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# ON THE FOURIER SINE AND KONTOROVICH – LEBEDEV GENERALIZED CONVOLUTION TRANSFORMS AND APPLICATIONS

# ПРО СИНУС-ПЕРЕТВОРЕННЯ ФУР'Є І ПЕРЕТВОРЕННЯ КОНТОРОВИЧА – ЛЕБЕДЄВА УЗАГАЛЬНЕНИХ ЗГОРТОК ТА ЇХ ЗАСТОСУВАННЯ

We study a generalized convolutions for the Fourier sine and Kontorovich – Lebedev transforms $(h \underset{F_s,K}{*} f)(x)$ in a two-parameter function space $L_p^{\alpha,\beta}(\mathbb{R}_+)$. We obtain several estimates for the norms and prove a Young-type inequality for this generalized convolution.

We impose necessary and sufficient conditions on the kernel $h$ to ensure that the generalized convolution transform

$$D_h : f \mapsto D_h[f] = \left(1 - \frac{d^2}{dx^2}\right)(h \underset{F_s,K}{*} f)(x)$$

is a unitary operator in $L_2(\mathbb{R}_+)$ (Watson-type theorem) and derive its inverse formula. Finally, we apply these results to an integrodifferential equation and obtain an estimate for the solution in the $L_p$-norm.

Вивчається узагальнена згортка для синус-перетворення Фур'є і перетворення Конторовича – Лебедєва $(h \underset{F_s,K}{*} f)(x)$ у двопараметричному просторі функцій $L_p^{\alpha,\beta}(\mathbb{R}_+)$. Отримано кілька оцінок для норм і встановлено нерівність типу Юнга для цієї узагальненої згортки. Введено необхідні та достатні умови для ядра $h$, за яких перетворення узагальненої згортки

$$D_h : f \mapsto D_h[f] = \left(1 - \frac{d^2}{dx^2}\right)(h \underset{F_s,K}{*} f)(x)$$

— це унітарний оператор в $L_2(\mathbb{R}_+)$ (теорема типу Ватсона). Отримано формулу для оберненого перетворення. Крім того, ці результати застосовано до інтегро-диференціального рівняння та отримано оцінку для його розв'язку в $L_p$-нормі.

**1. Introduction.** The Kontorovich – Lebedev integral transform was introduced by M. J. Kontorovich and N. N. Lebedev during 1938 – 1939 (see [8, 14])

$$(Kf)(y) = \frac{2}{\pi^2}\int\limits_0^\infty K_{iy}(x)f(x)\frac{dx}{x}, \qquad y > 0.$$

Here, the transform kernel contains the Macdonald function $K_\nu(x)$ (see [2]) of the pure imaginary index $\nu = iy$. There are several integral representations for the Macdonald function, and the following one is very useful subsequently [2, 8, 17]:

$$K_{iy}(x) = \int\limits_0^\infty e^{-x\cosh u}\cos yu\, du, \qquad x > 0. \tag{1.1}$$

The inverse Kontorovich – Lebedev transform is of the form [8, 14]

$$f(x) = K^{-1}[g](x) = \int\limits_0^\infty y\sinh(\pi y)K_{iy}(x)g(y)\, dy.$$

Here, $g(y) = (Kf)(y)$.

A generalized convolution for the Fourier sine and the Kontorovich – Lebedev integral transforms has been studied in [12]:

$$\big(h \underset{F_s,K}{*} f\big)(x) = \frac{1}{\pi^2} \int\limits_{\mathbb{R}_+^2} \frac{1}{u} \left[e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}\right] h(u)f(v)\,dudv, \qquad x > 0. \tag{1.2}$$

Here, the Fourier sine integral transform is defined by [5, 11]

$$(F_s f)(y) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(x) \sin xy\,dx.$$

The existence of the generalized convolution (1.2) for two functions in $L_1(\mathbb{R}_+)$ with weight and its application to solving integral equations of generalized convolution type were studied in [12]. Namely, for $h(x) \in L_1\Big(\mathbb{R}_+, x^{-3/2}\,dx\Big), f(x) \in L_1(\mathbb{R}_+)$, the following factorization equality holds (see [12]):

$$F_s(h \underset{F_s,K}{*} f)(y) = (Kh)(y)(F_s f)(y) \qquad \forall\, y > 0. \tag{1.3}$$

In any convolution $h * f$ of two functions $h$ and $f$, if we fix a function $h$ and let $f$ vary in a certain function space, then we can define convolution transforms of the form $f \to D(h * f)$, where $D$ is a certain (differential) operator. The most well-known integral transforms constructed by that way are the Watson transforms that are related to the Mellin convolution and the Mellin transform [11]

$$f(x) \longmapsto g(x) = \int\limits_0^\infty k(xy)f(y)\,dy.$$

Recently, several authors have been interested in the convolution transforms of this type [3, 7, 13, 15]. In this paper, we will study the transforms $f \to D(h \underset{F_s,K}{*} f)$, where $h \underset{F_s,K}{*} f$ is the generalized convolution (1.2). The case $D$ is the identity operator is considered in Section 2, where we study operator properties for the generalized convolution (1.2) in the two parameter Lebesgue space $L_p^{\alpha,\beta}(\mathbb{R}_+)$. In particular, we obtain the Young theorem and the Young inequality for this generalized convolution. In Section 3, for the differential operators $D = I - \dfrac{d^2}{dx^2}$, we derive a necessary and sufficient condition such that the corresponding transforms are unitary on $L_2(\mathbb{R}_+)$, and we draw the inverse transforms (a Watson-type theorem). Finally, in Section 4, we obtain the solution in closed form of an integrodifferential equation related to the generalized convolution (1.2), and an $L_p$-norm estimate of the solution with respect to the data.

**2. Generalized convolution operator properties.** In this section, we will prove several norm properties of the generalized convolution (1.2). Throughout the paper, we are interested in the following family of two parameter Lebesgue spaces.

**Definition 2.1** [16]. *For* $\alpha \in \mathbb{R},\ 0 < \beta \leq 1,$ *we denote by* $L_p^{\alpha,\beta}(\mathbb{R}_+)$ *the normed space of all measurable functions* $f(x)$ *on* $\mathbb{R}_+$ *such that*

$$\int\limits_0^\infty |f(x)|^p K_0(\beta x) x^\alpha \, dx < \infty$$

*with the norm*

$$\|f\|_{L_p^{\alpha,\beta}(\mathbb{R}_+)} = \left( \int\limits_0^\infty |f(x)|^p K_0(\beta x) x^\alpha \, dx \right)^{\frac{1}{p}}.$$

The boundedness of the generalized convolution (1.2) on the space $L_1(\mathbb{R}_+)$ is shown in the following theorem.

**Theorem 2.1.** *Let* $h \in L_1^{-1,\beta}(\mathbb{R}_+)$ *and* $f \in L_1(\mathbb{R}_+),\ 0 < \beta < 1.$ *Then the generalized convolution* (1.2) *exists for almost all* $x > 0,$ *belongs to* $L_1(\mathbb{R}_+)$, *and the following estimate holds:*

$$\|(h \underset{F_s,K}{*} f)\|_{L_1(\mathbb{R}_+)} \leq \frac{2}{\pi^2} \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+)} \|f\|_{L_1(\mathbb{R}_+)}.$$

*Moreover, the factorization property* (1.3) *holds true. Furthermore, convolution* (1.2) *belongs to* $C_0^1(\mathbb{R}_+)$, *and the following Parseval-type equality takes place, for all* $x > 0$:

$$(h \underset{F_s,K}{*} f)(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty (Kh)(y)(F_s f)(y) \sin xy \, dy. \tag{2.1}$$

***Proof.*** By using formula (1.1), we obtain

$$\frac{1}{2} \int\limits_0^\infty (e^{-u\cosh(x+v)} + e^{-u\cosh(x-v)}) \, dx = K_0(u). \tag{2.2}$$

Recalling that $K_0(u) \leq K_0(\beta u),\ 0 < \beta \leq 1$ [14], we have

$$\|(h \underset{F_s,K}{*} f)\|_{L_1(\mathbb{R}_+)} \leq \frac{2}{\pi^2} \int\limits_{\mathbb{R}_+^2} \frac{|h(u)|}{u} K_0(u) |f(v)| \, dudv \leq$$

$$\leq \frac{2}{\pi^2} \int\limits_{\mathbb{R}_+^2} \frac{|h(u)|}{u} K_0(\beta u) |f(v)| \, dudv = \frac{2}{\pi^2} \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+)} \|f\|_{L_1(\mathbb{R}_+)}.$$

It shows that $(h \underset{F_s,K}{*} f)(x)$ belongs to $L_1(\mathbb{R}_+)$. We now prove the Parseval-type equality. By using formula 2.16.48.19 in [9]

$$\int\limits_0^\infty \cos by \, K_{iy}(u) dy = \frac{\pi}{2} e^{-u\cosh b},$$

we get

$$(h \underset{F_s,K}{*} f)(x) = \frac{1}{\pi^2} \int\limits_{\mathbb{R}_+^2} \frac{1}{u}[e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}]h(u)f(v)\,dudv =$$

$$= \frac{1}{\pi^2} \int\limits_{\mathbb{R}_+^3} \frac{2}{\pi}\frac{1}{u} h(u)f(v)K_{iy}(u)[\cos(x-v)y - \cos(x+v)y]\,dydudv =$$

$$= \frac{4}{\pi^3} \int\limits_{\mathbb{R}_+^3} \frac{1}{u} h(u)f(v)K_{iy}(u)\sin xy\ \sin vy\,dydudv.$$

By using the uniform estimate [14]

$$|K_{iy}(u)| \leq e^{-\delta y} K_0(u\cos\delta), \qquad 0 \leq \delta < \frac{\pi}{2},$$

with $\delta = \arccos\beta$, we have

$$\int\limits_{\mathbb{R}_+^3} \left| \frac{1}{u} h(u)f(v)K_{iy}(u)\sin xy\ \sin vy \right| dydudv \leq$$

$$\leq \int\limits_0^\infty \frac{1}{u}|h(u)|K_0(\beta u)\,du \int\limits_0^\infty |f(v)|\,dv \int\limits_0^\infty e^{-y\ \arccos\beta}\,dy =$$

$$= \frac{1}{\arccos\beta}\ \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+)}\ \|f\|_{L_1(\mathbb{R}_+)} < \infty.$$

It means that we can apply Fubini's theorem to obtain

$$(h \underset{F_s,K}{*} f)(x) = \frac{4}{\pi^3} \int\limits_{\mathbb{R}_+^3} \frac{1}{u} h(u)f(v)K_{iy}(u)\sin xy\ \sin vy\,dydudv =$$

$$= \sqrt{\frac{2}{\pi}} \int\limits_0^\infty \left( \frac{2}{\pi^2} \int\limits_0^\infty \frac{1}{u} K_{iy}(u)h(u)du \right) \left( \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f(v)\sin vy\,dv \right) \sin xy\,dy =$$

$$= \sqrt{\frac{2}{\pi}} \int\limits_0^\infty (Kh)(y)(F_s f)(y)\sin xy\,dy.$$

That is the Parseval identity (2.1). Since

$$|(Kh)(y)| \leq \|h\|_{L_1^{-1,\beta}(\mathbb{R}_+)} e^{-y\ \arccos\beta}, \qquad |(F_s f)(y)| \leq \|f\|_{L_1(\mathbb{R}_+)},$$

it follows that $(1+y)(Kh)(y)(F_s f)(y) \in L_1(\mathbb{R}_+)$. Thus, the Parseval identity (2.1) shows that $(h \underset{F_s,K}{*} f)(x)$ is the Fourier sine transform of a function from $L_1(\mathbb{R}_+)$, differentiable, and, therefore, belongs to $C_0^1(\mathbb{R}_+)$.

Theorem 2.1 is proved.

**Theorem 2.2.** *Let $1 < p < \infty$ be a real number and $q$ be its conjugate exponent, i.e., $1/p + 1/q = 1$. Then, for any $h \in L_p^{-p,\beta}(\mathbb{R}_+)$ and $f \in L_q(\mathbb{R}_+)$, the generalized convolution $h \underset{F_s,K}{*} f$ is a bounded function on $\mathbb{R}_+$. Moreover, $h \underset{F_s,K}{*} f$ belongs to $L_r^{\alpha,\gamma}(\mathbb{R}_+)$, $1 \le r < \infty$, $\alpha > -1$, $0 < \gamma \le 1$, and*

$$\|(h \underset{F_s,K}{*} f)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \le C_{\alpha,\gamma}^{1/r} \|h\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|f\|_{L_q(\mathbb{R}_+)}, \tag{2.3}$$

*where*

$$C_{\alpha,\gamma} = \frac{2^{r+\alpha-1}}{\pi^{2r}\gamma^{\alpha+1}} \Gamma^2\left(\frac{\alpha+1}{2}\right).$$

***Proof.*** By using the integral representation (2.2) for the function $K_0(u)$, Hölder's inequality, and the fact that $e^{-u\cosh(x+v)} + e^{-u\cosh(x-v)} \le 2e^{-u}$ for all positives $u$, $v$, and $x$, we get

$$|(h \underset{F_s,K}{*} f)(x)| \le \frac{1}{\pi^2} \int\limits_{\mathbb{R}_+^2} \left|\frac{h(u)}{u}\right| |f(v)| [e^{-u\cosh(x+v)} + e^{-u\cosh(x-v)}]\, dudv \le$$

$$\le \frac{1}{\pi^2} \left( \int\limits_{\mathbb{R}_+^2} \left|\frac{h(u)}{u}\right|^p [e^{-u\cosh(x+v)} + e^{-u\cosh(x-v)}]\, dudv \right)^{\frac{1}{p}} \times$$

$$\times \left( \int\limits_{\mathbb{R}_+^2} |f(v)|^q [e^{-u\cosh(x+v)} + e^{-u\cosh(x-v)}]\, dudv \right)^{\frac{1}{q}} \le$$

$$\le \frac{2}{\pi^2} \left( \int\limits_0^\infty \left|\frac{h(u)}{u}\right|^p K_0(u)\, du \right)^{\frac{1}{p}} \|f\|_{L_q(\mathbb{R}_+)}.$$

Therefore, the generalized convolution is a bounded function. Moreover, in view of formula (2.16.2.2) in [9] we get

$$\|(h \underset{F_s,K}{*} f)\|_{L_r^{\alpha,\gamma}(\mathbb{R}_+)} \le \frac{2}{\pi^2} \|h\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|f\|_{L_q(\mathbb{R}_+)} \left( \int\limits_0^\infty x^\alpha K_0(\gamma x)\, dx \right)^{\frac{1}{r}} =$$

$$= \frac{2}{\pi^2} (2\gamma)^{-1/r} \left(\frac{\gamma}{2}\right)^{-\alpha/r} \Gamma^{2/r}\left(\frac{\alpha+1}{2}\right) \|h\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|f\|_{L_q(\mathbb{R}_+)}, \qquad \alpha > -1.$$

It yields (2.3).

Theorem 2.2 is proved.

By a similar argument as in the proof of Theorem 2.1, one can easily prove the following lemma.

**Lemma 2.1.** *Let $h \in L_2^{-2,\beta}(\mathbb{R}_+)$, $0 < \beta < 1$, and $f \in L_2(\mathbb{R}_+)$. Then the generalized convolution* (1.2) *satisfies the factorization equality* (1.3). *Furthermore, the following generalized Parseval identity holds*:

$$(h \underset{F_s,K}{*} f)(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty (Kh)(y)(F_s f)(y) \sin xy \, dy, \tag{2.4}$$

*where the integral is understood in* $L_2(\mathbb{R}_+)$ *norm, if necessary.*

Next, we will prove a Young-type theorem for the generalized convolution (1.2).

**Theorem 2.3** (Young-type theorem)**.** *Let* $p, q, r$ *be real numbers in* $(1, \infty)$ *such that* $1/p + 1/q + 1/r = 2$ *and let* $f \in L_p^{-p,\beta}(\mathbb{R}_+), 0 < \beta < 1,\ g \in L_q(\mathbb{R}_+),\ h \in L_r(\mathbb{R}_+)$. *Then*

$$\left| \int\limits_0^\infty (f \underset{F_s,K}{*} g)(x)\, h(x)\, dx \right| \le \frac{2^{\frac{p-1}{p}}}{\pi^2} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}.$$

***Proof.*** Let $p_1, q_1, r_1$ be the conjugate exponents of $p, q, r,$ respectively, it means

$$\frac{1}{p} + \frac{1}{p_1} = \frac{1}{q} + \frac{1}{q_1} = \frac{1}{r} + \frac{1}{r_1} = 1.$$

Then $\dfrac{1}{p_1} + \dfrac{1}{q_1} + \dfrac{1}{r_1} = 1$. Put

$$F(x, u, v) = |g(v)|^{\frac{q}{p_1}} |h(x)|^{\frac{r}{p_1}} |e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}|^{\frac{1}{p_1}},$$

$$G(x, u, v) = \left| \frac{f(u)}{u} \right|^{\frac{p}{q_1}} |h(x)|^{\frac{r}{q_1}} |e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}|^{\frac{1}{q_1}},$$

$$H(x, u, v) = \left| \frac{f(u)}{u} \right|^{\frac{p}{r_1}} |g(v)|^{\frac{q}{r_1}} |e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}|^{\frac{1}{r_1}}.$$

We have

$$F(x, u, v) G(x, u, v) H(x, u, v) = \left| \frac{f(u)}{u} \right| |g(v)| |h(x)| |e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}|. \tag{2.5}$$

Furthermore, in the space $L_{p_1}(\mathbb{R}_+^3)$ we obtain

$$\|F\|^{p_1}_{L_{p_1}(\mathbb{R}^3_+)} = \int\limits_{\mathbb{R}^3_+} |g(v)|^q |h(x)|^r |e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}| \, dudvdx \le$$

$$\le 2 \int\limits_{\mathbb{R}^3_+} |g(v)|^q |h(x)|^r e^{-u} \, dudvdx =$$

$$= 2 \|g\|^q_{L_q(\mathbb{R}_+)} \|h\|^r_{L_r(\mathbb{R}_+)}. \tag{2.6}$$

On the other hand, by the fact that $K_0(u) \le K_0(\beta u)$, for $0 < \beta < 1$,

$$\|G\|^{p_1}_{L_{q_1}(\mathbb{R}^3_+)} = \int\limits_{\mathbb{R}^3_+} \left| \frac{f(u)}{u} \right|^p |h(x)|^r |e^{-u\cosh(x+v)} - e^{-u\cosh(x-v)}| \, dudvdx \le$$

$$\leq \int\limits_{\mathbb{R}_+^2} \left|\frac{f(u)}{u}\right|^p K_0(\beta u)|h(x)|^r\, dudx =$$

$$= \|f\|^p_{L_p^{-p,\beta}(\mathbb{R}_+)} \|h\|^r_{L_r(\mathbb{R}_+)}, \tag{2.7}$$

and, similarly,

$$\|H\|^{r_1}_{L_{r_1}(\mathbb{R}_+^3)} = \int\limits_{\mathbb{R}_+^3} \left|\frac{f(u)}{u}\right|^p |g(v)|^q |e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}|\, dudvdx \leq$$

$$\leq \int\limits_{\mathbb{R}_+^2} \left|\frac{f(u)}{u}\right|^p K_0(\beta u)|g(v)|^r\, dudv =$$

$$= \|f\|^p_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|^q_{L_q(\mathbb{R}_+)}. \tag{2.8}$$

Hence, from (2.6), (2.7) and (2.8), we have

$$\|F\|_{L_{p_1}(\mathbb{R}_+^3)} \|G\|_{L_{q_1}(\mathbb{R}_+^3)} \|H\|_{L_{r_1}(\mathbb{R}_+^3)} \leq 2^{\frac{p-1}{p}} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}. \tag{2.9}$$

From (2.5) and (2.9), by the three-function form of the Hölder inequality [1], we have

$$\left|\int\limits_0^\infty (f \underset{F_s,K}{*} g)(x)\, h(x)\, dx\right| \leq \frac{1}{\pi^2} \int\limits_{\mathbb{R}_+^3} \left|\frac{f(u)}{u}\right| |g(v)||h(x)||e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)}|\, dudvdx =$$

$$= \frac{1}{\pi^2} \int\limits_{\mathbb{R}_+^3} F(x,u,v) G(x,u,v) H(x,u,v)\, dudvdx \leq$$

$$\leq \frac{1}{\pi^2} \|F\|_{L_{p_1}(\mathbb{R}_+^3)} \|G\|_{L_{q_1}(\mathbb{R}_+^3)} \|H\|_{L_{r_1}(\mathbb{R}_+^3)} \leq$$

$$\leq \frac{2^{\frac{p-1}{p}}}{\pi^2} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)} \|h\|_{L_r(\mathbb{R}_+)}.$$

Theorem 2.3 is proved.

The following Young-type inequality is a direct corollary of the above theorem.

***Corollary* 2.1** (Young-type inequality). *Let $1 < p, q, r < \infty$ be such that $1/p + 1/q = 1 + 1/r$ and let $f \in L_p^{-p,\beta}(\mathbb{R}_+), 0 < \beta < 1,\ g \in L_q(\mathbb{R}_+)$. Then the generalized convolution* (1.2) *is well-defined in $L_r(\mathbb{R}_+)$. Moreover, the following inequality holds*:

$$\|(f \underset{F_s,K}{*} g)\|_{L_r(\mathbb{R}_+)} \leq \frac{2^{\frac{p-1}{p}}}{\pi^2} \|f\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)}. \tag{2.10}$$

**3. A Watson-type theorem.** An important class of integral transforms is unitary transforms. In this section, for $D = I - \dfrac{d^2}{dx^2}$, we give a necessary and sufficient condition for a kernel $h$ such that the generalized convolution transform

$$D_h\colon \quad f \mapsto g = D_h[f] = \left(1 - \frac{d^2}{dx^2}\right)(h \underset{F_s,K}{*} f)(x)$$

is a unitary operator in $L_2(\mathbb{R}_+)$, and derive its inverse formula.

**Theorem 3.1.** *Let $h \in L_2^{-2,\beta}(\mathbb{R}_+)$, $0 < \beta < 1$. Then the condition*

$$|(Kh)(y)| = \frac{1}{1+y^2} \tag{3.1}$$

*is necessary and sufficient to ensure that the transformation $f \to g$, given by formula*

$$g(x) = \frac{1}{\pi^2}\left(1 - \frac{d^2}{dx^2}\right)\int\limits_{\mathbb{R}_+^2} \frac{1}{u}(e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)})h(u)f(v)\,dudv \tag{3.2}$$

*is unitary in $L_2(\mathbb{R}_+)$. Moreover, the inverse transformation can be written in the conjugate symmetric form*

$$f(x) = \frac{1}{\pi^2}\left(1 - \frac{d^2}{dx^2}\right)\int\limits_{\mathbb{R}_+^2} \frac{1}{u}(e^{-u\cosh(x-v)} - e^{-u\cosh(x+v)})\overline{h}(u)f(v)\,dudv. \tag{3.3}$$

***Proof.*** *Sufficiency.* Suppose that the function $h$ satisfies condition (3.1). Applying Lemma 2.1, it is easy to see that the generalized convolution transform (3.2) can be written in the form

$$g(x) = \sqrt{\frac{2}{\pi}}\left(1 - \frac{d^2}{dx^2}\right)\int\limits_0^\infty (Kh)(y)(F_sf)(y)\sin xy\,dy,$$

or, equivalently,

$$g(x) = \left(1 - \frac{d^2}{dx^2}\right)F_s\Big[(Kh)(y)(F_sf)(y)\Big](x).$$

It is well-known that $h(y)$, $yh(y)$, $y^2h(y) \in L_2(\mathbb{R}_+)$ if and only if $(Fh)(x)$, $\dfrac{d}{dx}(Fh)(x)$, $\dfrac{d^2}{dx^2}(Fh)(x) \in L_2(\mathbb{R}_+)$ (Theorem 68 [11, p. 92]). Moreover,

$$\left(1 - \frac{d^2}{dx^2}\right)(F_sh)(x) = F_s\Big[(1+y^2)h(y)\Big](x). \tag{3.4}$$

Condition (3.1) shows that $(1+y^2)(Kh)(y)$ is bounded. Therefore $(1+y^2)(Kh)(y)(F_sf)(y) \in$ $\in L_2(\mathbb{R}_+)$, and formula (3.4) yields

$$g(x) = F_s\Big[(1+y^2)(Kh)(y)(F_sf)(y)\Big](x) \in L_2(\mathbb{R}_+).$$

Applying the Fourier sine transform to both sides of the above equation, we have

$$(F_s g)(y) = (1 + y^2)(Kh)(y)(F_s f)(y).$$

Besides, from the Plancherel theorem for the Fourier sine transform $\|F_s f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}$, and condition (3.1), it is easy to see that $\|f\|_{L_2(\mathbb{R}_+)} = \|g\|_{L_2(\mathbb{R}_+)}$, which implies that transform (3.2) is unitary. Again from condition (3.1) we obtain

$$(K\overline{h})(y)(F_s g)(y) = (F_s f)(y).$$

Thus, in the same manner as above it corresponds to (3.3) and the inversion formula of transform (3.2) follows.

*Necessity.* Suppose that transform (3.2) is unitary in $L_2(\mathbb{R}_+)$ and the inversion formula is defined by (3.3). Then, by using the Parseval-type identity (2.4), the Plancherel theorem for the Fourier sine transform, and formula 4.5.68 in [2], we obtain

$$\|g\|_{L_2(\mathbb{R}_+)} = \|(Kh)(y)(F_s f)(y)\|_{L_2(\mathbb{R}_+)} = \|F_s f\|_{L_2(\mathbb{R}_+)} = \|f\|_{L_2(\mathbb{R}_+)}.$$

The middle equality holds for all $f \in L_2(\mathbb{R}_+)$ if and only if $h$ satisfies the condition (3.1).

Theorem 3.1 is proved.

**4. A class of integrodifferential equations.** Not many integrodifferential equations can be solved in closed form despite their useful applications (see [4]). In particular, no applications of convolution type transforms for solving integrodifferential equations were found in recent investigations [3, 7, 13, 15]. In this section, we apply the Fourier sine and Kontorovich – Lebedev generalized convolution to investigate a class of integrodifferential equations, which seems to be difficult to be solved in closed form by using other techniques.

To introduce a class of integrodifferential equation, we recall the generalized convolution for the Fourier sine and Fourier cosine transforms, which is of the form (see [5])

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty [f(x+y) - f(|x-y|)] g(y)\, dy, \qquad x > 0. \tag{4.1}$$

For $f, g \in L_1(\mathbb{R}_+)$, we have $f \underset{1}{*} g \in L_1(\mathbb{R}_+)$, and the following factorization equality holds:

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)(F_c g)(y).$$

Here, the Fourier cosine transform is defined by [5, 11]

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos xy\, dx.$$

We consider the integrodifferential equation

$$\begin{gathered} f(x) - f''(x) + (D_h f)(x) = (h \underset{F_s,K}{*} g)(x), \\ f(0) = 0, \\ \lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0. \end{gathered} \tag{4.2}$$

Here, $h \in L_1^{-1,\beta}(\mathbb{R}_+)$, $0 < \beta < 1$, $g \in L_1(\mathbb{R}_+)$ are given functions, and $f \in C^2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$ is the unknown function.

In order to get a solution of the above problem, note that, for $f \in C^2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$, such that $f(0) = 0, \lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$, we have

$$(F_s f'')(y) = \sqrt{\frac{2}{\pi}} \int\limits_0^\infty f''(x) \sin xy \, dx =$$

$$= \sqrt{\frac{2}{\pi}} \left\{ f'(x) \sin xy \Big|_{x=0}^\infty - y \int\limits_0^\infty f'(x) \cos xy \, dx \right\} =$$

$$= -\sqrt{\frac{2}{\pi}} y \left\{ f(x) \cos xy \Big|_{x=0}^\infty + y \int\limits_0^\infty f(x) \sin xy \, dx \right\} = -y^2 (F_s f)(y). \tag{4.3}$$

**Lemma 4.1.** *Let* $f \in C_0^1(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$. *Then* $g(x) = (f(y) \underset{1}{*} e^{-y})(x)$ *is twice differentiable,* $g(0) = 0$, *and* $\lim_{x\to\infty} g(x) = \lim_{x\to\infty} g'(x) = 0$.

***Proof.*** We have

$$g(0) = \int\limits_0^\infty [f(0+y) - f(|0-y|)] \, e^{-y} \, dy = 0.$$

On the other hand,

$$g(x) = \int\limits_0^\infty [f(x+y) - f(|x-y|)] \, e^{-y} \, dy =$$

$$= \int\limits_0^\infty f(x+y) \, e^{-y} \, dy - \int\limits_0^x f(x-y) \, e^{-y} \, dy - \int\limits_x^\infty f(y-x) \, e^{-y} \, dy =$$

$$= e^x \int\limits_x^\infty f(u) \, e^{-u} \, du - e^{-x} \int\limits_0^x f(u) \, e^u \, du - e^{-x} \int\limits_0^\infty f(u) \, e^{-u} \, du =$$

$$= I_1(x) - I_2(x) - I_3(x). \tag{4.4}$$

Clearly, $I_3(x) \to 0$ as $x \to \infty$. For $I_1(x)$ we obtain

$$|I_1(x)| \le \int\limits_x^\infty |f(u) \, e^{-(u-x)}| \, du \le \int\limits_x^\infty |f(u)| \, du \to 0 \quad \text{as} \quad x \to \infty.$$

For any $\epsilon > 0$ choose $N$ large enough such that $\int_N^\infty |f(u)| \, du < \epsilon$. Then, for $x \to \infty$,

$$|I_2(x)| \leq e^{-x} \int\limits_0^N |f(u)|\, e^u\, du + \int\limits_N^x |f(u)|\, du \leq e^{-x} \int\limits_0^N |f(u)|\, e^u\, du + \epsilon \to \epsilon.$$

Thus, $I_2(x) \to 0$, and, therefore, $g(x) \to 0$ as $x \to \infty$.

Next, from (4.4) we get

$$g'(x) = e^x \int\limits_x^\infty f(u)\, e^{-u}\, du + e^{-x} \int\limits_0^x f(u)\, e^u\, du + e^{-x} \int\limits_0^\infty f(u)\, e^{-u}\, du - 2f(x) =$$

$$= I_1(x) + I_2(x) + I_3(x) - 2f(x). \tag{4.5}$$

Since $f \in C_0(\mathbb{R}_+)$, then $\lim_{x\to\infty} f(x) = 0$, and, therefore, $\lim_{x\to\infty} g'(x) = 0$. From formula (4.5) it is clear that $g$ is twice differentiable.

Lemma 4.1 is proved.

**Theorem 4.1.** *Suppose that the following condition holds:*

$$1 + (Kh)(y) \neq 0 \qquad \forall\, y > 0. \tag{4.6}$$

*Then problem* (4.2) *has a unique solution* $f \in C^2(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$:

$$f(x) = ((\ell \underset{F_s,K}{*} g) \underset{1}{*} m)(x).$$

*Here,* $m(x) = \sqrt{\dfrac{\pi}{2}} e^{-x}$ *and* $\ell \in L_1^{-1,\beta}(\mathbb{R}_+)$ *is defined by*

$$(K\ell)(y) = \frac{(Kh)(y)}{1 + (Kh)(y)},$$

*the generalized convolution* $(\cdot \underset{F_s,K}{*} \cdot)$ *and the convolution* $(\cdot \underset{1}{*} \cdot)$ *are defined by* (1.2), (4.1), *respectively.*

***Proof.*** Equation (4.2) can be rewritten in the form

$$f(x) - f''(x) + \left(1 - \frac{d^2}{dx^2}\right) \left\{(h \underset{F_s,K}{*} f)(x)\right\} = (h \underset{F_s,K}{*} g)(x). \tag{4.7}$$

Applying the Fourier sine transform to both sides of (4.7), and by virtue of the factorization equality (1.3) and formula (4.3), we obtain

$$(1 + y^2)(F_s f)(y) + (1 + y^2)(Kh)(y)(F_s f)(y) = (Kh)(y)(F_s g)(y),$$

or, equivalently,

$$(1 + y^2)(1 + (Kh)(y))(F_s f)(y) = (Kh)(y)(F_s g)(y).$$

From the condition (4.6) we get

$$(F_s f)(y) = \frac{1}{1 + y^2} \frac{(Kh)(y)}{1 + (Kh)(y)} (F_s g)(y).$$

By condition (4.6) the function $\varphi(y) = \dfrac{(Kh)(y)}{1+(Kh)(y)}$ satisfies conditions of the Wiener – Levy theorem for the Kontorovich – Lebedev transform [14], and, therefore, there exists a unique function $\ell \in L_1^{-1,\beta}(\mathbb{R}_+)$ such that

$$(K\ell)(y) = \frac{(Kh)(y)}{1+(Kh)(y)}.$$

Moreover, note that $\dfrac{1}{1+y^2} = (F_c m)(y)$ with $m(x) = \sqrt{\dfrac{\pi}{2}} e^{-x}$, we have

$$(F_s f)(y) = \sqrt{\frac{\pi}{2}}(F_c m)(y)(K\ell)(y)(F_s g)(y) =$$

$$= \sqrt{\frac{\pi}{2}}(F_c m)(y) F_s \left[ (\ell \underset{F_s,K}{*} g) \right](y) =$$

$$= \sqrt{\frac{\pi}{2}} F_s \left[ \left( \left( \ell \underset{F_s,K}{*} g \right) \underset{1}{*} m \right) \right](y).$$

This implies $f(x) = ((\ell \underset{F_s,K}{*} g) \underset{1}{*} m)(x)$. Since $\ell \in L_1^{-1,\beta}(\mathbb{R}_+)$ and $g \in L_1(\mathbb{R}_+)$, then by Theorem 2.1 we have $\ell \underset{F_s,K}{*} g \in C_0^1(\mathbb{R}_+) \cap L_1(\mathbb{R}_+)$. Together with $m \in L_1(\mathbb{R}_+)$ it yields $f = (\ell \underset{F_s,K}{*} g) \underset{1}{*} m \in$ $\in L_1(\mathbb{R}_+)$. Lemma 4.1 implies $f(0) = 0$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} f'(x) = 0$, and $f \in C^2(\mathbb{R}_+)$.

Theorem 4.1 is proved.

***Remark.*** For $p, q, r > 1$ satisfying $\dfrac{1}{p} + \dfrac{1}{q} = 1 + \dfrac{1}{r}$, the following inequality holds [10]:

$$\|(f \underset{1}{*} g)\|_{L_r(\mathbb{R}_+)} \le \|f\|_{L_p(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)}, \qquad f \in L_p(\mathbb{R}_+), \quad g \in L_q(\mathbb{R}_+).$$

Combining with inequality (2.10), if we assume that $\ell \in L_p^{-p,\beta}(\mathbb{R}_+)$, $g \in L_q(\mathbb{R}_+)$, $h \in L_r(\mathbb{R}_+)$, and $s > 1$, such that $\dfrac{1}{p} + \dfrac{1}{q} + \dfrac{1}{r} = \dfrac{1}{s} + 2$, we obtain an estimate for the solution of the problem (4.2) in the space $L_s(\mathbb{R}_+)$ as follows:

$$\|f\|_{L_s(\mathbb{R}_+)} = \left\| \left( (\ell \underset{F_s,K}{*} g) \underset{1}{*} m \right) \right\|_{L_s(\mathbb{R}_+)} \le \frac{2^{\frac{p-2}{2p}}}{\pi^{3/2} r^{1/r}} \|\ell\|_{L_p^{-p,\beta}(\mathbb{R}_+)} \|g\|_{L_q(\mathbb{R}_+)}.$$

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