2) *χ_P is homogeneous and $R_0 = 1$.*

As it is well-known the irreducible characters of the extraspecial group P are uniquely determined by their restriction $Z(P)$ so that $\chi_P = e\theta$ for some faithful irreducible GA-invariant character θ of P and some positive integer e, since $Z(P) \leq Z(GA)$. The coprimeness condition $(|P|, |RA_{p'}|) = 1$ enables us to extend θ in a unique way to an irreducible character $\overline{\theta}$ of $GA_{p'}$ such that $\det(\overline{\theta})(x) = 1$ for each $x \in RA_{p'}$ by [10] (8.16). On the other hand $\theta_1 = \theta \times 1_{R_0}$ is an irreducible $P \times R_0$-character with $R_0 \leq \operatorname{Ker} \theta_1$. We can extend θ_1 uniquely to $\overline{\theta}_1 \in Irr(GA_{p'}/R_0)$ with $\det(\overline{\theta}_1)(x) = 1$ for each $x \in RA_{p'}/R_0$. The uniqueness of this extension implies $R_0 \leq \operatorname{Ker} \overline{\theta}$. Notice that $(\overline{\theta}_1)_P = \theta = \overline{\theta}_P$

and also that the set $\{\varphi : \varphi \in \mathrm{Irr}(GA_{p'})$ such that $\varphi_P = \theta\}$ is A_p-invariant, because $\theta^a = \theta$ for each $a \in A_p$. Since $\det(\overline{\theta}^a)(x) = 1$ for each $a \in A_p$, the uniqueness of $\overline{\theta}$ gives $\overline{\theta}^a = \overline{\theta}$. It follows from [10] (Corollary 11.22) that $\overline{\theta}$ is extendible to an irreducible GA-character, say $\overline{\overline{\theta}}$. Now $\overline{\overline{\theta}}_G = \overline{\theta}$, $\overline{\overline{\theta}}_P = \theta$ and $R_0 \leq \mathrm{Ker}\,\overline{\theta} = G \cap \mathrm{Ker}\,\overline{\overline{\theta}}$. If $\overline{\overline{\theta}}(1) < \chi_1$ or $R_0 \neq 1$, by induction applied to the group GA/R_0 over $\overline{\overline{\theta}}$ we see that $\overline{\overline{\theta}}_A$ contains the regular A-character. Since χ is a constituent of $\overline{\overline{\theta}}_P|^{GA}$, there exists $\beta \in \mathrm{Irr}(GA/P)$ such that $\chi = \overline{\overline{\theta}} \cdot \beta$ by [10] (6.17) and hence $\chi_A = \overline{\overline{\theta}}_A \cdot \beta_A$. We conclude that χ_A contains the regular A-character, while $\overline{\overline{\theta}}_A$ does. Therefore without loss of generality we may assume that $R_0 = 1$ as claimed.

(3) *Theorem follows.*

Theorem 1.3 in [11] applied to the group PR over χ shows that one of the following holds:

(i) χ_R contains the regular R-character;

(ii) $p = 2$ and r is a Fermat prime.

By hypothesis (d) we see that (i) follows, that is χ_R contains a copy of every irreducible R-character. On the other hand we can regard $\mathrm{Irr}(R/\Phi(R))$ as a faithful $\mathbb{F}_r(A)$-module which is isomorphic to $R/\Phi(R)$ and hence apply the proposition above to get a linear character ν of R such that $C_A(\nu) = 1$. Let V be a GA-module affording χ and let W be the homogeneous component of V_R corresponding to ν. Since the stabilizer in A of W is trivial, V_A contains the regular A-module. Therefore, χ_A contains the regular A-character.

The theorem is proved.

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